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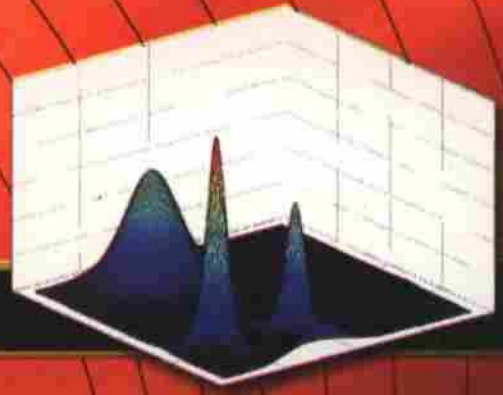
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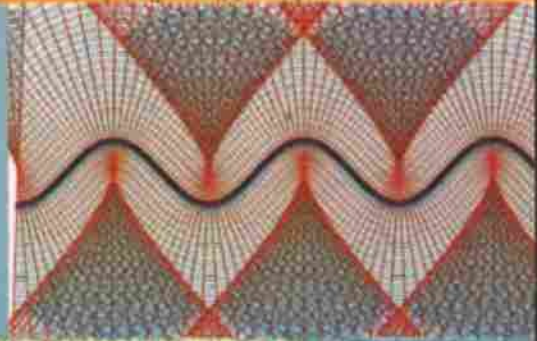
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Edition



Higher Engineering Mathematics

Dr. B.S. Grewal



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Differential Equations of First Order

1. Definitions. 2. Practical approach to differential equations. 3. Formation of a differential equation. 4. Solution of a differential equation—Geometrical meaning—5. Equations of the first order and first degree. 6. Variables separable. 7. Homogeneous equations. 8. Equations reducible to homogeneous form. 9. Linear equations. 10. Bernoulli's equation. 11. Exact equations. 12. Equations reducible to exact equations. 13. Equations of the first order and higher degree. 14. Clairut's equation. 15. Objective Type of Questions.

11.1 DEFINITIONS

(1) A **differential equation** is an equation which involves differential coefficients or differentials.

Thus (i) $e^x dx + e^y dy = 0$

(ii) $\frac{d^2x}{dt^2} + n^2x = 0$

(iii) $y = x \frac{dy}{dx} + \frac{x}{dy/dx}$

(iv) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \frac{d^2y}{dx^2} = c$

(v) $\frac{dx}{dt} - wy = a \cos pt, \frac{dy}{dt} + wx = a \sin pt$

(vi) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$

(vii) $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ are all examples of differential equations.

(2) An **ordinary differential equation** is that in which all the differential coefficients have reference to a single independent variable. Thus the equations (i) to (v) are all ordinary differential equations.

A **partial differential equation** is that in which there are two or more independent variables and partial differential coefficients with respect to any of them. Thus the equations (vi) and (vii) are partial differential equations.

(3) The **order** of a differential equation is the order of the highest derivative appearing in it.

The **degree** of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Thus, from the examples above,

(i) is of the first order and first degree ;

(ii) is of the second order and first degree ;

(iii) written as $y \frac{dy}{dx} = x \left(\frac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree ;

and (iv) written as $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = c^2 \left(\frac{d^2y}{dx^2}\right)^2$ is of the second order and second degree.

11.2 PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc., and as such play a very important role in all modern scientific and engineering studies.

The approach of an engineering student to the study of differential equations has got to be practical unlike that of a student of mathematics, who is only interested in solving the differential equations without knowing as to how the differential equations are formed and how their solutions are physically interpreted.

Thus for an applied mathematician, the study of a differential equation consists of three phases :

- (i) formulation of differential equation from the given physical situation, called **modelling**.
- (ii) solutions of this differential equation, evaluating the arbitrary constants from the given conditions, and
- (iii) physical interpretation of the solution.

11.3 FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. It will, however, be seen later that the partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

Example 11.1. Form the differential equation of simple harmonic motion given by $x = A \cos(nt + \alpha)$.

Solution. To eliminate the constants A and α differentiating it twice, we have

$$\frac{dx}{dt} = -nA \sin(nt + \alpha) \quad \text{and} \quad \frac{d^2x}{dt^2} = -n^2A \cos(nt + \alpha) = -n^2x$$

Thus
$$\frac{d^2x}{dt^2} + n^2x = 0$$

is the desired differential equation which states that the acceleration varies as the distance from the origin.

Example 11.2. Obtain the differential equation of all circles of radius a and centre (h, k) .

(Andhra, 1999)

Solution. Such a circle is $(x - h)^2 + (y - k)^2 = a^2$... (i)

where h and k , the coordinates of the centre, and a are the constants.

Differentiate it twice, we have

$$x - h + (y - k) \frac{dy}{dx} = 0 \quad \text{and} \quad 1 + (y - k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Then
$$y - k = -\frac{1 + (dy/dx)^2}{d^2y/dx^2}$$

and
$$x - h = -(y - k) \frac{dy}{dx} = \frac{\frac{dy}{dx} \left[1 + \left(\frac{dy}{dx}\right)^2 \right]}{d^2y/dx^2}$$

Substituting these in (i) and simplifying, we get $[1 + (dy/dx)^2]^3 = a^2 (d^2y/dx^2)^2$... (ii)
as the required differential equation

Writing (ii) in the form
$$\frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = a,$$

it states that the radius of curvature of a circle at any point is constant.

Example 11.3. Obtain the differential equation of the coaxial circles of the system $x^2 + y^2 + 2ax + c^2 = 0$ where c is a constant and a is a variable. (J.N.T.U., 2003)

Solution. We have $x^2 + y^2 + 2ax + c^2 = 0$... (i)

Differentiating w.r.t. x , $2x + 2y \frac{dy}{dx} + 2a = 0$

or
$$2a = -2 \left(x + y \frac{dy}{dx} \right)$$

Substituting in (i), $x^2 + y^2 - 2 \left(x + y \frac{dy}{dx} \right) x + c^2 = 0$

or
$$2xy \frac{dy}{dx} = y^2 - x^2 + c^2$$

which is the required differential equation.

11.4 (1) SOLUTION OF A DIFFERENTIAL EQUATION

A *solution (or integral)* of a differential equation is a relation between the variables which satisfies the given differential equation.

For example, $x = A \cos (nt + \alpha)$... (1)

is a solution of $\frac{d^2x}{dt^2} + n^2x = 0$ [Example 11.1] ... (2)

The **general (or complete) solution** of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example, $x = A \cos (nt + \pi/4)$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad \dots (3)$$

are said to be linearly independent if $c_1 y_1 + c_2 y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be linearly dependent.

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), then their linear combination $c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants, is also a solution of (3).

Example 11.4. Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.

Solution. Let the general solution of the required differential equation be $y = c_1 e^x + c_2 x e^x$... (i)

Differentiating (i) w.r.t. x , we get

$$y_1 = c_1 e^x + c_2 (e^x + x e^x)$$

$\therefore y - y_1 = c_2 e^x$... (ii)

Again differentiating (ii) w.r.t. x , we obtain

$$y_1 - y_2 = c_2 e^x \quad \dots (iii)$$

Subtracting (iii) from (ii), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \quad \dots (1)$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of $dy/dx (= m)$ when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0 . Take a neighbouring

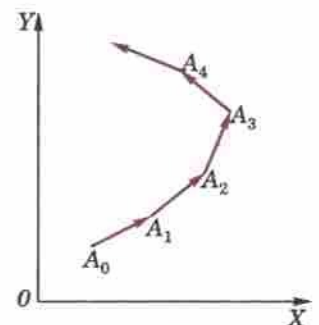


Fig. 11.1

point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0 . Let the corresponding value of m at A_1 be m_1 . Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points $A_0, A_1, A_2, A_3 \dots$ are chosen very near one another, the broken curve $A_0A_1A_2A_3 \dots$ approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point $A_0(x_0, y_0)$. Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. The equation of each such curve is thus a **particular solution** of the differential equation (1). The equation of the whole family of such curves is the **general solution** of (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

PROBLEMS 11.1

Form the differential equations from the following equations :

- 1. $y = ax^3 + bx^2$, (Bhopal, 2008)
- 2. $y = C_1 \cos 2x + C_2 \sin 2x$, (P.T.U., 2003)
- 3. $xy = Ae^x + Be^{-x} + x^2$. (U.P.T.U., 2005)
- 4. $y = e^x (A \cos x + B \sin x)$.
- 5. $y = ae^{2x} + be^{-3x} + ce^x$.

Find the differential equations of :

- 6. A family of circles passing through the origin and having centres on the x-axis. (J.N.T.U., 2006)
- 7. All circles of radius 5, with their centres on the y-axis.
- 8. All parabolas with x-axis as the axis and $(a, 0)$ as focus.
- 9. If $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ are two solutions of $y'' + 4y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.
- 10. Determine the differential equation whose set of independent solutions is $\{e^x, xe^x, x^2e^x\}$ (U.P.T.U., 2002)
- 11. Obtain the differential equation of the family of parabolas $y = x^2 + c$ and sketch those members of the family which pass through $(0, 0)$, $(1, 1)$, $(0, 1)$ and $(1, -1)$ respectively.

11.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations :

- (i) Equations where variables are separable, (ii) Homogeneous equations,
- (iii) Linear equations, (iv) Exact equations.

In other cases, the particular solution may be determined numerically (Chapter 31).

11.6 VARIABLES SEPARABLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable*. Thus the general form of such an equation is $f(y) dy = \phi(x) dx$

Integrating both sides, we get $\int f(y) dy = \int \phi(x) dx + c$ as its solution.

Example 11.5. Solve $dy/dx = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$. (V.T.U., 2008)

Solution. Given equation is $x(2 \log x + 1) dx = (\sin y + y \cos y) dy$

Integrating both sides, $2 \int (\log x \cdot x + x) dx = \int \sin y dy + \int y \cos y dy + c$

or $2 \left[\left(\log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right) + \frac{x^2}{2} \right] = -\cos y + \left[y \sin y - \int \sin y \cdot 1 dy + c \right]$

or
$$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is $2x^2 \log x - y \sin y = c$.

Example 11.6. Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Solution. Given equation is $\frac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$ or $e^{2y} dy = (e^{3x} + x^2) dx$

Integrating both sides, $\int e^{2y} dy = \int (e^{3x} + x^2) dx + c$

or
$$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \quad \text{or} \quad 3e^{2y} = 2(e^{3x} + x^3) + 6c.$$

Example 11.7. Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

(V.T.U., 2005)

Solution. Putting $x+y=t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\frac{dt}{dx} - 1 = \sin t + \cos t$

or
$$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \int \frac{dt}{1 + \sin t + \cos t} + c$.

or
$$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c \quad \text{[Putting } t = 2\theta]$$

$$= \int \frac{2d\theta}{2 \cos^2 \theta + 2 \sin \theta \cos \theta} + c = \int \frac{\sec^2 \theta}{1 + \tan \theta} d\theta + c$$

$$= \log(1 + \tan \theta) + c$$

Hence the solution is $x = \log \left[1 + \tan \frac{1}{2}(x+y) \right] + c$.

Example 11.8. Solve $dy/dx = (4x+y+1)^2$, if $y(0) = 1$.

Solution. Putting $4x+y+1=t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 4$.

\therefore the given equation becomes $\frac{dt}{dx} - 4 = t^2$ or $\frac{dt}{dx} = 4 + t^2$

Integrating both sides, we get $\int \frac{dt}{4+t^2} = \int dx + c$

or
$$\frac{1}{2} \tan^{-1} \frac{t}{2} = x + c \quad \text{or} \quad \frac{1}{2} \tan^{-1} \left[\frac{1}{2}(4x+y+1) \right] = x + c$$

or
$$4x+y+1 = 2 \tan 2(x+c)$$

When $x=0, y=1 \quad \therefore \frac{1}{2} \tan^{-1}(1) = c$ i.e. $c = \pi/8$.

Hence the solution is $4x+y+1 = 2 \tan(2x + \pi/4)$.

Example 11.9. Solve $\frac{y}{x} \frac{dy}{dx} + \frac{x^2+y^2-1}{2(x^2+y^2)+1} = 0$.

(V.T.U., 2003)

Solution. Putting $x^2+y^2=t$, we get $2x + 2y \frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{y}{x} \frac{dy}{dx} = \frac{1}{2x} \frac{dt}{dx} - 1$.

Therefore the given equation becomes $\frac{1}{2x} \frac{dt}{dx} - 1 + \frac{t-1}{2t+1} = 0$

or
$$\frac{1}{2x} \frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x \, dx = \frac{2t+1}{t+2} dt$$

or
$$2x \, dx = \left(2 - \frac{3}{t+2} \right) dt$$

Integrating, we get $x^2 = 2t - 3 \log(t+2) + c$

or
$$x^2 + 2y^2 - 3 \log(x^2 + y^2 + 2) + c = 0$$

$[\because t = x^2 + y^2]$

which is the required solution.

PROBLEMS 11.2

Solve the following differential equations :

1. $y \sqrt{1-x^2} \, dy + x \sqrt{1-y^2} \, dx = 0.$

2. $(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0.$

3. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0. \quad (P.T.U., 2003)$

4. $\frac{y}{x} \frac{dy}{dx} = \sqrt{(1+x^2+y^2+x^2y^2)}. \quad (V.T.U., 2011)$

5. $e^x \tan y \, dx + (1-e^x) \sec^2 y \, dy = 0. \quad (V.T.U., 2009)$

6. $\frac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0. \quad (J.N.T.U., 2006)$

7. $x \frac{dy}{dx} + \cot y = 0$ if $y = \pi/4$ when $x = \sqrt{2}.$

8. $(xy^2 + x) \, dx + (yx^2 + y) \, dy = 0.$

9. $\frac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}.$

10. $y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right).$

11. $(x+1) \frac{dy}{dx} + 1 = 2e^{-y}. \quad (Madras, 2000 S)$

12. $(x-y)^2 \frac{dy}{dx} = a^2,$

13. $(x+y+1)^2 \frac{dy}{dx} = 1. \quad (Kurukshetra, 2005)$

14. $\sin^{-1}(dy/dx) = x+y \quad (V.T.U., 2010)$

15. $\frac{dy}{dx} = \cos(x+y+1) \quad (V.T.U., 2003)$

16. $\frac{dy}{dx} - x \tan(y-x) = 1.$

17. $x^4 \frac{dy}{dx} + x^3 y + \operatorname{cosec}(xy) = 0.$

11.7 HOMOGENEOUS EQUATIONS

are of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y (see page 205).

To solve a homogeneous equation (i) Put $y = vx$, then $\frac{dy}{dx} = v + x \frac{dv}{dx},$

(ii) Separate the variables v and x , and integrate.

Example 11.10. Solve $(x^2 - y^2) \, dx - xy \, dy = 0.$

Solution. Given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ which is homogeneous in x and $y.$... (i)

Put $y = vx$, then $\frac{dy}{dx} = v + x \frac{dv}{dx}.$ \therefore (i) becomes $v + x \frac{dv}{dx} = \frac{1 - v^2}{v}$

or
$$x \frac{dv}{dx} = \frac{1 - v^2}{v} - v = \frac{1 - 2v^2}{v}.$$

Separating the variables, $\frac{v}{1-2v^2} dv = \frac{dx}{x}$

Integrating both sides, $\int \frac{v dv}{1-2v^2} = \int \frac{dx}{x} + c$

$$\text{or} \quad -\frac{1}{4} \int \frac{-4v}{1-2v^2} dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4} \log(1-2v^2) = \log x + c$$

$$\text{or} \quad 4 \log x + \log(1-2v^2) = -4c \quad \text{or} \quad \log x^4(1-2v^2) = -4c \quad [\text{Put } v = y/x]$$

$$\text{or} \quad x^4(1-2y^2/x^2) = e^{-3c} = c'$$

Hence the required solution is $x^2(x^2 - 2y^2) = c'$.

Example 11.11. Solve $(x \tan y/x - y \sec^2 y/x) dx - x \sec^2 y/x dy = 0$.

(V.T.U., 2006)

Solution. The given equation may be rewritten as

$$\frac{dy}{dx} = \left(\frac{y}{x} \sec^2 \frac{y}{x} - \tan \frac{y}{x} \right) \cos^2 y/x \quad \dots(i)$$

which is a homogeneous equation. Putting $y = vx$, (i) becomes $v + x \frac{dv}{dx} = (v \sec^2 v - \tan v) \cos^2 v$

$$\text{or} \quad x \frac{dv}{dx} = v - \tan v \cos^2 v - v$$

Separating the variables $\frac{\sec^2 v}{\tan v} dv = -\frac{dx}{x}$

Integrating both sides $\log \tan v = -\log x + \log c$

$$\text{or} \quad x \tan v = c \quad \text{or} \quad x \tan y/x = c.$$

Example 11.12. Solve $(1 + e^{x/y}) dx + e^{x/y}(1 - x/y) dy = 0$.

(P.T.U., 2006; Rajasthan, 2005; V.T.U., 2003)

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1 - x/y)}{1 + e^{x/y}} \quad \dots(i)$$

which is a homogeneous equation. Putting $x = vy$ so that (i) becomes

$$v + y \frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y \frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v = -\frac{v+e^v}{1+e^v}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1+e^v}{v+e^v} dv = \frac{d(v+e^v)}{v+e^v}$$

Integrating both sides, $-\log y = \log(v+e^v) + c$

$$\text{or} \quad y(v+e^v) = e^{-c} \quad \text{or} \quad x + ye^{x/y} = c' \quad (\text{say})$$

which is the required solution.

PROBLEMS 11.3

Solve the following differential equations :

1. $(x^2 - y^2) dx = 2xy dy$

2. $(x^2y - 2xy^2) dx - (x^3 - 3x^2y) dy = 0$. (Bhopal, 2008)

3. $x^2y dx - (x^3 + y^3) dy = 0$. (V.T.U., 2010)

4. $y dx - x dy = \sqrt{x^2 + y^2} dx$. (Raipur, 2005)

5. $y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$

6. $(3xy - 2ay^2) dx + (x^2 - 2axy) dy = 0$. (S.V.T.U., 2009)

[Equations solvable like homogeneous equations: When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting $y/x = v$].

$$7. \frac{dy}{dx} = \frac{y}{x} + \sin \frac{y}{x}. \quad (\text{V.T.U., 2000 S})$$

$$8. ye^{xy} dx = (xe^{xy} + y^2) dy. \quad (\text{V.T.U., 2006})$$

$$9. xy (\log x/y) dx + [y^2 - x^2 \log (x/y)] dy = 0.$$

$$10. x dx + \sin^2 (y/x) (y dx - x dy) = 0.$$

$$11. x \cos \frac{y}{x} (y dx + x dy) = y \sin \frac{y}{x} (x dy - y dx).$$

11.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

$$\text{The equations of the form } \frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \quad \dots(1)$$

can be reduced to the homogeneous form as follows :

Case I. When $\frac{a}{a'} \neq \frac{b}{b'}$

Putting $x = X + h, y = Y + k$, (h, k being constants)
so that $dx = dX, dy = dY$, (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad \dots(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0$, and $a'h + b'k + c' = 0$

$$\text{so that } \frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

$$\text{or } h = \frac{bc' - b'c}{ab' - b'a}, k = \frac{ca' - c'a}{ab' - ba'} \quad \dots(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$ which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\frac{a}{a'} = \frac{b}{b'}$.

i.e., $ab' - b'a = 0$, the above method fails as h and k become infinite or indeterminate.

$$\text{Now } \frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \text{ (say)}$$

$\therefore a' = am, b' = bm$ and (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} \quad \dots(4)$$

Put $ax + by = t$, so that $a + b \frac{dy}{dx} = \frac{dt}{dx}$

$$\text{or } \frac{dy}{dx} = \frac{1}{b} \left(\frac{dt}{dx} - a \right) \therefore (4) \text{ becomes } \frac{1}{b} \left(\frac{dt}{dx} - a \right) = \frac{t + c}{mt + c'}$$

$$\text{or } \frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 11.13. Solve $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$. (Raipur, 2005)

Solution. Given equation is $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ [Case $\frac{a}{a'} \neq \frac{b}{b'}$] ... (i)

Putting $x = X + h$, $y = Y + k$, (h, k being constants) so that $dx = dX$, $dy = dY$, (i) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)} \quad \dots(ii)$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1$, $k = 3$.

\therefore (ii) becomes $\frac{dY}{dX} = \frac{Y + X}{Y - X}$ which is homogeneous in X and Y(iii)

\therefore put $Y = vX$, then $\frac{dY}{dX} = v + X \frac{dv}{dX}$

\therefore (iii) becomes $v + X \frac{dv}{dX} = \frac{v+1}{v-1}$ or $X \frac{dv}{dX} = \frac{v+1}{v-1} - v = \frac{1+2v-v^2}{v-1}$

or
$$\frac{v-1}{1+2v-v^2} dv = \frac{dX}{X}$$

Integrating both sides, $-\frac{1}{2} \int \frac{2-2v}{1+2v-v^2} dv = \int \frac{dX}{X} + c$.

or
$$-\frac{1}{2} \log(1+2v-v^2) = \log X + c$$

or
$$\log \left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2} \right) + \log X^2 = -2c$$

or
$$\log(X^2 + 2XY - Y^2) = -2c \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2c} = c' \quad \dots(iv)$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (iv) becomes

$$(x+1)^2 + 2(x+1)(y-3) - (y-3)^2 = c'$$

or $x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$ which is the required solution.

Example 11.14. Solve $(3y + 2x + 4) dx - (4x + 6y + 5) dy = 0$.

(Madras, 2000 S)

Solution. Given equation is $\frac{dy}{dx} = \frac{(2x+3y)+4}{2(2x+3y)+5}$...(i)

Putting $2x + 3y = t$ so that $2 + 3 \frac{dy}{dx} = \frac{dt}{dx}$ \therefore (i) becomes $\frac{1}{3} \left(\frac{dt}{dx} - 2 \right) = \frac{t+4}{2t+5}$

or
$$\frac{dt}{dx} = 2 + \frac{3t+12}{2t+5} = \frac{7t+22}{2t+5} \quad \text{or} \quad \frac{2t+5}{7t+22} dt = dx$$

Integrating both sides, $\int \frac{2t+5}{7t+22} dt = \int dx + c$

or
$$\int \left(\frac{2}{7} - \frac{9}{7} \cdot \frac{1}{7t+22} \right) dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49} \log(7t+22) = x + c$$

Putting $t = 2x + 3y$, we have $14(2x + 3y) - 9 \log(14x + 21y + 22) = 49x + 49c$

or $21x - 42y + 9 \log(14x + 21y + 22) = c'$ which is the required solution.

PROBLEMS 11.4

Solve the following differential equations :

1. $(x - y - 2) dx + (x - 2y - 3) dy = 0$. (Rajasthan, 2006)

2. $(2x + y - 3) dy = (x + 2y - 3) dx$. (V.T.U., 2009 S ; Madras, 2000)

3. $(2x + 5y + 1) dx - (5x + 2y - 1) dy = 0$. (J.N.T.U., 2000)

4. $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$.

5. $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

6. $(4x - 6y - 1) dx + (3y - 2x - 2) dy = 0$.

7. $(x + 2y)(dx - dy) = dx + dy$.

(Bhopal, 2002 S ; V.T.U., 2001)

11.9 LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus the standard form of a linear equation of the first order, commonly known as Leibnitz's linear equation,* is

$$\frac{dy}{dx} + Py = Q \quad \text{where, } P, Q \text{ are the functions of } x. \quad \dots(1)$$

To solve the equation, multiply both sides by $e^{\int Pdx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int Pdx} + y(e^{\int Pdx} P) = Qe^{\int Pdx} \quad \text{i.e.,} \quad \frac{d}{dx}(ye^{\int Pdx}) = Qe^{\int Pdx}$$

Integrating both sides, we get $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$ as the required solution.

Obs. The factor $e^{\int Pdx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that $\text{I.F.} = e^{\int Pdx}$
and the solution is $y(\text{I.F.}) = \int Q(\text{I.F.}) dx + c$.

Example 11.15. Solve $(x + 1) \frac{dy}{dx} - y e^{3x} (x + 1)^2$.

Solution. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x} (x + 1) \text{ which is Leibnitz's equation.} \quad \dots(i)$$

Here $P = -\frac{1}{x+1}$ and $\int Pdx = -\int \frac{dx}{x+1} = -\log(x+1) = \log(x+1)^{-1}$

\therefore I.F. = $e^{\int Pdx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$

Thus the solution of (1) is $y(\text{I.F.}) = \int [e^{3x} (x + 1)](\text{I.F.}) dx + c$

or $\frac{y}{x+1} = \int e^{3x} dx + c = \frac{1}{3} e^{3x} + c$ or $y = \left(\frac{1}{3} e^{3x} + c\right)(x + 1)$.

Example 11.16. Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right) \frac{dx}{dy} = 1$.

Solution. Given equation can be written as $\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \quad \dots(i)$

\therefore I.F. = $e^{\int x^{1/2} dx} = e^{2\sqrt{x}}$

Thus solution of (i) is $y(\text{I.F.}) = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} (\text{I.F.}) dx + c$

or $ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} dx + c$

or $ye^{2\sqrt{x}} = \int x^{-1/2} dx + c$ or $ye^{2\sqrt{x}} = 2\sqrt{x} + c$.

* See footnote p. 139.

Example 11.17. Solve $3x(1-x^2)y^2 \frac{dy}{dx} + (2x^2-1)y^3 = ax^3$

(Rajasthan, 2006)

Solution. Putting $y^3 = z$ and $3y^2 \frac{dy}{dx} = \frac{dz}{dx}$, the given equation becomes

$$x(1-x^2) \frac{dz}{dx} + (2x^2-1)z = ax^3, \quad \text{or} \quad \frac{dz}{dx} + \frac{2x^2-1}{x-x^3}z = \frac{ax^3}{x-x^3} \quad \dots(i)$$

which is Leibnitz's equation in z

$$\therefore \quad \text{I.F.} = \exp \left(\int \frac{2x^2-1}{x-x^3} dx \right)$$

$$\begin{aligned} \text{Now} \quad \int \frac{2x^2-1}{x-x^3} dx &= \int \left(-\frac{1}{x} - \frac{1}{2} \frac{1}{1+x} + \frac{1}{2} \cdot \frac{1}{1-x} \right) dx = -\log x - \frac{1}{2} \log(1+x) - \frac{1}{2} \log(1-x) \\ &= -\log [x\sqrt{(1-x^2)}] \end{aligned}$$

$$\therefore \quad \text{I.F.} = e^{-\log [x\sqrt{(1-x^2)}]} = [x\sqrt{(1-x^2)}]^{-1}$$

Thus the solution of (i) is

$$z(\text{I.F.}) = \int \frac{ax^3}{x-x^3} (\text{I.F.}) dx + c$$

$$\begin{aligned} \text{or} \quad \frac{z}{[x\sqrt{(1-x^2)}]} &= a \int \frac{x^3}{x(1-x^2)} \cdot \frac{1}{x\sqrt{(1-x^2)}} dx + c = a \int x(1-x^2)^{-3/2} dx \\ &= -\frac{a}{2} \int (-2x)(1-x^2)^{-3/2} dx + c = a(1-x^2)^{-1/2} + c \end{aligned}$$

Hence the solution of the given equation is

$$y^3 = ax + cx \sqrt{(1-x^2)}. \quad [\because z = y^3]$$

Example 11.18. Solve $y(\log y) dx + (x - \log y) dy = 0$.

(U.P.T.U., 2000)

Solution. We have $\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$... (i)

which is a Leibnitz's equation in x

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{y \log y} dy} = e^{\log(\log y)} = \log y$$

Thus the solution of (i) is $x(\text{I.F.}) = \int \frac{1}{y} (\text{I.F.}) dy + c$

$$x \log y = \int \frac{1}{y} \log y dy + c = \frac{1}{2} (\log y)^2 + c$$

$$\text{i.e.,} \quad x = \frac{1}{2} \log y + c (\log y)^{-1}.$$

Example 11.19. Solve $(1+y^2) dx = (\tan^{-1} y - x) dy$. (Bhopal, 2008 ; V.T.U., 2008 ; U.P.T.U., 2005)

Solution. This equation contains y^2 and $\tan^{-1} y$ and is, therefore, not a linear in y ; but since only x occurs, it can be written as

$$(1+y^2) \frac{dx}{dy} = \tan^{-1} y - x \quad \text{or} \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

which is a Leibnitz's equation in x .

$$\therefore \quad \text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Thus the solution is $x(\text{I.F.}) = \int \frac{\tan^{-1} y}{1+y^2} (\text{I.F.}) dy + c$

or
$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y} dy + c$$

$$= \int te^t dt + c = t \cdot e^t - \int 1 \cdot e^t dt + c$$

$$= t \cdot e^t - e^t + c = (\tan^{-1} y - 1) e^{\tan^{-1} y} + c$$

or
$$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Put $\tan^{-1} y = t$
 $\therefore \frac{dy}{1+y^2} = dt$

(Integrating by parts)

Example 11.20. Solve $r \sin \theta d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) dr = 0$.

Solution. Given equation can be rewritten as

$$\sin \theta \frac{d\theta}{dr} + \frac{1}{r} (1 - 2r^2) \cos \theta = -r^2 \quad \dots(i)$$

Put $\cos \theta = y$ so that $-\sin \theta d\theta/dr = dy/dr$

Then (i) becomes $-\frac{dy}{dr} + \left(\frac{1}{r} - 2r\right) y = -r^2$ or $\frac{dy}{dr} + \left(2r - \frac{1}{r}\right) y = r^2$

which is a Leibnitz's equation \therefore I.F. = $e^{\int (2r - 1/r) dr} = e^{r^2 - \log r} = \frac{1}{r} e^{r^2}$

Thus its solution is $y \left(\frac{1}{r} e^{r^2}\right) = \int r^2 \cdot e^{r^2} \cdot \frac{1}{r} dr + c$

or $y e^{r^2} / r = \frac{1}{2} \int e^{r^2} 2r dr + c = \frac{1}{2} e^{r^2} + c$

or $2e^{r^2} \cos \theta = re^{r^2} + 2cr$ or $r(1 + 2ce^{-r^2}) = 2 \cos \theta.$

PROBLEMS 11.5

Solve the following differential equations :

- | | |
|---|---|
| 1. $\cos^2 x \frac{dy}{dx} + y = \tan x.$ | 2. $x \log x \frac{dy}{dx} + y = \log x^2.$ (V.T.U., 2011) |
| 3. $2y' \cos x + 4y \sin x = \sin 2x$, given $y = 0$ when $x = \pi/3$. (V.T.U., 2003) | |
| 4. $\cosh x \frac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x.$ (J.N.T.U., 2003) | |
| 5. $(1 - x^2) \frac{dy}{dx} - xy = 1$ (V.T.U., 2010) | 6. $(1 - x^2) \frac{dy}{dx} + 2xy = x \sqrt{1 - x^2}$ (Nagpur, 2009) |
| 7. $\frac{dy}{dx} = \frac{x + y \cos x}{1 + \sin x}.$ | 8. $dr + (2r \cot \theta + \sin 2\theta) d\theta = 0.$ (J.N.T.U., 2003) |
| 9. $\frac{dy}{dx} + 2xy = 2e^{-x^2}$ (P.T.U., 2005) | 10. $(x + 2y^3) \frac{dy}{dx} = y.$ (Marathwada, 2008) |
| 11. $\sqrt{1 - y^2} dx = (\sin^{-1} y - x) dy.$ | 12. $ye^3 dx = (y^3 + 2xe^y) dy.$ |
| 13. $(1 + y^2) dx + (x - e^{-\tan^{-1} y}) dy = 0.$ (V.T.U., 2006) | 14. $e^{-y} \sec^2 y dy = dx + x dy.$ |

11.10 BERNOULLI'S EQUATION

The equation $\frac{dy}{dx} + Py = Qy^n$...(1)

where P, Q are functions of x , is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation*.

*Named after the Swiss mathematician *Jacob Bernoulli* (1654–1705) who is known for his basic work in probability and elasticity theory. He was professor at Basel and had amongst his students his youngest brother *Johann Bernoulli* (1667–1748) and his nephew *Niklaus Bernoulli* (1687–1759). Johann is known for his basic contributions to Calculus while Niklaus had profound influence on the development of Infinite series and probability. His son *Daniel Bernoulli* (1700–1782) is known for his contributions to kinetic theory of gases and fluid flow.

To solve (1), divide both sides by y^n , so that $y^{-n} \frac{dy}{dx} + Py^{1-n} = Q$... (2)

Put $y^{1-n} = z$ so that $(1-n)y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$.

\therefore (2) becomes $\frac{1}{1-n} \frac{dz}{dx} + Pz = Q$ or $\frac{dz}{dx} + P(1-n)z = Q(1-n)$,

which is Leibnitz's linear in z and can be solved easily.

Example 11.21. Solve $x \frac{dy}{dx} + y = x^3 y^6$.

Solution. Dividing throughout by xy^6 , $y^{-6} \frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$... (i)

Put $y^{-5} = z$, so that $-5y^{-6} \frac{dy}{dx} = \frac{dz}{dx}$ \therefore (i) becomes $-\frac{1}{5} \frac{dz}{dx} + \frac{z}{x} = x^2$

or $\frac{dz}{dx} - \frac{5}{x}z = -5x^2$ which is Leibnitz's linear in z (ii)

$$\text{I.F.} = e^{-\int (5/x) dx} = e^{-5 \log x} = e^{\log x^{-5}} = x^{-5}$$

\therefore the solution of (ii) is $z (\text{I.F.}) = \int (-5x^2) (\text{I.F.}) dx + c$ or $zx^{-5} = \int (-5x^2) x^{-5} dx + c$

or $y^{-5} x^{-5} = -5 \cdot \frac{x^{-2}}{-2} + c$ [$\because z = y^{-5}$]

Dividing throughout by $y^{-5} x^{-5}$, $1 = (2.5 + cx^2) x^3 y^5$ which is the required solution.

Example 11.22. Solve $xy(1 + xy^2) \frac{dy}{dx} = 1$.

(Nagpur, 2009)

Solution. Rewriting the given equation as

$$\frac{dx}{dy} - yx = y^3 x^2$$

and dividing by x^2 , we have

$$x^{-2} \frac{dx}{dy} - yx^{-1} = y^3$$
 ... (i)

Putting $x^{-1} = z$ so that $-x^{-2} \frac{dx}{dy} = \frac{dz}{dy}$ (i) becomes

$$\frac{dz}{dy} + yz = -y^3 \text{ which is Leibnitz's linear in } z.$$

Here $\text{I.F.} = e^{\int y dy} = e^{y^2/2}$

\therefore the solution is $z (\text{I.F.}) = \int (-y^3) (\text{I.F.}) dy + c$

or $ze^{y^2/2} = - \int y^2 \cdot e^{\frac{1}{2}y^2} \cdot y dy + c$ Put $\frac{1}{2} y^2 = t$
so that $y dy = dt$

$$= -2 \int t \cdot e^t dt + c$$
 [Integrate by parts]

$$= -2 [t \cdot e^t - \int 1 \cdot e^t dt] + c = -2 [te^t - e^t] + c = (2 - y^2) e^{y^2/2} + c$$

or $z = (2 - y^2) + ce^{-\frac{1}{2}y^2}$ or $1/x = (2 - y^2) + ce^{-\frac{1}{2}y^2}$.

Note. General equation reducible to Leibnitz's linear is $f'(y) \frac{dy}{dx} + Pf(y) = Q$... (A)

where P, Q are functions of x . To solve it, put $f(y) = z$.

Example 11.23. Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$. (V.T.U., 2011 ; Marathwada, 2008 ; J.N.T.U., 2005)

Solution. Dividing throughout by $\cos^2 y$, $\sec^2 y \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$

or $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$ which is of the form (A) above. ...(i)

\therefore put $\tan y = z$ so that $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$ \therefore (i) becomes $\frac{dz}{dx} + 2xz = x^3$.

This is Leibnitz's linear equation in z . \therefore I.F. = $e^{\int 2x dx} = e^{x^2}$

\therefore the solution is $ze^{x^2} = \int e^{x^2} x^3 dx + c = \frac{1}{2}(x^2 - 1)e^{x^2} + c$.

Replacing z by $\tan y$, we get $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$ which is the required solution.

Example 11.24. Solve $\frac{dz}{dx} + \left(\frac{z}{x}\right) \log z = \frac{z}{x}(\log z)^2$.

Solution. Dividing by z , the given equation becomes

$$\frac{1}{z} \frac{dz}{dx} + \frac{1}{x} \log z = \frac{1}{x} (\log z)^2 \quad \dots(i)$$

Put $\log z = t$ so that $\frac{1}{z} \frac{dz}{dx} = \frac{dt}{dx}$. \therefore (i) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2} \frac{dt}{dx} + \frac{1}{x} \cdot \frac{1}{t} = \frac{1}{x} \quad \dots(ii)$$

This being Bernoulli's equation, put $1/t = v$ so that (ii) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x} v = -\frac{1}{x}$$

This is Leibnitz's linear in v . \therefore I.F. = $e^{-\int 1/x dx} = 1/x$

\therefore the solution is $v \cdot \frac{1}{x} = -\int \frac{1}{x} \cdot \frac{1}{x} dx + c = \frac{1}{x} + c$

Replacing v by $1/\log z$, we get $(x \log z)^{-1} = x^{-1} + c$ or $(\log z)^{-1} = 1 + cx$ which is the required solution.

PROBLEMS 11.6

Solve the following equations :

- | | |
|--|--|
| 1. $\frac{dy}{dx} = y \tan x - y^2 \sec x$. (P.T.U., 2005) | 2. $r \sin \theta - \cos \theta \frac{dr}{d\theta} = r^2$. (V.T.U., 2005) |
| 3. $2xy' = 10x^3y^5 + y$. | 4. $(x^3y^2 + xy) dx = dy$. (B.P.T.U., 2005) |
| 5. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$. (Bhullai, 2005) | 6. $x(x - y) dy + y^2 dx = 0$. (I.S.M., 2001) |
| 7. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$. (Bhopal, 2009) | 8. $e^x \left(\frac{dy}{dx} + 1 \right) = e^x$. (V.T.U., 2009) |
| 9. $\sec^2 y \frac{dy}{dx} + x \tan y = x^3$. | 10. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$. (Sambalpur, 2002) |
| 11. $\frac{dy}{dx} = \frac{y}{x - \sqrt{xy}}$. (V.T.U., 2011) | 12. $(y \log x - 2) y dx - x dy = 0$. (V.T.U., 2006) |

11.11 EXACT DIFFERENTIAL EQUATIONS

(1) **Def.** A differential equation of the form $M(x, y) dx + N(x, y) dy = 0$ is said to be **exact** if its left hand member is the exact differential of some function $u(x, y)$ i.e., $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) **Theorem.** The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy = du \quad \dots(1)$$

where u is some function of x and y .

$$\text{But} \quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \dots(2)$$

\therefore equating coefficients of dx and dy in (1) and (2), we get $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\text{But} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{(Assumption)}$$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e., if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

$$\text{Then} \quad \frac{\partial}{\partial x} \left(\int Mdx \right) = \frac{\partial u}{\partial x}, \quad \text{i.e.,} \quad M = \frac{\partial u}{\partial x} \quad \left\{ \begin{array}{l} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ (given)} \\ \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \end{array} \right. \quad \dots(3)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right)$$

Integrating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \quad \text{where } f(y) \text{ is a function of } y \text{ alone.} \quad \dots(4)$$

$$\therefore \quad Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{ \frac{\partial u}{\partial y} + f(y) \right\} dy \quad \text{[By (3) and (4)]}$$

$$= \left\{ \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right\} + f(y) dy = du + f(y) dy = d \left[u + \int f(y) dy \right] \quad \dots(5)$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) **Method of solution.** By (5), the equation $Mdx + Ndy = 0$ becomes $d \left[u + \int f(y) dy \right] = 0$

$$\text{Integrating} \quad u + \int f(y) dy = 0.$$

$$\text{But} \quad u = \int_{y \text{ constant}} Mdx \quad \text{and} \quad f(y) = \text{terms of } N \text{ not containing } x.$$

\therefore The solution of $Mdx + Ndy = 0$ is

$$\int_{(y \text{ cons.})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

provided $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 11.25. Solve $(y^2 e^{xy^2} + 4x^3) dx + (2xy e^{xy^2} - 3y^2) dy = 0$.

(V.T.U., 2006)

Solution. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy e^{xy^2} - 3y^2$

$$\therefore \frac{\partial M}{\partial y} = 2y e^{xy^2} + y^2 e^{xy^2} \cdot 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

i.e.,
$$\int_{(y \text{ const.})} (y^2 e^{y^2 x} + 4x^3) dx + \int (-3y^2) dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. Solve $\left\{ y \left(1 + \frac{1}{x} \right) + \cos y \right\} dx + (x + \log x - x \sin y) dy = 0$.

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y \left(1 + \frac{1}{x} \right) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

$$\int_{(y \text{ const.})} \left\{ \left(1 + \frac{1}{x} \right) y + \cos y \right\} dx = c \quad \text{or} \quad (x + \log x) y + x \cos y = c.$$

Example 11.27. Solve $(1 + 2xy \cos x^2 - 2xy) dx + (\sin x^2 - x^2) dy = 0$.

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y \text{ const.})} (1 + 2xy \cos x^2 - 2xy) dx = c \quad \text{or} \quad x + y \left[\int \cos x^2 \cdot 2x dx - \int 2x dx \right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. Solve $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$.

(Kurukshetra, 2005)

Solution. Given equation can be written as

$$(y \cos x + \sin y + y) dx + (\sin x + x \cos y + x) dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

$$\therefore \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

i.e.,
$$\int_{(y \text{ const.})} (y \cos x + \sin y + y) dx + \int (0) dx = c \quad \text{or} \quad y \sin x + (\sin y + y) x = c.$$

Example 11.29. Solve $(2x^2 + 3y^2 - 7) x dx - (3x^2 + 2y^2 - 8) y dy = 0$.

(U.P.T.U., 2005)

Solution. Given equation can be written as

$$\frac{y dy}{x dx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $\frac{y dy + x dx}{y dy - x dx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$ [By componendo & dividendo]

or $\frac{x dx + y dy}{x^2 + y^2 - 3} = 5 \cdot \frac{x dx - y dy}{x^2 - y^2 - 1}$

Integrating both sides, we get

$$\int \frac{2x dx + 2y dy}{x^2 + y^2 - 3} = 5 \int \frac{2x dx - 2y dy}{x^2 - y^2 - 1} + c$$

or $\log(x^2 + y^2 - 3) = 5 \log(x^2 - y^2 - 1) + \log c'$ [Writing $c = \log c'$]

or $x^2 + y^2 - 3 = c' (x^2 - y^2 - 1)^5$

which is the required solution.

PROBLEMS 11.7

Solve the following equations :

1. $(x^2 - ay) dx = (ax - y^2) dy$.

2. $(x^2 + y^2 - a^2) x dx + (x^2 - y^2 - b^2) y dy = 0$

(Kurukshetra, 2005)

3. $(x^2 - 4xy - 2y^2) dx + (y^2 - 4xy - 2x^2) dy = 0$.

4. $(x^4 - 2xy^2 + y^4) dx - (2x^2y - 4xy^3 + \sin y) dy = 0$

5. $ye^{xy} dx + (xe^{xy} + 2y) dy = 0$

6. $(5x^4 + 3x^2y^2 - 2xy^3) dx + (2x^3y - 3x^2y^2 - 5y^4) dy = 0$

(V.T.U., 2008)

7. $(3x^2 + 6xy^2) dx + (6x^2y + 4y^3) dy = 0$

8. $\frac{2x}{y^3} dx + \frac{y^2 - 3x^2}{y^4} dy = 0$

9. $y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0$

10. $(\sec x \tan x \tan y - e^x) dx + \sec x \sec^2 y dy = 0$

(Marathwada, 2008)

11. $(2xy + y - \tan y) dx + x^2 - x \tan^2 y + \sec^2 y dy = 0$.

(Nagpur, 2009)

11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) **I.F. found by inspection.** In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$x dy + y dx = d(xy)$$

$$\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right); \frac{x dy - y dx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{x dy - y dx}{y^2} = -d\left(\frac{x}{y}\right); \frac{x dy - y dx}{x^2 + y^2} = d\left(\tan^{-1} \frac{y}{x}\right)$$

$$\frac{x dy - y dx}{x^2 - y^2} = d\left(\frac{1}{2} \log \frac{x+y}{x-y}\right).$$

Example 11.30. Solve $y(2xy + e^x) dx = e^x dy$.

(Kurukshetra, 2005)

Solution. It is easy to note that the terms $ye^x dx$ and $e^x dy$ should be put together.

$$\therefore (ye^x dx - e^x dy) + 2xy^2 dx = 0$$

Now we observe that the term $2xy^2 dx$ should not involve y^2 . This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x dx - e^x dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\frac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. If $Mdx + Ndy = 0$ be a homogeneous equation in x and y , then $1/(Mx + Ny)$ is an integrating factor ($Mx + Ny \neq 0$).

Example 11.31. Solve $(x^2y - 2xy^2) dx - (x^3 - 3x^2y) dy = 0$.

(Osmania, 2003 S)

Solution. This equation is homogeneous in x and y .

$$\therefore \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0 \quad \text{which is exact.}$$

\therefore the solution is $\int_{(y \text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x) dy = c$ or $\frac{x}{y} - 2 \log x + 3 \log y = c$.

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)x dy = 0$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor ($Mx - Ny \neq 0$).

Example 11.32. Solve $(1 + xy)ydx + (1 - xy)x dy = 0$.

(S.V.T.U., 2008)

Solution. The given equation is of the form $f_1(xy)ydx + f_2(xy)x dy = 0$

Here $M = (1 + xy)y$, $N = (1 - xy)x$.

$$\therefore \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1 + xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right) dy = 0, \quad \text{which is an exact equation.}$$

\therefore the solution is $\int_{(y \text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x) dy = c$

$$\text{or} \quad \frac{1}{2y} \left(-\frac{1}{x}\right) + \frac{1}{2} \log x - \frac{1}{2} \log y = c \quad \text{or} \quad \log \frac{x}{y} - \frac{1}{xy} = c'$$

(4) In the equation $Mdx + Ndy = 0$,

(a) if $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ be a function of x only = $f(x)$ say, then $e^{\int f(x)dx}$ is an integrating factor.

(b) if $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ be a function of y only = $F(y)$ say, then $e^{\int F(y)dy}$ is an integrating factor.

Example 11.33. Solve $(xy^2 - e^{1/x^3})dx - x^2y dy = 0$.

(S.V.T.U., 2009 ; Mumbai, 2007)

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \quad \text{which is a function of } x \text{ only.}$$

$$\therefore \text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4} , we get $\left(\frac{y^2}{x^3} - \frac{1}{4^4} e^{1/x^3}\right) dx - \frac{y}{x^2} dy = 0$

which is an exact equation.

$$\therefore \text{the solution is } \int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x) dy = c.$$

or
$$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx + 0 = c$$

or
$$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4}) dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

Example 11.34. Solve $(xy^3 + y) dx + 2(x^2y^2 + x + y^4) dy = 0$.

Solution. Here $M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$

$$\therefore \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{y(xy^2 + 1)} (4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\therefore \text{I.F.} = e^{\int 1/y dy} = e^{\log y} = y$$

Multiplying throughout by y , it becomes $(xy^4 + y^2) dx + (2x^2y^3 + 2xy + 2y^5) dy = 0$, which is an exact equation.

$$\therefore \text{its solution is } \int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x) dy = 0$$

or
$$\int_{(y \text{ const})} (xy^4 + y^2) dx + \int 2y^5 dy = c \text{ or } \frac{1}{2} x^2 y^4 + xy^2 + \frac{1}{3} y^6 = c.$$

Example 11.35. Solve $(y \log y) dx + (x - \log y) dy = 0$

(U.P.T.U., 2004)

Solution. Here $M = y \log y$ and $N = x - \log y$

$$\therefore \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{y \log y} (1 - \log y - 1) = -\frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\therefore \text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y dx + \frac{1}{y} (x - \log y) dy = 0$$

which is an exact equation

$$\left[\because \frac{\partial}{\partial y} (\log y) = \frac{\partial}{\partial x} \left(\frac{x - \log y}{y} \right) \right]$$

$$\therefore \text{its solution is } \int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x) dy = c$$

or
$$\log y \int dx + \int \left(\frac{-\log y}{y} \right) dy = c \text{ or } x \log y - \frac{1}{2} (\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b (mydx + nxdy) + x^a y^b (m'ydx + n'xdy) = 0,$$

an integrating factor is $x^h y^k$

where
$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

Example 11.36. Solve $y(xy + 2x^2y^3) dx + x(xy - x^2y^2) dy = 0$. (Hissar, 2005 ; Kurukshetra, 2005)

Solution. Rewriting the equation as $xy(ydx + xdy) + x^2y^2(2ydx - xdy) = 0$ and comparing with

$$x^a y^b (m y dx + n x dy) + x^a y^{b'} (m' y dx + n' x dy) = 0,$$

we have $a = b = 1, m = n = 1; a' = b' = 2, m' = 2, n' = -1$.

$$\therefore \text{I.F.} = x^h y^k.$$

where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

i.e.

$$\frac{1+h+1}{1} = \frac{1+k+1}{1}, \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or

$$h - k = 0, h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. \therefore I.F. = $1/x^3 y^3$.

Multiplying throughout by $1/x^3 y^3$, it becomes

$$\left(\frac{1}{x^2 y} + \frac{2}{x}\right) dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right) dy = 0, \text{ which is an exact equation.}$$

$$\therefore \text{The solution is } \int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

or

$$\frac{1}{y} \left(-\frac{1}{x}\right) + 2 \log x - \log y = c \quad \text{or} \quad 2 \log x - \log y - 1/xy = c.$$

PROBLEMS 11.8

Solve the following equations :

1. $x dy - y dx + a(x^2 + y^2) dx = 0$.

2. $x dx + y dy = \frac{a^2(x dy - y dx)}{x^2 + y^2}$. (U.P.T.U., 2005)

3. $y dx - x dy + \log x dx = 0$.

4. $\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$.

5. $(x^3 y^2 + x) dy + (x^2 y^3 - y) dx = 0$.

6. $(x^2 y^2 + xy + 1) y dx + (x^2 y^2 - xy + 1) x dy = 0$.

7. $(y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0$.

8. $(4xy + 3y^2 - x) dx + x(x + 2y) dy = 0$ (Mumbai, 2006)

9. $x^4 \frac{dy}{dx} + x^3 y + \operatorname{cosec}(xy) = 0$.

10. $(y - xy^2) dx - (x + x^2 y) dy = 0$ (Mumbai, 2006)

11. $y dx - x dy + 3x^2 y^2 e^{x^3} dx = 0$. (Kurukshetra, 2006)

12. $(y^2 + 2x^2 y) dx + (2x^3 - xy) dy = 0$. (Rajasthan, 2005)

13. $2y dx + x(2 \log x - y) dy = 0$. (P.T.U., 2005)

11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p . Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the n th degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0 \quad \dots(1)$$

where P_1, P_2, \dots, P_n are functions of x and y .

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)] [p - f_2(x, y)] \dots [p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y), p = f_2(x, y), \dots, p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0, F_2(x, y, c) = 0, \dots, F_n(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \cdot F_2(x, y, c) \dots F_n(x, y, c) = 0.$$

Example 11.37. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Solution. Given equation is $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$ where $p = \frac{dy}{dx}$ or $p^2 + p \left(\frac{y}{x} - \frac{x}{y} \right) - 1 = 0$.

Factorising $(p + y/x)(p - x/y) = 0$.

Thus we have $p + y/x = 0$... (i) and $p - x/y = 0$... (ii)

From (i), $\frac{dy}{dx} + \frac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $d(xy) = 0$. Integrating, $xy = c$.

From (ii), $\frac{dy}{dx} - \frac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 11.38. Solve $p^2 + 2py \cot x = y^2$.

(Bhopal, 2008 ; Kerala, 2005)

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $p + y \cot x = \pm y \operatorname{cosec} x$

i.e., $p = y(-\cot x + \operatorname{cosec} x)$... (i)

or $p = y(-\cot x - \operatorname{cosec} x)$... (ii)

From (i), $\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\frac{dy}{y} = (\operatorname{cosec} x - \cot x) dx$

Integrating, $\log y = \log \tan \frac{x}{2} - \log \sin x + \log c = \log \frac{c \tan x/2}{\sin x}$

or $y = \frac{c}{2 \cos x^2/2}$ or $y(1 + \cos x) = c$... (iii)

From (ii), $\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$ or $\frac{dy}{y} = -(\cot x + \operatorname{cosec} x) dx$

Integrating, $\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$

or $y = \frac{c}{2 \sin^2 \frac{x}{2}}$ or $y(1 - \cos x) = c$... (iv)

Thus combining (iii) and (iv), the required general solution is

$$y(1 \pm \cos x) = c.$$

PROBLEMS 11.9

Solve the following equations :

$$1. y \left(\frac{dy}{dx} \right)^2 + (x-y) \frac{dy}{dx} - x = 0. \quad 2. p(p+y) = x(x+y). \quad (V.T.U., 2011) \quad 3. y = x[p + \sqrt{(1+p^2)}].$$

$$4. xy \left(\frac{dy}{dx} \right)^2 - (x^2 + y^2) \frac{dy}{dx} + xy = 0. \quad 5. p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0. \quad (\text{Madras, 2003})$$

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \quad \dots(1)$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi \left(x, p, \frac{dp}{dx} \right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F(x, p, c) = 0$ (2)

The elimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c), y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 11.39. Solve $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$... (i)

Differentiating both sides with respect to x, $\frac{dy}{dx} = p = 2 \left(p + x \frac{dp}{dx} \right) + \frac{p^2 + 2xp \frac{dp}{dx}}{1 + x^2 p^4}$

or
$$p + 2x \frac{dp}{dx} + \left(p + 2x \frac{dp}{dx} \right) \cdot \frac{p}{1 + x^2 p^4} = 0 \text{ or } \left(p + 2x \frac{dp}{dx} \right) \left(1 + \frac{p}{1 + x^2 p^4} \right) = 0$$

This gives $p + 2x dp/dx = 0$.

Separating the variables and integrating, we have $\int \frac{dx}{x} + 2 \int \frac{dp}{p} = \text{a constant}$

or
$$\log x + 2 \log p = \log c \text{ or } \log xp^2 = \log c$$

whence
$$xp^2 = c \text{ or } p = \sqrt{(c/x)} \quad \dots(ii)$$

Eliminating p from (i) and (ii), we get $y = 2\sqrt{(c/x)}x + \tan^{-1} c$

or $y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (i).

Obs. The significance of the factor $1 + p/(1 + x^2 p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (i) whereas we are interested only in finding general solution.

Caution. Sometimes one is tempted to write (ii) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x} \right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is incorrect.

Example 11.40. Solve $y = 2px + p^n$.

(Bhopal, 2009)

Solution. Given equation is $y = 2px + p^n$... (i)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x \frac{dp}{dx} + np^{n-1} \frac{dp}{dx} \text{ or } p \frac{dx}{dp} + 2x = -np^{n-1}$$

or
$$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \quad \dots(ii)$$

This is Leibnitz's linear equation in x and p. Here I.F. = $e^{\int \frac{2}{p} dp} = e^{\log p^2} = p^2$

∴ the solution of (ii) is

$$x(\text{I.F.}) = \int (-np^{n-2}) \cdot (\text{I.F.}) dp + c \quad \text{or} \quad xp^2 = -n \int p^n dp + c = -\frac{np^{n+1}}{n+1} + c$$

$$\text{or} \quad x = cp^{-2} - \frac{np^{n-1}}{n+1} \quad \dots(\text{iii})$$

$$\text{Substituting this value of } x \text{ in (i), we get } y = \frac{2c}{p} + \frac{1-n}{1+n} p^n \quad \dots(\text{iv})$$

The equations (iii) and (iv) taken together, with parameter p , constitute the general solution (i).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's equation*, are solvable for y and lead to Leibnitz's equation in dx/dp .

PROBLEMS 11.10

Solve the following equations :

1. $y = x + a \tan^{-1} p.$
2. $y + px = x^4 p^2.$ (S.V.T.U., 2007)
3. $x^2 \left(\frac{dy}{dx} \right)^4 + 2x \frac{dy}{dx} - y = 0.$
4. $xp^2 + x = 2yp.$
5. $y = xp^2 + p.$
6. $y = p \sin p + \cos p.$

Case III. Equations solvable for x . If the given equation on solving for x , takes the form

$$x = f(y, p) \quad \dots(1)$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi \left(y, p, \frac{dp}{dy} \right)$$

Now it may be possible to solve the new differential equation in y and p . Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y .

Example 11.41. Solve $y = 2px + y^2 p^3.$

(Bhopal, 2008)

Solution. Given equation, on solving for x , takes the form $x = \frac{y - y^2 p^3}{2p}$

$$\text{Differentiating with respect to } y, \frac{dx}{dy} \left(= \frac{1}{p} \right) = \frac{1}{2} \cdot \frac{p \left(1 - 2y \cdot p^3 - y^2 3p^2 \frac{dp}{dy} \right) - (y - y^2 p^3) \frac{dp}{dy}}{p^2}$$

$$\text{or} \quad 2p = p - 2yp^4 - 3y^2 p^3 \frac{dp}{dy} - y \frac{dp}{dy} + y^2 p^3 \frac{dp}{dy}$$

$$\text{or} \quad p + 2yp^4 + 2y^2 p^3 \frac{dp}{dy} + y \frac{dp}{dy} = 0 \quad \text{or} \quad p(1 + 2yp^3) + y \frac{dp}{dy} (1 + 2yp^3) = 0.$$

$$\text{or} \quad \left(p + y \frac{dp}{dy} \right) (1 + 2yp^3) = 0 \quad \text{This gives } p + y \frac{dp}{dy} = 0. \quad \text{or} \quad \frac{d}{dy} (py) = 0.$$

$$\text{Integrating} \quad py = c. \quad \dots(i)$$

Thus eliminating from the given equation and (i), we get $y = 2 \frac{c}{y} x + \frac{c^3}{y^3} y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

PROBLEMS 11.11

Solve the following equations :

1. $p^3 - 4xyp + 8y^2 = 0$. (Kanpur, 1996)

2. $p^3y + 2px = y$.

3. $x - yp = ap^2$. (Andhra, 2000)

4. $p = \tan \left(x - \frac{p}{1+p^2} \right)$. (S.V.T.U., 2008)

11.14 CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ... (1)Differentiating with respect to x , we have $p = p + x \frac{dp}{dx} + f'(p) \frac{dp}{dx}$

or $[x + f'(p)] \frac{dp}{dx} = 0 \quad \therefore \frac{dp}{dx} = 0$, or $x + f'(p) = 0$

$\frac{dp}{dx} = 0$, gives $p = c$... (2)

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$... (3)
as the general solution of (1).Hence the solution of the Clairaut's equation is obtained on replacing p by c .**Obs.** If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. This is the **singular solution** of (1) which gives the envelope of the family of straight lines (3).

To obtain the singular solution, we proceed as follows :

(i) Find the general solution by replacing p by c i.e., (3)(ii) Differentiate this w.r.t. c giving $x + f'(c) = 0$ (4)(iii) Eliminate c from (3) and (4) which will be the singular solution.**Example 11.42.** Solve $p = \sin(y - xp)$. Also find its singular solutions.**Solution.** Given equation can be written as

$$\sin^{-1} p = y - xp \text{ or } y = px + \sin^{-1} p \text{ which is the Clairaut's equation.}$$

 \therefore its solution is $y = cx + \sin^{-1} c$.To find the singular solution, differentiate (i) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \quad \dots(ii)$$

To eliminate c from (i) and (ii), we rewrite (ii) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (i), we get

$$y = N(x^2 - 1) + \sin^{-1} \{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.**Example 11.43.** Solve $(px - y)(py + x) = a^2p$.

(V.T.U., 2011 ; J.N.T.U., 2006)

Solution. Put

$$x^2 = u \text{ and } y^2 = v \text{ so that } 2xdx = du \text{ and } 2ydy = dv$$

$$\therefore p = \frac{dy}{dx} = \frac{dv}{y} \bigg/ \frac{du}{x} = \frac{x}{y} P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy Alexis Claude Clairaut (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

Then the given equation becomes $\left(\frac{xP}{y} \cdot x - y\right)\left(\frac{xP}{y} \cdot y + x\right) = a^2 \frac{xP}{y}$

or $(uP - v)(P + 1) = a^2P$ or $uP - v = \frac{a^2P}{P + 1}$

or $\therefore v = uP - a^2P/(P + 1)$, which is Clairaut's form.

\therefore its solution is $v = uc - a^2c/(c + 1)$, i.e., $y^2 = cx^2 - a^2c/(c + 1)$.

PROBLEMS 11.12

1. Find the general and singular solution of the equations :

(i) $xp^2 - yp + a = 0$. (J.N.T.U., 2006)

(ii) $p = \log(px - y)$.

(iii) $y = px + \sqrt{(a^2p^2 + b^2)}$ (W.B.T.U., 2005)

(iv) $\sin px \cos y = \cos px \sin y + p$ (P.T.U., 2006)

Solve the following equations :

2. $y + 2\left(\frac{dy}{dx}\right)^2 = (x + 1)\frac{dy}{dx}$.

3. $(y - px)(p - 1) = p$.

4. $(x - a)\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - y = 0$.

5. $x^2(y - px) = yp^2$.

6. $(px + y)^2 = py^2$.

7. $(px - y)(x + py) = 2p$.

11.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.13

Fill up the blanks or choose the correct answer in the following problems :

1. $y = cx - c^2$, is the general solution of the differential equation

(i) $(y')^2 - xy' + y = 0$

(ii) $y'' = 0$

(iii) $y' = c$

(iv) $(y')^2 + xy' + y = 0$.

2. The differential equation having a basis for its solution as $\sinh 6x$ and $\cosh 6x$ is

(i) $y'' + 36y = 0$

(ii) $y'' - 36y = 0$

(iii) $y'' + 6y = 0$

(iv) none of these.

3. The differential equation $(dx/dy)^2 + 5y^{1/3} = x$ is

(i) linear of degree 3

(ii) non-linear of order 1 and degree 6

(iii) non-linear of order 1 and degree 2.

4. The differential equation $ydx/dy + 1 = y$, $y(0) = 1$, has

(i) a unique solution

(ii) two solutions

(iii) infinite number of solutions

(iv) no solution

5. Solution of $(x^2 + y^2)dy = xy dx$ is

6. Solution of $(3x - 2y)dx = xdy$ is

7. Solution of $dy/dx - y = 2xy^2 e^{-x}$ is

8. The differential equation $(y^2 e^{xy^2} + 6x)dx + (2xye^{xy^2} - 4y)dy = 0$ is

(i) linear, homogeneous and exact

(ii) non-linear, homogeneous and exact

(iii) non-linear, non-homogeneous and exact

(iv) non-linear, non-homogeneous and inexact.

9. Solution of $x dx + y dy + \frac{x dy - y dx}{x^2 + y^2}$ is

10. Solution of $dy/dx = \frac{x^3 + y^3}{xy^2}$ is

11. The differential equation $(x + x^8 + ay^2)dx + (y^8 - y + bxy)dy = 0$ is exact if

(i) $b = 2a$

(ii) $a = b$

(iii) $a \neq 2b$

(iv) $a = 1, b = 3$.

12. Solution of $xy(1 + xy^2)dy = dx$ is

13. Solution of $xp^2 - yp + a = 0$ is

14. The differential equation $p = \log(px - y)$ has the solution

15. Solution of $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ is

16. The order of the differential equation $(1 + y_1^2)^{3/2}/y_2 = c$ is
17. The general solution of $\frac{1}{x^2 y^2} (x dy + y dx) = 0$ is
18. Integrating factor of the differential equation $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is
(a) e^{y^3} (b) y^3 (c) x^3 (d) $-y^3$. (V.T.U., 2009)
19. Solution of the equation $\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec} \frac{y}{x}$ is
(a) $\cos(y/x) - \log x = c$ (b) $\cos(y/x) + \log x = c$
(c) $\cos^2(y/x) + \log x = c$ (d) $\cos^2(y/x) - \log x = c$. (V.T.U., 2010)
20. Solution of $x \sqrt{1+x^2} + y \sqrt{1+y^2} dy/dx = 0$ is
21. Solution of $dy/dx + y = 0$ given $y(0) = 5$ is
22. The substitution that transforms the equation $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$ to homogeneous form is
23. Integrating factor of $xy' + y = x^3 y^6$ is
24. Solution of the exact differential equation $Mdx + Ndy = 0$ is
25. Solution of $(2x^3 y^2 + x^4) dx + (x^4 y + y^4) dy = 0$ is
26. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \tan 2x$ is
27. Degree of the differential equation $\left(\frac{d^2 y}{dx^2}\right)^2 + x \left(\frac{dy}{dx}\right)^5 x^2 y = 0$ is
(a) 2 (b) 0 (c) 3 (d) 5 (Bhopal, 2008)
28. Integrating factor of the differential equation $\frac{dy}{dx} + y \cos x = \frac{\sin 2x}{2}$ is
(a) $e^{\sin^2 x}$ (b) $e^{\sin^3 x}$ (c) $e^{\sin x}$ (d) $\sin x$
29. The differential equation of the family of circles with centre as origin is (Nagarjuna, 2008)
30. Solution of $x e^{-x^2} dx + \sin y dy = 0$ is (Nagarjuna, 2008)
31. Solution of $p = \sin(y - xp)$ is
(a) $y = \frac{c}{x} + \sin^{-1} c$ (b) $y = cx + \sin c$ (c) $y = cx + \sin^{-1} c$ (d) $y = x + \sin^{-1} c$ (V.T.U., 2011)
32. Differential equation obtained by eliminating A and B from $y = A \cos x + B \sin x$ is $d^2 y/dx^2 - y = 0$ (True or False)
33. $(x^3 - 3xy^2) dx + (y^3 - 2x^2 y) dy = 0$ is an exact differential equation. (True or False)

Fourier Transforms

1. Introduction. 2. Definition. 3. Fourier integrals — Fourier sine and cosine integral — Complex forms of Fourier integral. 4. Fourier transform — Fourier sine and cosine transforms — Finite Fourier sine and cosine transforms. 5. Properties of F-transforms. 6. Convolution theorem for F-transforms. 7. Parseval's identity for F-transforms. 8. Relation between Fourier and Laplace transforms. 9. Fourier transforms of the derivatives of a function— 10. Inverse Laplace transforms by method of residues. 11. Application of transforms to boundary value problems. 12. Objective Type of Questions.

22.1 INTRODUCTION

In the previous chapter, the reader has already been acquainted with the use of Laplace transforms in the solution of ordinary differential equations. In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used in the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained *e.g.*, conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, free and forced vibrations of a membrane, transmission lines etc. Some of these applications will be illustrated in the last section.

22.2 DEFINITION

The integral transform of a function $f(x)$ denoted by $I[f(x)]$, is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x) dx$$

where $K(s, x)$ is called the *kernel* of the transform and is a known function of s and x . The function $f(x)$ is called the *inverse transform* of $\bar{f}(s)$.

Three simple examples of a kernel are as follows :

(i) When $K(s, x) = e^{-sx}$, it leads to the **Laplace transform** of $f(x)$, *i.e.*,

$$\bar{f}(s) = \int_0^{\infty} f(x) e^{-sx} dx.$$

[Chap. 21]

(ii) When $K(s, x) = e^{isx}$, we have the **Fourier transform** of $f(x)$, *i.e.*,

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

(iii) When $K(s, x) = x^{s-1}$, it gives the Mellin transform of $f(x)$ i.e.,

$$M(s) = \int_0^{\infty} f(x) x^{s-1} dx.$$

Other special transforms arise when the kernel is a sine or a cosine function or a Bessel's function. These lead to *Fourier sine* or *cosine transforms* and the *Hankel transform* respectively.

In order to introduce the *Fourier transforms*, we shall first derive the Fourier integral theorem.

22.3 (1) FOURIER INTEGRAL THEOREM

Consider a function $f(x)$ which satisfies the Dirichlet's conditions (Art. 10.3) in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \quad \dots(1)$$

where $a_0 = \frac{1}{c} \int_{-c}^c f(t) dt$, $a_n = \frac{1}{c} \int_{-c}^c f(t) \cos \frac{n\pi t}{c} dt$, and $b_n = \frac{1}{c} \int_{-c}^c f(t) \sin \frac{n\pi t}{c} dt$.

Substituting the values of a_0 , a_n and b_n in (1), it takes the form

$$f(x) = \frac{1}{2c} \int_{-c}^c f(t) dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^c f(t) \cos \frac{n\pi(t-x)}{c} dt \quad \dots(2)$$

If we assume that $\int_{-\infty}^{\infty} |f(x)| dx$ converges, the first term on the right side of (2) approaches 0 as $c \rightarrow \infty$, since

$$\left| \frac{1}{2c} \int_{-c}^c f(t) dt \right| \leq \frac{1}{2c} \int_{-\infty}^{\infty} |f(t)| dt$$

The second term on the right side of (2) tends to

$$\begin{aligned} \text{Lt}_{c \rightarrow \infty} \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^c f(t) \cos \frac{n\pi(t-x)}{c} dt \\ = \text{Lt}_{\delta\lambda \rightarrow 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda(t-x) dt, \text{ on writing } \pi/c = \delta\lambda \end{aligned}$$

This is of the form $\text{Lt}_{\delta\lambda \rightarrow 0} \sum_{n=1}^{\infty} F(n\delta\lambda)$, i.e., $\int_0^{\infty} F(\lambda) d\lambda$

Thus as $c \rightarrow \infty$, (2) becomes $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad \dots(3)$

which is known as the **Fourier integral** of $f(x)$.

Obs. We have given a heuristic demonstration of the Fourier integral theorem which simply helps in deriving the result (3). It cannot however, be taken as a rigorous proof for that would, involve a proof of the convergence of the Fourier integral which is beyond the scope of this book. When $f(x)$ satisfies the above-mentioned conditions, equation (3) holds good at a point of continuity. If however, x is point of discontinuity, we replace $f(x)$ by $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

(2) Fourier sine and cosine integrals. Expanding $\cos \lambda(t-x)$, (3) may be written as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t dt d\lambda + \frac{1}{\pi} \int_0^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t dt d\lambda \quad \dots(4)$$

If $f(x)$ is an odd function, $f(t) \cos \lambda t$ is also an odd function while $f(t) \sin \lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda \quad \dots(5)$$

which is known as the *Fourier sine integral*.

Similarly, if $f(x)$ is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \int_0^{\infty} f(t) \cos \lambda t dt d\lambda \quad \dots(6)$$

which is known as the *Fourier cosine integral*.

Obs. A function $f(x)$ defined in the interval $(0, \infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral, merely looking upon it as an odd or even function in $(-\infty, \infty)$ on the lines of half-range Fourier series.

(3) Complex form of Fourier integrals. Equation (3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad \dots(7)$$

because $\cos \lambda(t-x)$ is an even function of λ . Also since $\sin \lambda(t-x)$ is an odd function of λ , we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t-x) dt d\lambda \quad \dots(8)$$

Now multiply (8) by i and add it to (7), so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(t-x)} dt d\lambda \quad \dots(9)$$

which is the *complex form of the Fourier integral*.

(4) Fourier integral representation of a function

Using (4), a function $F(x)$ may be represented by a Fourier integral as

$$F(x) = \frac{1}{\pi} \int_0^{\infty} [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] d\lambda$$

where $A(\lambda) = \int_{-\infty}^{\infty} f(t) \cos \lambda t dt$; $B(\lambda) = \int_{-\infty}^{\infty} f(t) \sin \lambda t dt \quad \dots(10)$

If $f(x)$ is an odd function, then [By (5)]

$$f(x) = \frac{1}{\pi} \int_0^{\infty} B(\lambda) \sin \lambda x d\lambda \text{ where } B(\lambda) = 2 \int_0^{\infty} f(t) \sin \lambda t dt \quad \dots(11)$$

If $f(x)$ is an even function, then [By (6)]

$$f(x) = \frac{1}{\pi} \int_0^{\infty} A(\lambda) \cos \lambda x d\lambda \text{ where } A(\lambda) = 2 \int_0^{\infty} f(t) \cos \lambda t dt \quad \dots(12)$$

Example 22.1. Express $f(x) = 1$ for $0 \leq x \leq \pi$,
 $= 0$ for $x > \pi$,

as a Fourier sine integral and hence evaluate

$$\int_0^{\infty} \frac{1 - \cos(\pi\lambda)}{\lambda} \sin(x\lambda) d\lambda. \quad \text{(Kottayam, 2005; J.N.T.U., 2004 S)}$$

Solution. The Fourier sine integral for $f(x) = \frac{2}{\pi} \int_0^{\infty} \sin(\lambda x) d\lambda \int_0^{\infty} f(t) \sin(\lambda t) dt$

$$\begin{aligned} &= \frac{2}{\pi} \int_0^{\infty} \sin(\lambda x) d\lambda \int_0^{\infty} \sin(\lambda t) dt \\ &= \frac{2}{\pi} \int_0^{\infty} \sin(\lambda x) d\lambda \left[\frac{-\cos(\lambda t)}{\lambda} \right]_0^{\infty} = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda \end{aligned}$$

$$\therefore \int_0^{\infty} \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & \text{for } 0 \leq x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2} \left[\frac{f(\pi-0) + f(\pi+0)}{2} \right] = \frac{\pi}{2} \cdot \frac{1+0}{2} = \frac{\pi}{4}$$

22.4 (1) FOURIER TRANSFORMS

Rewriting (9) of § 22.3 as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} ds \int_{-\infty}^{\infty} f(t)e^{ist} dt,$$

it follows that if

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{ist} dt \quad \dots(1)$$

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \quad \dots(2)$$

The function $F(s)$, defined by (1), is called the **Fourier transform** of $f(x)$. Also the function $f(x)$, as given by (2), is called the **inverse Fourier transform** of $F(s)$. Sometimes, we call (2) as an *inversion formula* corresponding to (1).

(2) Fourier sine and cosine transforms. From (5) of § 22.3, it follows that if

$$F_s(s) = \int_0^{\infty} f(x) \sin sx dx \quad \dots(3)$$

then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx ds \quad \dots(4)$$

The function $F_s(s)$, as defined by (3), is known as the **Fourier sine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by (4) is called the **inverse Fourier sine transform** of $F_s(s)$.

Similarly, it follows from (6) of § 22.3 that if

$$F_c(s) = \int_0^{\infty} f(x) \cos sx dx \quad \dots(5)$$

then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos sx ds \quad \dots(6)$$

The function $F_c(s)$ as defined by (5) is known as the **Fourier cosine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by (6), is called the **inverse Fourier cosine transform** of $F_c(s)$.

(3) Finite Fourier sine and cosine transforms. These transforms are useful for such a boundary-value problem in which at least two of the boundaries are parallel and separated by a finite distance.

The **finite Fourier sine transform** of $f(x)$, in $0 < x < c$, is defined as

$$F_s(n) = \int_0^c f(x) \sin \frac{n\pi x}{c} dx \quad \dots(7)$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier sine transform** of $F_s(n)$ which is given by

$$f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{c} \quad \dots(8)$$

The **finite Fourier cosine transform** of $f(x)$, in $0 < x < c$, is defined as

$$F_c(n) = \int_0^c f(x) \cos \frac{n\pi x}{c} dx \quad \dots(9)$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier cosine transform** of $F_c(n)$ which is given by

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{c} \quad \dots(10)$$

Obs. The finite Fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

22.5 PROPERTIES OF FOURIER TRANSFORMS

(1) Linear property. If $F(s)$ and $G(s)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F[af(x) + bg(x)] = aF(s) + bG(s)$$

where a and b are constants.

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x) dx$

$$\begin{aligned} \therefore F[af(x) + bg(x)] &= \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)] dx = a \int_{-\infty}^{\infty} e^{isx} f(x) dx + b \int_{-\infty}^{\infty} e^{isx} g(x) dx \\ &= aF(s) + bG(s) \end{aligned}$$

(2) Change of scale property. If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), a \neq 0$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$...(i)

$$\begin{aligned} \therefore F[f(ax)] &= \int_{-\infty}^{\infty} e^{isx} f(ax) dx \quad \left| \begin{array}{l} \text{Put } ax = t \\ \text{so that } dx = dt/a \end{array} \right. \\ &= \int_{-\infty}^{\infty} e^{ist/a} f(t) dt / a = \frac{1}{a} \int_{-\infty}^{\infty} e^{i(s/a)t} f(t) dt = \frac{1}{a} F\left(\frac{s}{a}\right) \end{aligned} \quad \text{[By (i)]}$$

Cor. If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of $f(x)$ respectively, then

$$F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right) \quad \text{and} \quad F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right).$$

(3) Shifting property. If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F[f(x-a)] = e^{isa} F(s)$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$...(i)

$$\begin{aligned} \therefore F[f(x-a)] &= \int_{-\infty}^{\infty} e^{isx} f(x-a) dx \quad \left| \begin{array}{l} \text{Put } x-a = t \\ \text{so that } dx = dt \end{array} \right. \\ &= \int_{-\infty}^{\infty} e^{is(t+a)} f(t) dt = e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t) dt = e^{isa} F(s) \end{aligned} \quad \text{[By (i)]}$$

(4) Modulation theorem. If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F[f(x) \cos ax] = \frac{1}{2} [F(s+a) + F(s-a)]$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$...(i)

$$\begin{aligned} \therefore F[f(x) \cos ax] &= \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax dx = \int_{-\infty}^{\infty} e^{isx} \cdot f(x) \cdot \frac{e^{iax} + e^{-iax}}{2} dx \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx \right] = \frac{1}{2} [F(s+a) + F(s-a)]. \end{aligned}$$

Cor. If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then

$$(i) F_s[f(x) \cos ax] = \frac{1}{2} [F_s(s+a) + F_s(s-a)] \quad \text{(Anna, 2008)}$$

$$(ii) F_c[f(x) \sin ax] = \frac{1}{2} [F_s(s+a) - F_s(s-a)]$$

$$(iii) F_s[f(x) \sin ax] = \frac{1}{2} [F_c(s-a) - F_c(s+a)]$$

Obs. This theorem is of great importance in radio and television where the harmonic carrier wave is modulated by an envelope.

Example 22.2. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$.

(V.T.U., 2010 ; S.V.T.U., 2009 ; U.P.T.U., 2008)

Solution. The Fourier transform of $f(x)$, i.e.,

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx} dx = \int_{-1}^1 (1) e^{isx} dx = \left[\frac{e^{isx}}{is} \right]_{-1}^1 = \frac{e^{is} - e^{-is}}{is}$$

Thus $F\{f(x)\} = F(s) = 2 \frac{\sin s}{s}$, $s \neq 0$. For $s = 0$, we have $F(s) = 2$.

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)e^{-isx} ds, \text{ or } \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} e^{-isx} ds = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \quad \therefore \int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

Example 22.3. Find the Fourier transform of:

$$f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0 & , |x| > 1 \end{cases}$$

Hence evaluate $\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx$. (V.T.U., 2011 S ; Anna, 2005 S ; Mumbai, 2005 S)

Solution. $F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)$, say

$$\begin{aligned} &= \int_{-\infty}^{-1} (0) e^{isx} dx + \int_{-1}^1 (1 - x^2) e^{isx} dx + \int_1^{\infty} (0) e^{isx} dx = \left[(1 - x^2) \frac{e^{isx}}{is} - (2x) \frac{e^{isx}}{(is)^2} + (-2) \frac{e^{isx}}{(is)^3} \right]_{-1}^1 \\ &= 2 \left(\frac{e^{is} + e^{-is}}{-s^2} \right) - 2 \left(\frac{e^{is} - e^{-is}}{-is^3} \right) = -\frac{4}{s^3} (s \cos s - \sin s) \end{aligned}$$

Now by inversion formula, we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)e^{-isx} ds \\ \text{or } &-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3} (s \cos s - \sin s) e^{-isx} ds = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \end{aligned}$$

Putting $x = 1/2$, we obtain

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3} (s \cos s - \sin s) e^{-is/2} ds = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \left(\cos \frac{s}{2} - i \sin \frac{s}{2} \right) ds = -\frac{3\pi}{8}$$

$$\int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \cdot \cos \frac{s}{2} ds = -\frac{3\pi}{8}$$

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cdot \cos \frac{x}{2} dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

Example 22.4. (a) Find the Fourier transform of $e^{-a^2 x^2}$, $a < 0$. Hence deduce that $e^{-x^2/2}$ is self reciprocal in respect of Fourier transform. (Madras, 2006 ; Kottayam, 2005)

(b) Find Fourier transform of (i) $e^{-2(x-3)^2}$ (ii) $e^{-x^2} \cos 3x$.

$$\begin{aligned} \text{Solution. (a)} \quad F(e^{-a^2 x^2}) &= \int_{-\infty}^{\infty} e^{-a^2 x^2} \cdot e^{isx} dx = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)} dx \\ &= \int_{-\infty}^{\infty} e^{-a^2(x - is/2a^2)^2} \cdot e^{-s^2/4a^2} dx \end{aligned}$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2} dt/a \quad [\text{Putting } a(x - is/2a^2) = t, dx = dt/a]$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi} \quad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \right]$$

Hence $F(e^{-a^2x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})} e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

i.e., Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ is self-reciprocal under the Fourier transform.

(b) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

\therefore by change of scale property, $F[f(2x)] = \frac{1}{2} F(s/2)$

i.e., $F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2/2} = \sqrt{2\pi} e^{-s^2/8}$

By shifting property $F[f(x-3)] = e^{i3s} F(s)$

$\therefore F[e^{-2(x-3)^2}] = e^{3is} \sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is-s^2/8)} \quad \dots(i)$

Also by modulation theorem,

$$F[f(x) \cos 2x] = \frac{1}{2} [F(s+a) + F(s-a)]$$

$$F(e^{-x^2} \cos 3x) = \frac{1}{2} \sqrt{2\pi} [e^{-(s+3)^2/2} + e^{-(s-3)^2/2}] \quad \dots(ii)$$

Example 22.5. Find the Fourier cosine transform of e^{-x^2} .

(V.T.U., 2010; Rajasthan, 2006)

Solution. We have $F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2} \cos sx \, dx = I$ (say)

Differentiating under the integral sign w.r.t. s ,

$$\begin{aligned} \frac{dI}{ds} &= - \int_0^{\infty} x e^{-x^2} \sin sx \, dx = \frac{1}{2} \int_0^{\infty} (\sin sx)(-2xe^{-x^2}) dx \\ &= \frac{1}{2} \left\{ \sin sx \cdot e^{-x^2} \Big|_0^{\infty} - s \int_0^{\infty} \cos sx \cdot e^{-x^2} dx \right\} \\ &= - \frac{s}{2} \int_0^{\infty} e^{-x^2} \cos sx \, dx = - \frac{s}{2} I \quad \text{or} \quad \frac{dI}{I} = - \int \frac{s}{2} ds + \log c \end{aligned}$$

or $\log I = - \frac{s^2}{4} + \log c = \log (ce^{-s^2/4})$

$\therefore I = ce^{-s^2/4}$ or $\int_0^{\infty} e^{-x^2} \cos sx \, dx = ce^{-s^2/4}$

Putting $s = 0$, $c = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. i.e. $I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$.

Hence $F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$.

Example 22.6. Find the Fourier sine transform of $e^{-|x|}$.

Hence show that $\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$, $m > 0$. (V.T.U., 2010; S.V.T.U., 2008; Kottayam, 2005)

Solution. x being positive in the interval $(0, \infty)$, $e^{-|x|} = e^{-x}$

\therefore Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$\begin{aligned} F_s[f(x)] &= \int_0^{\infty} f(x) \sin sx \, dx = \int_0^{\infty} e^{-x} \sin sx \, dx \\ &= \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right]_0^{\infty} = \frac{s}{1+s^2} \end{aligned}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s[f(x)] \sin sx \, dx \quad \text{or} \quad e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{s}{1+s^2} \sin sx \, ds$$

or changing x to m ,

$$e^{-m} = \frac{2}{\pi} \int_0^{\infty} \frac{s \sin ms}{1+s^2} \, ds = \frac{2}{\pi} \int_0^{\infty} \frac{x \sin mx}{1+m^2} \, dx$$

$$\text{Hence} \quad \int_0^{\infty} \frac{x \sin mx}{1+m^2} \, dx = \frac{\pi e^{-m}}{2}.$$

Example 22.7. Find the Fourier cosine transform of $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$ (J.N.T.U., 2006)

Solution. Fourier cosine transform of $f(x)$ i.e., $F_c[f(x)]$

$$\begin{aligned} &= \int_0^{\infty} f_c(x) \cos sx \, dx = \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + \int_2^{\infty} 0 \cdot dx \\ &= \left[x \frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2} \right) \right]_0^1 + \left[(2-x) \frac{\sin sx}{s} - (-1) \frac{-\cos sx}{s^2} \right]_1^2 \\ &= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right) \\ &= \frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}. \end{aligned}$$

Example 22.8. Find the Fourier sine transform of e^{-ax}/x . (V.T.U., 2010 S ; P.T.U., 2006 ; Rohtak, 2005)

Solution. Let $f(x) = e^{-ax}/x$, then its Fourier sine transform

$$\text{i.e.} \quad F_s[f(x)] = \int_0^{\infty} f(x) \sin sx \, dx = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx = F(s), \text{ say}$$

Differentiating both sides w.r.t. s , we get

$$\frac{d}{ds} \{F(s)\} = \int_0^{\infty} \frac{x e^{-ax} \cos sx}{x} \, dx = \int_0^{\infty} e^{-ax} \cos sx \, dx = \frac{a}{s^2 + a^2}$$

Integrating w.r.t. s , we obtain $F(s) = \int_0^{\infty} \frac{a}{s^2 + a^2} \, ds = \tan^{-1} \frac{s}{a} + c$

But $F(s) = 0$, when $s = 0$; $\therefore c = 0$. Hence $F(s) = \tan^{-1}(s/a)$.

Example 22.9. Find the Fourier cosine transform of $f(x) = 1/(1+x^2)$. (V.T.U., 2011 S ; Anna, 2009)

Hence derive Fourier sine transform of $\phi(x) = x/(1+x^2)$. (V.T.U., 2009 S)

Solution. $F_c[f(x)] = \int_0^{\infty} \frac{\cos sx}{1+x^2} \, dx = I$, say ...(i)

\therefore $\frac{dI}{ds} = \int_0^{\infty} \frac{-x \sin sx}{1+x^2} \, dx = - \int_0^{\infty} \frac{x^2 \sin sx}{x(1+x^2)} \, dx$...(ii)

$$= - \int_0^{\infty} \frac{[(1+x^2)-1] \sin sx}{x(1+x^2)} dx = - \int_0^{\infty} \frac{\sin sx}{x} dx + \int_0^{\infty} \frac{\sin sx}{x(1+x^2)} dx$$

or

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^{\infty} \frac{\sin sx}{x(1+x^2)} dx \quad \dots(iii)$$

 \therefore

$$\frac{d^2 I}{ds^2} = \int_0^{\infty} \frac{x \cos sx}{x(1+x^2)} dx = I$$

or

$$\frac{d^2 I}{ds^2} - I = 0 \quad \text{or} \quad (D^2 - 1)I = 0, \text{ where } D = \frac{dI}{ds}$$

Its solution is

$$I = c_1 e^s + c_2 e^{-s} \quad \dots(iv)$$

 \therefore

$$dI/ds = c_1 e^s - c_2 e^{-s} \quad \dots(v)$$

When $s = 0$, (i) and (iv) give $c_1 + c_2 = \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$

Also when $s = 0$, (iii) and (v) give $c_1 - c_2 = -\pi/2$.

Solving these, $c_1 = 0$, $c_2 = \pi/2$.

Thus from (i) and (iv), we have $F_c[f(x)] = I = (\pi/2)e^{-s}$

Now

$$\begin{aligned} F_s[\phi(x)] &= \int_0^{\infty} \frac{x \sin sx}{1+x^2} dx = -\frac{dI}{ds}, \text{ from (ii)} \\ &= (\pi/2)e^{-s}, \text{ from (v), with } c_1 = 0, c_2 = \pi/2. \end{aligned}$$

Example 22.10. Find the Fourier sine and cosine transform of x^{n-1} , $n > 0$.

(Madras, 2006)

Solution. We know that $F_s(x^{n-1}) = \int_0^{\infty} x^{n-1} \sin sx dx \quad \dots(i)$

and

$$F_c(x^{n-1}) = \int_0^{\infty} x^{n-1} \cos sx dx \quad \dots(ii)$$

 \therefore

$$\begin{aligned} F_c(x^{n-1}) + i F_s(x^{n-1}) &= \int_0^{\infty} (\cos sx + i \sin sx) x^{n-1} dx \\ &= \int_0^{\infty} e^{isx} x^{n-1} dx = \int_0^{\infty} e^{-t} \left(-\frac{t}{is}\right)^{n-1} \left(-\frac{dt}{is}\right) \quad [\text{Where } isx = -t] \\ &= \left(-\frac{1}{i}\right)^n \int_0^{\infty} e^{-t} t^{n-1} dt = \frac{(i)^{2n}}{(i)^n s^n} \Gamma(n) = \frac{(i)^n}{s^n} \Gamma(n) \\ &= \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^n \Gamma(n)/s^n = \left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right) \Gamma(n)/s^n \end{aligned}$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2} \quad \text{and} \quad F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2}.$$

Example 22.11. (a) Show that $F_s[x f(x)] = -\frac{d}{ds} \{F_c(s)\}$; $F_c[x f(x)] = \frac{d}{ds} \{F_s(s)\}$.

(b) Find the Fourier sine and cosine transform of $x e^{-ax}$.

(Madras, 2006)

Solution. (a) $\frac{d}{ds} \{F_c(s)\} = \frac{d}{ds} \left\{ \int_0^{\infty} f(x) \cos sx dx \right\} = \int_0^{\infty} f(x) (-x \sin sx) dx$
 $= - \int_0^{\infty} \{x f(x)\} \sin sxdx = -F_s[x f(x)] \quad \dots(i)$

$$\begin{aligned} \frac{d}{ds} \{F_s(s)\} &= \frac{d}{ds} \left\{ \int_0^{\infty} f(x) \sin sx dx \right\} = \int_0^{\infty} f(x) (x \cos sx) dx \\ &= \int_0^{\infty} \{x f(x)\} \cos sxdx = F_c[x f(x)] \quad \dots(ii) \end{aligned}$$

(b) We have
$$F_s(e^{-ax}) = \int_0^{\infty} e^{-ax} \sin sx \, dx = \frac{e^{-ax}}{a^2 + s^2} \Big|_{-a \sin sx - s \cos sx}^{\infty}$$

$$= \frac{s}{a^2 + s^2} \quad \dots(iii)$$

and
$$F_c(e^{-ax}) = \int_0^{\infty} e^{-ax} \cos sx \, dx = \frac{e^{-ax}}{a^2 + s^2} \Big|_{-a \cos sx + s \sin sx}^{\infty}$$

$$= \frac{a}{a^2 + s^2} \quad \dots(iv)$$

Now
$$F_c(xe^{-ax}) = -\frac{d}{ds} \{F_c(e^{-ax})\} \quad \text{[by (i)]}$$

$$= -\frac{d}{ds} \left(\frac{a}{a^2 + s^2} \right) = \frac{2as}{(a^2 + s^2)^2} \quad \text{[by (iv)]}$$

$$F_c(xe^{-ax}) = \frac{d}{ds} \{F_s(e^{-ax})\} \quad \text{[by (ii)]}$$

$$= \frac{d}{ds} \left(\frac{s}{a^2 + s^2} \right) = \frac{(a^2 + s^2) - s(2s)}{(a^2 + s^2)^2} = \frac{a^2 - s^2}{(a^2 + s^2)^2} \quad \text{[by (iii)]}$$

Example 22.12. If the Fourier sine transform of $f(x) = \frac{1 - \cos n\pi}{n^2\pi^2}$ ($0 \leq x \leq \pi$), find $f(x)$. (Delhi, 2002)

Solution. We have $f(x) =$ inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{\pi} = \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1 - \cos n\pi}{n^2\pi^2} \right\} \sin nx$$

$$= \frac{2}{\pi^3} \sum_{n=1}^{\infty} \left\{ \frac{1 - \cos n\pi}{n^2} \right\} \sin nx.$$

Example 22.13. Solve the integral equation*

$$\int_0^{\infty} f(\theta) \cos \alpha\theta \, d\theta = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ \theta, & \alpha > 1 \end{cases}$$

Hence evaluate $\int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt$. (V.T.U., 2011 S ; Kurukshetra, 2005)

Solution. We have
$$\int_0^{\infty} f(\theta) \cos \alpha\theta \, d\theta = F_c(\alpha)$$

$$\therefore F_c(\alpha) = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases} \quad \dots(i)$$

By the inversion formula, we have

$$f(\theta) = \frac{2}{\pi} \int_0^{\infty} F_c(\alpha) \cos \alpha\theta \, d\alpha = \frac{2}{\pi} \int_0^1 (1 - \alpha) \cos \alpha\theta \, d\alpha \quad \text{[Integrating by parts]}$$

$$= \frac{2}{\pi} \left[(1 - \alpha) \frac{\sin \alpha\theta}{\theta} \Big|_0^1 - \int_0^1 (-1) \frac{\sin \alpha\theta}{\theta} \, d\alpha \right] = \frac{2}{\pi\theta} \left[-\frac{\cos \alpha\theta}{\theta} \Big|_0^1 \right] = \frac{2(1 - \cos \theta)}{\pi\theta^2}$$

Now
$$F_c(\alpha) = \int_0^{\infty} f(\theta) \cos \alpha\theta \, d\theta = \int_0^{\infty} \frac{2(1 - \cos \theta)}{\pi\theta^2} \cos \alpha\theta \, d\theta \quad \dots(ii)$$

∴ From (i) and (ii), we have

$$\frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos \theta}{\theta^2} \cos \alpha \theta \, d\theta = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Now letting $\alpha \rightarrow 0$, we get $\frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos \theta}{\theta^2} \, d\theta = 1$

(V.T.U., 2008)

or

$$\int_0^{\infty} \frac{2 \sin^2 \theta/2}{\theta^2} \, d\theta = \pi/2$$

(Put $\theta/2 = t$, so that $d\theta = 2dt$)

$$\therefore \int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt = \pi/2.$$

PROBLEMS 22.1

1. Express the function $f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ as a Fourier integral.

Hence evaluate $\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} \, d\lambda$.

(Kottayam, 2005)

2. Find the Fourier integral representation for

(i) $f(x) = \begin{cases} 1 - x^2, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| > 1 \end{cases}$ (Mumbai, 2008)

(ii) $f(x) = \begin{cases} e^{ax}, & \text{for } x \leq 0, a > 0 \\ e^{-ax}, & \text{for } x \geq 0, a < 0 \end{cases}$

3. Using the Fourier integral representation, show that

(i) $\int_0^{\infty} \frac{\omega \sin x\omega}{1 + \omega^2} \, d\omega = \frac{\pi}{2} e^{-x} \quad (x > 0)$

(ii) $\int_0^{\infty} \frac{\cos \omega x}{1 + \omega^2} \, d\omega = \frac{\pi}{2} e^{-x} \quad (x \geq 0)$ (U.P.T.U., 2008)

(iii) $\int_0^{\infty} \frac{\sin \omega \cos x\omega}{\omega} \, d\omega = \frac{\pi}{2}$ when $0 \leq x < 1$.

(iv) $\int_0^{\infty} \frac{\sin \pi \alpha \sin \alpha \theta}{1 - \alpha^2} \, d\alpha = \begin{cases} \frac{1}{2} \pi \sin \theta, & 0 \leq \theta \leq \pi \\ 0, & \theta > \pi \end{cases}$

4. Find the Fourier transforms of

(i) $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

(W.B.T.U., 2005 ; Madras, 2003 ; P.T.U., 2003)

Hence evaluate $\int_{-\infty}^{\infty} \frac{\sin ax}{x} \, dx$

(Mumbai, 2009)

(ii) $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$

(S.V.T.U., 2008)

5. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$

(V.T.U., 2007)

Hence deduce that $\int_0^{\infty} \frac{\sin t - t \cos t}{t^3} \, dt = \frac{\pi}{4}$.

(Anna, 2009)

6. Given $F(e^{-x^2}) = \sqrt{\pi} e^{-s^2/4}$, find the Fourier transform of

(i) $e^{-x^2/3}$

(ii) $e^{-4(x-3)^2}$

7. Find the Fourier sine and cosine transforms of $f(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 0, & x \geq 2 \end{cases}$

(V.T.U., 2008)

8. Using the Fourier sine transform of e^{-ax} ($a > 0$), show that $\int_0^{\infty} \frac{x \sin kx}{a^2 + x^2} \, dx = \frac{\pi}{2} e^{-ak}$ ($k > 0$).

Hence obtain the Fourier sine transform of $x/(a^2 + x^2)$.

(Rohtak, 2006 ; Madras, 2003 S)

9. Find the Fourier cosine transform of e^{-ax} .

(Anna, 2009)

Hence evaluate $\int_0^{\infty} \frac{\cos \lambda x}{x^2 + a^2} \, dx$.

(V.T.U., 2003 S)

10. If the Fourier sine transform of $f(x)$ is e^{-as}/s , find $f(x)$. Hence obtain the inverse Fourier sine transform of $1/s$.

(Mumbai, 2009)

11. Find the Fourier cosine transform of e^{-x^2} and hence evaluate Fourier sine transform of xe^{-x^2} .
12. Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a > 0$ and hence prove that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform. (Anna, 2009)
13. Find the Fourier sine transform of (i) $\frac{1}{x(x^2 + a^2)}$. (Rohtak, 2006)
(ii) $[e^{-ax}/x]$, $a > 0$ (U.P.T.U., 2008)
14. Obtain Fourier sine transform of
(i) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ (Madras, 2000) (ii) $f(x) = \begin{cases} 4x, & \text{for } 0 < x < 1 \\ 4 - x, & \text{for } 1 < x < 4 \\ 0, & \text{for } x > 4 \end{cases}$ (V.T.U., 2006)
15. Find the Fourier cosine transform of $(1 - x/\pi)^2$. (P.T.U., 2006)
16. Find the finite Fourier sine and cosine transforms of $f(x) = 2x$, $0 < x < 4$. (V.T.U., 2011)
17. Find the finite sine transform of $f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \leq c \leq \pi$.
18. Show that the inverse finite Fourier sine transform of $F_s(n) = \frac{1}{\pi} \left\{ 1 + \cos n\pi - 2 \cos \frac{n\pi}{2} \right\}$ is
 $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$ (V.T.U., 2008)
19. Solve the integral equation $\int_0^\infty f(x) \sin tx \, dx = \begin{cases} 1, & 0 \leq t < 1, \\ 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ (Kottayam, 2005)
20. Solve the integral equation $\int_0^\infty f(x) \cos \alpha x \, dx = e^{-\alpha}$. (S.V.T.U., 2009 ; Rohtak, 2004)

22.6 (1) CONVOLUTION

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f * g = \int_{-\infty}^{\infty} f(u) g(x - u) du = h(x).$$

(2) **Convolution theorem for Fourier transforms.** The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, i.e.,

$$F\{f(x) * g(x)\} = F\{f(x)\} \cdot F\{g(x)\}$$

We have

$$\begin{aligned} F\{f(x) * g(x)\} &= F\left\{\int_{-\infty}^{\infty} f(u) g(x - u) du\right\} \\ &= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} f(u) g(x - u) du\right\} e^{isx} dx = \int_{-\infty}^{\infty} f(u) \left\{\int_{-\infty}^{\infty} g(x - u) \cdot e^{isx} dx\right\} du \\ &\quad \text{[Changing the order of integration]} \\ &= \int_{-\infty}^{\infty} f(u) \left\{\int_{-\infty}^{\infty} e^{is(x-u)} \cdot g(x - u) d(x - u)\right\} e^{isu} du \\ &= \int_{-\infty}^{\infty} e^{isu} f(u) \left\{\int_{-\infty}^{\infty} e^{ist} g(t) dt\right\} du \text{ where } x - u = t \\ &= \int_{-\infty}^{\infty} e^{isu} f(u) du \cdot F\{g(t)\} = \int_{-\infty}^{\infty} e^{isx} f(x) dx \cdot F\{g(x)\} = F\{f(x)\} \cdot F\{g(x)\} \end{aligned}$$

22.7 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively, then

$$(i) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \bar{G}(s) ds = \int_{-\infty}^{\infty} f(x) \bar{g}(x) dx \quad (ii) \frac{1}{2\pi} \int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

where bar implies the complex conjugate.

$$\begin{aligned} (i) \int_{-\infty}^{\infty} f(x) \bar{g}(x) dx &= \int_{-\infty}^{\infty} f(x) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) e^{isx} ds \right\} dx \quad [\text{Using the inversion formula for Fourier transform}] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) \left\{ \int_{-\infty}^{\infty} f(x) e^{isx} dx \right\} ds \quad [\text{Changing the order of integration}] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) F(s) ds, \text{ by definition of F-transform.} \end{aligned}$$

(ii) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \bar{F}(s) ds = \int_{-\infty}^{\infty} f(x) \bar{f}(x) dx \text{ or } \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Obs. The following Parseval's identities for Fourier cosine and sine transforms can be proved as above :

$$\begin{aligned} (i) \frac{2}{\pi} \int_0^{\infty} F_c(s) G_c(s) ds &= \int_0^{\infty} f(x) g(x) dx & (ii) \frac{2}{\pi} \int_0^{\infty} F_s(s) G_s(s) ds &= \int_0^{\infty} f(x) g(x) dx \\ (iii) \frac{2}{\pi} \int_0^{\infty} |F_c(s)|^2 ds &= \int_0^{\infty} |f(x)|^2 dx & (iv) \frac{2}{\pi} \int_0^{\infty} |F_s(s)|^2 ds &= \int_0^{\infty} |f(x)|^2 dx. \end{aligned}$$

Example 22.14. Using Parseval's identities, prove that

$$\begin{aligned} (i) \int_0^{\infty} \frac{dt}{(a^2 + t^2)(b^2 + t^2)} &= \frac{\pi}{2ab(a+b)} \quad (\text{S.V.T.U., 2009 ; J. in 2008}) \\ (ii) \int_0^{\infty} \frac{t^2}{(t^2 + 1)^2} dt &= \frac{\pi}{4} & (iii) \int_0^{\infty} \frac{\sin at}{t(a^2 + t^2)} dt &= \frac{\pi}{2} \cdot \frac{1 - e^{-a^2}}{a^2}. \end{aligned}$$

Solution. (i) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Then $F_c(s) = \frac{a}{a^2 + s^2}$, $G_c(s) = \frac{b}{b^2 + s^2}$

Now using Parseval's identity for Fourier cosine transforms, i.e.,

$$\frac{2}{\pi} \int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} f(x) g(x) dx \quad \dots(1)$$

We have
$$\frac{2}{\pi} \int_0^{\infty} \frac{ab}{(a^2 + s^2)(b^2 + s^2)} ds = \int_0^{\infty} e^{-(a+b)x} dx$$

or
$$\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(a^2 + s^2)(b^2 + s^2)} = \left| \frac{e^{-(a+b)x}}{-(a+b)} \right|_0^{\infty} = \frac{1}{a+b}$$

Thus
$$\int_0^{\infty} \frac{dt}{(a^2 + t^2)(b^2 + t^2)} = \frac{\pi}{2ab(a+b)}$$

(ii) Let $f(x) = \frac{x}{x^2 + 1}$ so that $F_s[f(x)] = \frac{\pi}{2} e^{-s}$

Now using Parseval's identity for sine transform, i.e.,

$$\frac{2}{\pi} \int_0^{\infty} [F_s\{f(x)\}]^2 ds = \int_0^{\infty} [f(x)]^2 dx$$

or
$$\int_0^{\infty} \left(\frac{x}{x^2 + 1} \right)^2 dx = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\pi}{2} e^{-s} \right)^2 ds = \frac{\pi}{2} \left| e^{-2s} / -2 \right|_0^{\infty} = \frac{\pi}{-4} (0 - 1) = \frac{\pi}{4}$$

Hence
$$\int_0^{\infty} \frac{t^2}{(t^2 + 1)^2} dt = \frac{\pi}{4}$$

(iii) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$. Then $F_c(s) = \frac{a}{a^2 + s^2}$, $G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\frac{2}{\pi} \int_0^{\infty} \frac{a \sin as}{s(a^2 + s^2)} ds = \int_0^a e^{-ax} \cdot 1 dx = \frac{1 - e^{-a^2}}{a}$

Thus $\int_0^{\infty} \frac{\sin at}{t(a^2 + t^2)} dt = \frac{\pi}{2a^2} (1 - e^{-a^2})$.

Example 22.15. Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} a - |x|, & \text{for } |x| < a \\ 0 & , \text{for } |x| > a \end{cases}$

Hence show that $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ and $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \pi/3$. (Anna, 2008)

Solution. Fourier transform of $f(x)$ i.e. $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-a}^a [a - |x|] e^{isx} dx$

$$= \int_{-a}^a [a - |x|] (\cos sx + i \sin sx) dx$$

$$= 2 \int_0^a (a - x) \cos sx dx + 0 \quad \left\{ \begin{array}{l} [a - |x|] \cos x \text{ in an even function} \\ [a - |x|] \sin x \text{ is an odd function} \end{array} \right.$$

$$= 2 \left[(a - x) \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right]_0^a = 2 \frac{1 - \cos as}{s^2} = 4 \frac{\sin^2 as/2}{s^2}$$

(i) By inversion formula,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4 \sin^2 as/2}{s^2} e^{-isx} ds$$

To evaluate $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$, put $x = 0$ and $a = 2$ so that

$$f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin s}{s}\right)^2 ds \quad \left[\because \frac{\sin s}{s} \text{ is an even function} \right]$$

$$\therefore \int_0^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4} f(0) = \frac{\pi}{2}. \quad [\because f(0) = a = 2]$$

(ii) Using Parseval's identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{4 \sin^2 as/2}{s^2}\right)^2 dx = \int_{-a}^a |[a - |x|]|^2 dx$$

$$\frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin as/2}{s}\right)^4 ds = 2 \int_0^a (a - x)^2 dx = 2 \left[\frac{(a - x)^3}{-3} \right]_0^a = \frac{2}{3} a^3$$

Putting $t = as/2$ and $dt = ads/2$

$$\frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin t}{2t/a}\right) \frac{2}{a} dt = \frac{2}{3} a^3 \quad \text{or} \quad \frac{2a^3}{\pi} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{2}{3} a^3$$

Hence $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$.

PROBLEMS 22.2

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$. (V.T.U., 2000 S)

2. Use Convolution theorem to find the inverse Fourier transform of $\frac{i}{(1+s^2)^2}$, given that $\frac{2}{(1+s^2)}$ is the Fourier transform of $e^{-|x|}$. (V.T.U., 2010 S)

3. Using Parseval's identity, show that

$$(i) \int_0^{\infty} \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}, \quad (\text{Hissar, 2007}) \quad (ii) \int_0^{\infty} \frac{t^2}{(4+t^2)(9+t^2)} dt = \frac{\pi}{10}. \quad (\text{Rohtak, 2003})$$

4. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$.

$$\text{Hence deduce that } \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}. \quad (\text{Anna, 2009})$$

5. Evaluate $\int_0^{\infty} \left(\frac{1-\cos x}{x}\right)^2 dx$.

22.8 RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

$$\text{If } f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases} \quad \dots(i)$$

then $F\{f(t)\} = L\{g(t)\}$.

$$\begin{aligned} \text{We have } F\{f(t)\} &= \int_{-\infty}^{\infty} e^{ist} f(t) dt = \int_{-\infty}^0 e^{ist} \cdot 0 \cdot dt + \int_0^{\infty} e^{ist} \cdot e^{-xt} g(t) dt \\ &= \int_0^{\infty} e^{(is-x)t} g(t) dt = \int_0^{\infty} e^{-pt} g(t) dt \quad \text{where } p = x - is \end{aligned}$$

Hence the Fourier transform of $f(t)$ [defined by (i)] is the Laplace transform of $g(t)$.

22.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x, t)] = \int_{-\infty}^{\infty} u e^{isx} dx$$

Then the Fourier transform of $\partial^2 u / \partial x^2$, i.e.

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx = \left[e^{isx} \frac{\partial u}{\partial x} - is e^{isx} \cdot u \right]_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} u e^{isx} dx,$$

on applying the general rule of integration by parts (p. 398). If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $\pm \infty$, then

$$\mathbf{F}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -s^2 \mathbf{F}[\mathbf{u}] \quad \dots(1)$$

Similarly in the case of Fourier sine and cosine transforms, we have

$$\mathbf{F}_s\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = s(\mathbf{u})_{\mathbf{x}=0} - s^2 \mathbf{F}_s[\mathbf{u}] \quad \dots(2)$$

$$\text{and } \mathbf{F}_c\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x}=0} - s^2 \mathbf{F}_c[\mathbf{u}] \quad \dots(3)$$

In general, the Fourier transform of the n th derivative of $f(x)$ is given by

$$\mathbf{F} \left[\frac{d^n f}{dx^n} \right] = (-is)^n \mathbf{F}[f(x)] \quad \dots(4)$$

provided the first $n - 1$ derivatives vanish as $x \rightarrow \pm \infty$.

$$\begin{aligned} \text{For } F[f^n(x)] &= \int_{-\infty}^{\infty} f^n(x) e^{isx} dx \\ &= \left[e^{isx} f^{n-1} - ise^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - \dots \right]_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f \cdot e^{isx} dx \end{aligned}$$

by the general rule of integration by parts, whence follows (4).

22.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt \quad \dots(1)$$

Multiply both sides by e^{xs} and integrate w.r.t. s within the limits $a - ir$ and $a + ir$. Then

$$\begin{aligned} \int_{a-ir}^{a+ir} e^{xs} \bar{f}(s) ds &= \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t) e^{-st} dt ds && [\text{Put } s = a - iu] \\ &= \int_{-r}^r e^{x(a-iu)} \int_0^{\infty} f(t) e^{-(a-iu)t} dt (-idu) = ie^{ax} \int_{-r}^r e^{-ixu} \int_0^{\infty} [e^{-at} f(t)] e^{iut} dt du \\ &= ie^{ax} \int_{-r}^r e^{-ixu} \int_{-\infty}^{\infty} \phi(t) e^{iut} dt du \end{aligned}$$

$$\text{where } \phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Proceeding to limits as $r \rightarrow \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs} \bar{f}(s) ds = ie^{ax} \cdot 2\pi\phi(x), \text{ by (2) of } \S 22.4 = 2\pi ie^{ax} \cdot e^{-ax} f(x) \text{ for } x > 0.$$

$$\text{Hence } f(x) = \int_{a-i\infty}^{a+i\infty} e^{xs} \bar{f}(s) ds \quad (x > 0) \quad \dots(2)$$

which is called the *complex inversion formula*. It provides a direct means for obtaining the inverse Laplace transform of a given function.

The integration in (2) is performed along a line LM parallel to the imaginary axis in the complex plane $z = x + iy$ such that all the singularities of $\bar{f}(s)$ lie to its left* (Fig. 22.1). Let us take a contour C which is composed of the line LM and the semi-circle C' (i.e., MNL). Then from (2)

$$\frac{1}{2\pi i} \int_{LM} e^{xs} \bar{f}(s) ds = \frac{1}{2\pi i} \int_C e^{xs} \bar{f}(s) ds - \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s) ds$$

The integral over C' tends to zero as $r \rightarrow \infty$ (under certain conditions†). Therefore,

$$\begin{aligned} f(x) &= \text{Lt}_{r \rightarrow \infty} \frac{1}{2\pi i} \int_C e^{xs} \bar{f}(s) ds \\ &= \text{sum of the residues of } e^{xs} \bar{f}(s) \text{ at the poles of } f(s) \quad \dots(3) \end{aligned}$$

[By §20.18]

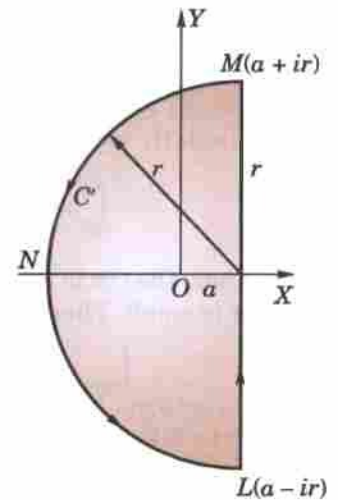


Fig. 22.1

* This has been so assumed simply to ensure the convergence of the integral (1).

† If positive constants A and k can be so found that $|\bar{f}(s)| < Ar^{-k}$ for every point on C' , then

$$\text{Lt}_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s) ds = 0.$$

(Jordan's Lemma)

Example 22.16. Evaluate $L^{-1} \left\{ \frac{1}{(s-1)(s^2+1)} \right\}$ by the method of residues.

Solution. Since $\left| \frac{1}{(s-1)(s^2+1)} \right| \sim \left| \frac{1}{s^3} \right|$ for $|s| \rightarrow \infty$, therefore,

$$L^{-1} \left[\frac{1}{(s-1)(s^2+1)} \right] = \text{sum of Res} \left[\frac{e^{xs}}{(s-1)(s^2+1)} \right] \text{ at the poles } s = 1, \pm i$$

$$\text{Now} \quad (\text{Res})_{s=1} = \text{Lt}_{s \rightarrow 1} \left[\frac{(s-1) \cdot e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^x}{2} \quad [\text{By } \S 20.19 (1)]$$

$$(\text{Res})_{s=i} = \text{Lt}_{s \rightarrow i} \left[\frac{(s-i) \cdot e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^{ix}}{(i-1)(i-1)} = -\frac{1}{2} \cdot \frac{e^{ix}}{1+i}$$

Changing i to $-i$, we get $(\text{Res})_{s=-i} = -\frac{1}{2} \cdot \frac{e^{-ix}}{1-i}$

$$\therefore L^{-1} \left[\frac{1}{(s-1)(s^2+1)} \right] = \frac{e^x}{2} - \frac{1}{2} \left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i} \right) = \frac{1}{2} (e^x - \sin x - \cos x).$$

Example 22.17. Prove that $L^{-1} \left(\frac{e^{-c\sqrt{s}}}{s} \right) = 1 - \text{erf} \left(\frac{c}{\sqrt{2x}} \right)$.

Solution. By the complex inversion formula,

$$L^{-1} \left(\frac{e^{-c\sqrt{s}}}{s} \right) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xs} \cdot \frac{e^{-c\sqrt{s}}}{s} ds.$$

Since $s = 0$ is a branch point of the integrand, we take a contour $LMNPQST$ as shown in Fig. 22.2, so that it doesn't include any singularity. Therefore, by Cauchy's theorem (§ 20.13), we have

$$\left\{ \int_{LM} + \int_{MN} + \int_{NP} + \int_{PQS} + \int_{ST} + \int_{TL} \right\} \times e^{xs} \frac{e^{-c\sqrt{s}}}{s} ds = 0 \quad \dots(i)$$

If $ON = \rho$ and $OP = \epsilon$, then along NP , $s = Re^{i\pi}$, therefore,

$$\int_{NP} = \int_{\rho}^{\epsilon} e^{-xR} \frac{e^{-ic\sqrt{R}}}{R} dR$$

Similarly along ST , $s = Re^{-i\pi}$, therefore,

$$\int_{ST} = \int_{\epsilon}^{\rho} e^{-xR} \frac{e^{ic\sqrt{R}}}{R} dR$$

Along the circle PQS , $s = \epsilon e^{i\theta}$. Also e^{xs} and $e^{-c\sqrt{s}}$ are both approximately 1 since ϵ is small. Therefore,

$$\int_{PQS} = \int_{\pi}^{-\pi} \frac{1}{\epsilon e^{i\theta}} \cdot \epsilon e^{i\theta} i d\theta = -2\pi i \text{ approximately.}$$

For $c > 0$, $|e^{-c\sqrt{s}}/s| < |s|^{-1}$.

But \int_{MN} and \int_{TL} both tend to zero as $r \rightarrow \infty$

Thus (i) takes the form

$$\int_{a-ir}^{a+ir} \frac{e^{xs-c\sqrt{s}}}{s} ds + \int_{\epsilon}^{\rho} e^{-xR} \frac{e^{ic\sqrt{R}} - e^{-ic\sqrt{R}}}{R} dR - 2\pi i = 0$$

Taking limits as $\epsilon \rightarrow 0$ and $\rho \rightarrow \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{xs-c\sqrt{s}}}{s} ds = 2\pi i - 2i \int_0^{\infty} e^{-xR} \frac{\sin c\sqrt{R}}{R} dR$$

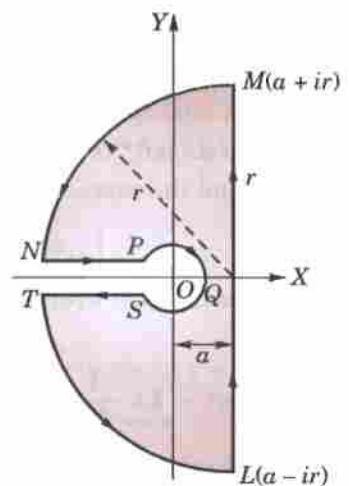


Fig. 22.2

or
$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs-c\sqrt{s}}}{s} ds = 1 - \frac{2}{\pi} \int_0^\infty e^{-t^2} \frac{\sin(ct/\sqrt{x})}{t} dt^*$$
, where $R = t^2/x$

$$= 1 - \frac{2}{\pi} \cdot \frac{\pi}{2} \operatorname{erf}\left(\frac{C}{2\sqrt{x}}\right) \text{ whence follows the result.}$$

PROBLEMS 22.3

Using the method of residues, evaluate the inverse Laplace transform of each of the following :

1. $\frac{1}{(s+1)(s-2)^2}$

2. $\frac{1}{(s-2)(s^2+1)}$

3. $\frac{1}{s^2(s^2-a^2)}$

4. $\frac{1}{(s-1)^2(s^2+1)}$

5. $\frac{1}{(s^2+1)^2}$

(V.T.U., 2008 S)

22.11 APPLICATION OF TRANSFORMS TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

(i) If in a problem $u(x, t)_{x=0}$ is given then we use infinite sine transform to remove $\partial^2 u / \partial x^2$ from the differential equation.

In case $[\partial u(x, t) / \partial x]_{x=0}$ is given then we employ infinite cosine transform to remove $\partial^2 u / \partial x^2$.

(ii) If in a problem $u(0, t)$ and $u(l, t)$ are given, then we use finite sine transform to remove $\partial^2 u / \partial x^2$ from the differential equation.

In case $(\partial u / \partial x)_{x=0}$ and $(\partial u / \partial x)_{x=l}$ are given, then we employ finite cosine transform to remove $\partial^2 u / \partial x^2$.

The method of solution is best explained through the following examples.

Heat conduction

Example 22.18. Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

(Osmania, 2003)

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t . We have to solve the heat-flow equation (§ 18.5)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (x > 0, t > 0) \quad \dots(i)$$

subject to the initial condition $u(x, 0) = f(x)$... (ii)

and the boundary condition $u(0, t) = 0$... (iii)

Taking Fourier sine transform of (1) and denoting $F_s[u(x, t)]$ by \bar{u}_s , we have

$$\frac{d\bar{u}_s}{dt} = c^2 [su(0, t) - s^2 \bar{u}_s] \quad \text{[By (2) of § 22.9]}$$

* We know that $\int_0^\infty e^{-t^2} \cos 2mt dt = \frac{1}{2} \sqrt{\pi} e^{-m^2}$ [Example 20.44]

Integrating both sides w.r.t. m from 0 to $c/2\sqrt{x}$.

$$\int_0^\infty e^{-t^2} \left| \frac{\sin 2mt}{2t} \right|_{t=0}^{c/2\sqrt{x}} dt = \frac{1}{2} \sqrt{\pi} \int_0^{c/2\sqrt{x}} e^{-m^2} dm$$

or
$$\int_0^\infty e^{-t^2} \frac{\sin(ct/\sqrt{x})}{t} dt = \frac{\pi}{2} \operatorname{erf}\left(\frac{c}{2\sqrt{x}}\right)$$
 [By § 7.18(1)]

or
$$\frac{d\bar{u}_s}{dt} + c^2 s^2 \bar{u}_s = 0 \quad \text{[By (iii)] ... (iv)]}$$

Also the Fourier sine transform of (ii) is $\bar{u}_s = \bar{f}(s)$ at $t = 0$(v)

Solving (iv) and using (v), we get $\bar{u}_s = \bar{f}_s(s)e^{-c^2 s^2 t}$

Hence taking its inverse Fourier sine transform, we obtain

$$u(x, t) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_s(s) e^{-c^2 s^2 t} \sin xs \, ds.$$

Example 22.19. Solve $\partial u / \partial t = 2\partial^2 u / \partial x^2$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$ ($x > 0$), $u(x, t)$ is bounded where $x > 0$, $t > 0$. (Rohtak, 2006)

Solution. Given $\partial u / \partial t = 2\partial^2 u / \partial x^2$, $x > 0$, $t > 0$...(i)

with boundary conditions: $u(0, t) = 0$, $u(x, t)$ is bounded ...(ii)

and initial condition $u(x, 0) = e^{-x}$, $x > 0$...(iii)

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (i) so that

$$\int_0^{\infty} \frac{\partial u}{\partial t} \sin px \, dx = 2 \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin px \, dx$$

or
$$\frac{d}{dt} \int_0^{\infty} u(x, t) \sin px \, dx = 2 \left[\left. \frac{\partial u}{\partial x} \sin px \right|_0^{\infty} - \int_0^{\infty} \frac{\partial u}{\partial x} \cdot p \cos px \, dx \right] \quad \text{(Integrating by parts)}$$

or
$$\frac{d\bar{u}_s}{dt} = -2p \int_0^{\infty} \frac{\partial u}{\partial x} \cos px \, dx$$
, if $\frac{\partial u}{\partial x} \rightarrow \frac{\partial u}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$ where $\bar{u}_s(p, t) = \int_0^{\infty} u(x, t) \sin px \, dx$

$$= -2p \left[\left. u(x, t) \cos px \right|_0^{\infty} - \int_0^{\infty} u(x, t) (-p \sin px) \, dx \right] \quad \text{[Again integrating by parts]}$$

$$= -2p [0 - u(0, t) + p \int_0^{\infty} u(x, t) \sin px \, dx] \quad [\because u(x, t) \rightarrow 0 \text{ as } x \rightarrow \infty \text{ by (ii)}]$$

$$= 2pu(0, t) - 2p^2 \bar{u}_s$$

or
$$\frac{d\bar{u}_s}{dt} = -2p^2 \bar{u}_s \quad \text{[By (ii)]}$$

Integrating $\int \frac{d\bar{u}_s}{\bar{u}_s} - \log c = -2p^2 \int dt$ or $\log \bar{u}_s - \log c = -2p^2 t$

$\therefore \bar{u}_s(p, t) = ce^{-2p^2 t}$...(iv)

Taking Fourier sine transform of both sides of (iii), we get

$$\int_0^{\infty} u(x, 0) \sin px \, dx = \int_0^{\infty} e^{-x} \sin px \, dx$$

or
$$\bar{u}_s(p, 0) = \left. \frac{e^{-x}}{1+p^2} (-\sin px - p \cos px) \right|_0^{\infty} = \frac{p}{1+p^2} \quad \text{...(v)}$$

Putting $t = 0$ in (iv) and using (v), we obtain $p/(1+p^2) = c$

Thus (iv) becomes $\bar{u}_s(p, t) = \frac{p}{1+p^2} e^{-2p^2 t}$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^{\infty} \frac{pe^{-2p^2 t}}{1+p^2} \sin px \, dp.$$

Example 22.20. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, ($x > 0$, $t > 0$) subject to the conditions

(i) $u = 0$, when $x = 0$, $t > 0$ (ii) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \leq 1, \text{ when } t = 0 \end{cases}$ (iii) $u(x, t)$ is bounded. (U.P.T.U., 2003 S)

Solution. Since $u(0, t) = 0$, we take Fourier sine transform of both sides of the given equation, we get

$$\int_0^{\infty} \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\frac{\partial}{\partial t} \int_0^{\infty} u \sin sx \, dx = -s^2 \bar{u}(s) + s u(0) \quad [\because u = 0, \text{ when } x = 0]$$

or
$$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad \text{or} \quad \frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \quad \text{or} \quad (D^2 + s^2) \bar{u} = 0 \text{ i.e., } D = \pm is$$

\therefore Its solution is $\bar{u}(s, t) = e^{-s^2 t}$... (1)

Since
$$\bar{u}(s, t) = \int_0^{\infty} u(x, t) \sin sx \, dx$$

\therefore
$$\bar{u}(s, 0) = \int_0^{\infty} u(x, 0) \sin sx \, dx = \int_0^1 1 \cdot \sin sx \, dx \quad [\text{By (ii)}]$$

$$= \frac{1 - \cos s}{s} \quad \dots (2)$$

From (1) and (2),
$$c = \bar{u}(s, 0) = \frac{1 - \cos s}{s}$$

Thus (1) gives
$$\bar{u}(s, t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \int_0^{\infty} \frac{1 - \cos s}{s} e^{-s^2 t} \, ds$$

which is the desired solution.

Example 22.21. Using finite Fourier transform, solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

given $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = 2x$ where $0 < x < 4$, $t > 0$.

(Rajasthan, 2006)

Solution. Since $u(0, t) = 0$, we take finite Fourier sine transform of both sides of the given equation

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi}{4} x \, dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi}{4} x \, dx$$

or
$$\frac{d}{dt} (\bar{u}_s) = F_s \left(\frac{\partial^2 u}{\partial x^2} \right)$$

$$= -\frac{n^2 \pi^2}{16} \bar{u}_s + \frac{n\pi}{4} [u(0, t) - (-1)^n u(4, t)]$$

$$= -\frac{n^2 \pi^2}{16} \bar{u}_s \quad [\because u(0, t) = 0, u(4, t) = 0.]$$

or
$$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2 \pi^2}{16} dt$$

Integrating both sides,
$$\log \bar{u}_s = -\frac{n^2 \pi^2}{16} t + c$$

or
$$\bar{u}_s(x, 0) = \alpha e^{-\frac{n^2 \pi^2 t}{16}} \quad \dots (i)$$

Putting $t = 0$,
$$a = \bar{u}_s(x, 0) = \int_0^4 u(x, 0) \sin \frac{n\pi x}{4} \, dx \quad [\because u(x, 0) = 2x]$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} \, dx = -\frac{32}{n\pi} \cos n\pi$$

Thus (i) gives,
$$\bar{u}_s(x, 0) = -\frac{32}{n\pi} \cos n\pi e^{-n^2\pi^2 t/16} = -\frac{32}{n\pi} (-1)^n e^{-n^2\pi^2 t/16}$$

Now taking inverse Fourier sine transform, we get

$$\begin{aligned} u(x, 0) &= \frac{2}{4} \sum_{n=1}^{\infty} \frac{32}{n\pi} (-1)^{n+1} e^{-n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right) \\ &= 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right). \end{aligned}$$

Example 22.22. If the initial temperature of an infinite bar is given by

$$\theta(x) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a, \end{cases}$$

determine the temperature at any point x and at any instant t .

(S.V.T.U., 2008 ; Rohtak, 2004)

Solution. To determine the temperature $\theta(x, t)$ at any point at any time, we have to solve the equation

$$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2} \quad (t > 0) \quad \dots(i)$$

subject to the initial condition $\theta(x, 0) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \quad \dots(ii)$

Taking Fourier transform of (i) and denoting $F[\theta(x, t)]$ by $\bar{\theta}$, we find

$$\frac{d\bar{\theta}}{dt} = -c^2 s^2 \bar{\theta} \quad \text{[by (1) of § 22.9] } \dots(iii)$$

Also the Fourier transform of (2) is

$$\bar{\theta}(s, 0) = \int_{-\infty}^{\infty} \theta(x, 0) e^{isx} dx = \int_{-a}^a \theta_0 e^{isx} dx = \theta_0 \frac{e^{isa} - e^{-isa}}{is} = 2\theta_0 \frac{\sin as}{s} \quad \dots(iv)$$

Solving (iii) and using (iv), we get $\bar{\theta} = \frac{2\theta_0 \sin as}{s} e^{-c^2 s^2 t}$

Hence taking its inverse Fourier transform, we get

$$\begin{aligned} \theta(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\theta_0 \sin as}{s} e^{-c^2 s^2 t} e^{-isx} ds = \frac{\theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-c^2 s^2 t} (\cos xs - i \sin xs) ds \\ &= \frac{2\theta_0}{\pi} \int_0^{\infty} \frac{\sin as}{s} e^{-c^2 s^2 t} \cos xs ds \quad \left\{ \begin{array}{l} \text{The second integral vanishes as} \\ \text{its integrand is an odd function} \end{array} \right. \\ &= \frac{\theta_0}{\pi} \int_0^{\infty} e^{-c^2 s^2 t} \frac{\sin(a+x)s + \sin(a-x)s}{s} ds \\ &= \frac{\theta_0}{\pi} \int_0^{\infty} e^{-v^2} \left\{ \sin \frac{(a+x)v}{c\sqrt{t}} + \sin \frac{(a-x)v}{c\sqrt{t}} \right\} \frac{dv}{v} \quad \text{where } v^2 = c^2 s^2 t \\ &= \frac{\theta_0}{\pi} \left\{ \operatorname{erf} \frac{(a+x)}{2c\sqrt{t}} + \operatorname{erf} \frac{(a-x)}{2c\sqrt{t}} \right\}. \end{aligned} \quad \text{[See footnote on p. 783]}$$

Example 22.23. A bar of length a is at zero temperature. At $t = 0$, the end $x = a$ is suddenly raised to temperature u_0 and the end $x = 0$ is insulated. Find the temperature at any point x of the bar at any time $t > 0$, assuming that the surface of the bar is insulated.

Solution. Here we have to solve the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < a, t > 0) \quad \dots(i)$$

subject to the conditions

$$u(x, 0) = 0 \quad \dots(ii); \quad u_x(0, t) = 0 \quad \dots(iii) \quad \text{and} \quad u(a, t) = u_0 \quad \text{(Rohtak, 2005) } \dots(iv)$$

The Laplace transform of (i), if $L[u(x, t)] = \bar{u}(x, s)$, is

$$s\bar{u} - u(x, 0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

Using (ii), we get $\frac{d^2\bar{u}}{dx^2} - \frac{s}{c^2}\bar{u} = 0$... (v)

Similarly the Laplace transform of (iii) and (iv) are

$$\bar{u}_x(0, s) = 0 \quad \dots(vi); \quad \bar{u}(a, s) = \frac{u_0}{s} \quad \dots(vii)$$

Solving (v), we have $\bar{u} = C_1 e^{x\sqrt{s}/c} + C_2 e^{-x\sqrt{s}/c}$

Using (vi), we find $C_1 = C_2$ so that

$$\bar{u} = C_1 (e^{\sqrt{sx}/c} + e^{-\sqrt{sx}/c}) = 2C_1 \cosh(\sqrt{sx}/c)$$

Now using (vii), we have $\bar{u} = \frac{u_0 \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)}$

By the inversion formula (3) § 22.10, we get

$$u(x, t) = \text{sum of the residues of } \left(\frac{e^{st} \cdot u_0 \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)} \right) \text{ at all the poles which occur at } s = 0$$

and $\cosh(\sqrt{sa}/c) = 0$ i.e., at $s = 0, \sqrt{sa}/c = \left(n - \frac{1}{2}\right)\pi i, n = 0, \pm 1, \pm 2, \dots$

or at $s = 0, s (= s_n) = -\frac{(2n-1)^2 c^2 \pi^2}{4a^2} = 0, 1, 2, \dots$

$$\text{Now } (\text{Res})_{s=0} = \text{Lt}_{s \rightarrow 0} \left\{ s \cdot \frac{u_0 e^{st} \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)} \right\} = u_0$$

$$\begin{aligned} (\text{Res})_{s=s_n} &= u_0 \text{Lt}_{s \rightarrow s_n} \left\{ (s - s_n) \cdot \frac{u_0 e^{st} \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)} \right\} \\ &= u_0 \text{Lt}_{s \rightarrow s_n} \left\{ \frac{s - s_n}{\cosh(\sqrt{sa}/c)} \right\} \cdot \text{Lt}_{s \rightarrow s_n} \left\{ \frac{e^{st} \cosh(\sqrt{sx}/c)}{s} \right\} \quad \left[\frac{0}{0} \text{ form} \right] \end{aligned}$$

$$= u_0 \text{Lt}_{s \rightarrow s_n} \frac{1}{\sinh(\sqrt{sa}/c) \cdot (a/2\sqrt{s/c})} \cdot \text{Lt}_{s \rightarrow s_n} \left\{ \frac{e^{st} \cosh(\sqrt{sx}/c)}{s} \right\}$$

$$= \frac{4u_0(-1)^n}{(2n-1)\pi} e^{-(2n-1)^2 \pi^2 c^2 t / 4a^2} \cos \frac{(2n-1)\pi x}{2a}$$

$$\text{Thus we get } u(x, t) = u_0 + \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2 \pi^2 c^2 t / 4a^2} \cos \frac{(2n-1)\pi x}{2a}.$$

Vibrations of a string

Example 22.24. An infinite string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$. Determine the displacement $y(x, t)$ of the string. (Rohtak, 2000)

Solution. The equation for the vibration of the string is

$$\partial^2 y / \partial t^2 = c^2 \partial^2 y / \partial x^2 \quad \dots(i)$$

and the initial conditions are

$$(\partial y / \partial t)_{t=0} = 0; y(x, 0) = f(x) \quad \dots(ii)$$

Multiplying (i) by e^{isx} and integrating w.r.t. x from $-\infty$ to ∞ , we get

$$\partial^2 Y / \partial t^2 = c^2(-s^2 Y) \quad \text{provided } y \text{ and } \partial y / \partial x \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$\therefore \text{ a solution of } d^2 Y / dt^2 + c^2 s^2 Y = 0 \text{ is } Y = A_1 \cos cst + A_2 \sin cst \quad \dots(iii)$$

Also Fourier transforms of (ii) are

$$\partial y / \partial t = 0 \quad \text{and} \quad Y = F(s) \text{ when } t = 0$$

Applying these to (iii), we get

$$A_2 = 0 \quad \text{and} \quad A_1 = F(s)$$

Thus

$$Y = F(s) \cos cst$$

Now taking inverse Fourier transforms, we get

$$\begin{aligned} y(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cos cst \cdot e^{-isx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \frac{e^{ics t} + e^{-ics t}}{2} \cdot e^{-isx} dx \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} [F(s)e^{-is(x-ct)} + F(s)e^{-is(x+ct)}] ds \\ &= \frac{1}{2} [f(x-ct) + f(x+ct)] \end{aligned} \quad \left[\because f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \right]$$

Example 22.25. An infinitely long string having one end at $x = 0$, is initially at rest along the x -axis. The end $x = 0$ is given a transverse displacement $f(t)$, $t > 0$. Find the displacement of any point of the string at any time.

Solution. Let $y(x, t)$ be the transverse displacement of any point x of the string at any time t . Then we have to solve the wave equation (§ 18.4)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (x > 0, t > 0) \quad \dots(i)$$

subject to the conditions $y(x, 0) = 0$, $y_t(x, 0) = 0$, $y(0, t) = f(t)$ and the displacement $y(x, t)$ is bounded.

The Laplace transform of (i), writing $L[y(x, t)] = \bar{y}(x, s)$ is

$$s^2 \bar{y} - sy(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2}$$

Using the first two conditions, we have

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \left[\frac{s}{c} \right]^2 \bar{y} \quad \dots(ii)$$

Similarly the Laplace transforms of the third and fourth conditions are

$$\bar{y}(0, s) = \bar{f}(s) \text{ at } x = 0 \quad \dots(iii) \quad \text{and} \quad \bar{y}(x, s) \text{ is bounded.} \quad \dots(iv)$$

Solving (ii), we get

$$\bar{y}(x, s) = C_1 e^{sx/c} + C_2 e^{-sx/c}$$

To satisfy condition (iv), we must have $C_1 = 0$

Using the condition (iii), we get $C_2 = \bar{f}(s)$.

$$\therefore \bar{y}(x, s) = \bar{f}(s) e^{-sx/c}$$

Using the complex inversion formula, we obtain

$$y = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{(t-x/c)s} \bar{f}(s) ds = f(t-x/c).$$

Example 22.26. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l-x)$, where μ is a constant and then released. Find the displacement of any point x of the string at any time $t > 0$. (V.T.U., M.E., 2006)

Solution. We have to solve the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (x > 0, t > 0)$

subject to the conditions $y(0, t) = 0, y(l, t) = 0$
and $y(x, 0) = \mu x(l - x), y_t(x, 0) = 0$

Now taking Laplace transform, writing $L[y(x, t)] = \bar{y}(x, s)$, we get

$$s^2 \bar{y} - s \bar{y}(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \quad \dots(i)$$

where $\bar{y}(0, s) = 0, \bar{y}(l, s) = 0 \quad \dots(ii)$

$$\therefore (i) \text{ reduces to } \frac{\partial^2 \bar{y}}{\partial x^2} - \left(\frac{s}{c}\right)^2 \bar{y} = -\frac{\mu s x(l-x)}{c^2}$$

Its solution is $\bar{y}(x, s) = c_1 \cosh(sx/c) + c_2 \sinh(sx/c) + \frac{\mu x(l-x)}{s} - \frac{2c^2 \mu}{s^3}$

Applying the conditions (ii), we get

$$c_1 = 2c^2 \mu / s^2 \quad \text{and} \quad c_2 = \frac{2c^2 \mu}{s^3} \left[\frac{1 - \cosh(sl/c)}{\sinh(sl/c)} \right] - \frac{2c^2 \mu}{s^3} \tanh(s/2c)$$

$$\text{Thus} \quad \bar{y}(x, s) = \frac{2c^2 \mu}{s^3} \left[\frac{\cosh\{s(2x-l)/2c\}}{\cosh(sl/2c)} \right] + \frac{\mu x(l-x)}{s} - \frac{2c^2 \mu}{s^3}$$

Now using the inversion formula (3) § 22.10, we get

$y(x, t) =$ sum of the residues of

$$2c^2 \mu \left[\frac{e^{st} \cosh\{s(2x-l)/2c\}}{s^3 \cosh(sl/2c)} \right] \text{ at all the poles} + \mu x(l-x) - c^2 \mu t^2$$

Proceeding exactly as in Example 22.23, we have,

sum of the residues of $2c^2 \mu \left[\frac{e^{st} \cosh\{s(2x-l)/2c\}}{s^3 \cosh sl/2c} \right]$ at all the poles

$$\begin{aligned} &= c^2 \mu \left[t^2 + \left(\frac{2x-l}{2c}\right)^2 - \left(\frac{l}{2c}\right)^2 \right] \\ &\quad - \frac{32c^2 \mu}{\pi^3} \left(\frac{l}{2c}\right)^2 \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{(2n-1)^3} \cos\left\{\frac{(2n-1)\pi(2x-l)}{2l}\right\} \cos\left\{\frac{(2n-1)\pi ct}{l}\right\} \right] \\ &= c^2 \mu t^2 - \mu x(l-x) + \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^3} \sin\frac{(2n-1)\pi x}{l} \cos\frac{(2n-1)\pi ct}{l} \right] \end{aligned}$$

$$\text{Hence} \quad y(x, t) = \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^3} \sin\frac{(2n-1)\pi x}{l} \cos\frac{(2n-1)\pi ct}{l} \right].$$

Transmission lines

Example 22.27. A semi-infinite transmission line of negligible inductance and leakage per unit length has its voltage and current equal to zero. A constant voltage v_0 is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) and at any instant.

Solution. Let $v(x, t)$ and $i(x, t)$ be the voltage and current at any point x and at any time t . If $L = 0$ and $G = 0$, then the transmission line equations [(1) and (2) of § 18.10] become

$$\frac{\partial v}{\partial x} = -Ri, \quad \frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t} \quad \text{i.e.,} \quad \frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t} \quad \dots(i)$$

The boundary conditions are $v(0, t) = v_0$ and $i(x, t)$ is finite for all x and t .

The initial conditions are $v(x, 0) = 0, i(x, 0) = 0. \quad \dots(ii)$

Laplace transforms of (i), are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0) \quad \text{or} \quad \frac{d^2\bar{v}}{dx^2} - RCs\bar{v} = 0 \quad \dots(iii)$$

Laplace transforms of the conditions in (ii), are

$$\bar{v}(0, s) = \frac{v_0}{s} \quad \text{at } x = 0 \quad \dots(iv)$$

and

$$\bar{v}(x, s) \text{ remains finite as } x \rightarrow \infty \quad \dots(v)$$

\therefore the solution of (iii) is

$$\bar{v}(x, s) = C_1 e^{\sqrt{RCs}x} + C_2 e^{-\sqrt{RCs}x}$$

To satisfy condition (v), we must have $C_1 = 0$.

Using the condition (iv), we get $C_2 = v_0/s$

Thus

$$\bar{v}(x, s) = \frac{v_0}{s} e^{-\sqrt{RCs}x}$$

Using the inversion formula, we obtain

$$\begin{aligned} v(x, t) &= v_0 L^{-1} \left\{ \frac{e^{-\sqrt{RC}x \cdot \sqrt{s}}}{s} \right\} = v_0 \operatorname{erfc} \left(x \frac{\sqrt{RC}}{2\sqrt{t}} \right) \quad [\text{By Ex. 22.17}] \\ &= v_0 \frac{x\sqrt{RC}}{2\sqrt{\pi}} \int_0^t u^{-3/2} e^{-(RCx^2/4u)} du \end{aligned}$$

\therefore since $i = -\frac{1}{R} \frac{\partial v}{\partial x}$, we obtain by differentiation,

$$i(x, t) = \frac{v_0 x}{2\sqrt{x}} \sqrt{\frac{C}{R}} t^{-3/2} e^{(-RCx^2/4t)}$$

Example 22.28. A transmission line of length l has negligible inductance and leakage. A constant voltage v_0 is applied at the sending end ($x = 0$) and is open circuited at the far end. Assuming the initial voltage and current to be zero, determine the voltage and current.

Solution. For a transmission line with $L = G = 0$, the voltage v and current i are given by the equations

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t} \quad \text{and} \quad \frac{\partial v}{\partial x} + Ri = 0 \quad \dots(i)$$

The boundary conditions are (for $t > 0$)

$$v = v_0 \quad \text{at } x = 0 \quad \text{and} \quad i = \frac{\partial v}{\partial x} = 0 \quad \text{at } x = l \quad \dots(ii)$$

The initial condition is $v = 0$ at $t = 0$ ($x > 0$)

Laplace transforms of (i) and (ii) are

$$\frac{\partial^2 \bar{v}}{\partial x^2} = RC(s\bar{v} - 0) \quad \dots(iii)$$

and

$$\bar{v} = v_0/s \quad \text{at } x = 0, \quad \frac{\partial \bar{v}}{\partial x} = 0 \quad \text{at } x = l \quad \dots(iv)$$

\therefore the solution of (iii) is

$$\bar{v} = c_1 \cosh \sqrt{RCs}x + c_2 \sinh \sqrt{RCs}x$$

Applying conditions (iv), it gives

$$v_0/s = c_1, \quad 0 = c_1 \sinh \sqrt{RCs}l + c_2 \cosh \sqrt{RCs}l$$

$$\therefore \bar{v} = \frac{v_0}{s} \left[\cosh \sqrt{RCs}x - \frac{\sinh \sqrt{RCs}l}{\cosh \sqrt{RCs}l} \sinh \sqrt{RCs}x \right]$$

$$= \frac{v_0}{s} \frac{\cosh pq\sqrt{s}}{\cosh p\sqrt{s}}$$

where $p = \sqrt{(RC)l}$ and $q = (l-x)/l$

By the inversion formula (3) § 22.10, we get

$$v(x, t) = \text{sum of the residues of } (e^{st} \bar{v}) \text{ at all poles of } e^{st} \bar{v}. \quad \dots(iv)$$

These poles are at $s = 0$ and $p\sqrt{s} = \pm i(2n-1)\pi/2 = \pm ipk$ (say)

$$\text{Now } \text{Res}(e^{st} \bar{v})_{s=0} = \text{Lt}_{s \rightarrow 0} \frac{se^{st} v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} = v_0$$

$$\text{and } \text{Res}(e^{st} \bar{v})_{s=-k^2} = \text{Lt}_{s \rightarrow -k^2} \frac{(s+k^2)e^{st} v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\begin{aligned} &= \text{Lt}_{s \rightarrow -k^2} \frac{v_0 \cdot e^{st} \cosh pq\sqrt{s} + (s+k^2)(\dots)}{\cosh p\sqrt{s} + s \sinh p\sqrt{s} \cdot \frac{1}{2} ps^{-1/2}} \\ &= \frac{v_0 e^{-k^2 t} \cosh(ipqk) + 0}{0 + 1/2(ipk) \sinh(ipk)} = \frac{2v_0 e^{-k^2 t} \cos(pqk)}{-pk \sin pk} \end{aligned}$$

Adding up all the residues, (iv) gives

$$v(x, t) = v_0 + \frac{4v_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-[(2n-1)^2 \pi^2 t / 4RCl^2]} \cos [(2n-1) \pi(l-x)/2l]$$

$$\begin{aligned} [\because pk &= (2n-1) \pi/2, -\sin pk = (-1)^n, pqk = \frac{1}{2}(2n-1) \pi(l-x)/l, \\ k^2 &= (2n-1)^2 \pi^2 / 4RCl^2] \end{aligned}$$

$$\text{Also } i = -\frac{1}{R} \frac{\partial v}{\partial x}. \quad [\text{By (i)}]$$

PROBLEMS 22.4

1. Solve the differential equation using Laplace transform method, $\frac{\partial y}{\partial t} = 3 \frac{\partial^2 y}{\partial x^2}$
where $y(\pi/2, t) = 0$, $(\partial y / \partial x)_{x=0} = 0$ and $y(x, 0) = 30 \cos 5x$. (U.P.T.U., 2005)

2. Using suitable transforms, solve the differential equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$, $0 \leq x \leq \pi$, $t \geq 0$.
where $V(0, t) = 0 = V(\pi, t)$ and $V(x, 0) = V_0$ constant.

3. The initial temperature along the length of an infinite bar is given by $u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. If the temperature $u(x, t)$ satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, find the temperature at any point of the bar at any point t . (Rohtak, 2006)

4. Use the complex form of the Fourier transform to show that

$$V = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \bar{f}(u) e^{t-(x-u)^2/4t} du$$

is the solution of the boundary value problem

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0; \quad V = f(x) \text{ when } t = 0. \quad (\text{U.P.T.U., 2008})$$

5. A semi-infinite solid ($x > 0$) is initially at temperature zero. At time $t = 0$, a constant temperature $\theta_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point x and at any time t , is given by $\theta(x, t) = \theta_0 \operatorname{erfc}(x/2c\sqrt{t})$.

6. A solid is initially at constant temperature θ_0 , while the ends $x = 0$ and $x = a$ are maintained at temperature zero. Determine the temperature at any point of the solid at any later time $t > 0$.
7. An infinite string is initially at rest along the x -axis. Its one end which is at $x = 0$, is given a periodic transverse displacement $a_0 \sin \omega t$, $t > 0$. Show that the displacement of any point of the string at any time is given by

$$y(x, t) = \begin{cases} a_0 \sin \omega(t - x/c), & t > x/c \\ 0, & t < x/c, \end{cases}$$

where c is the wave velocity.

8. An infinite string has an initial transverse displacement $y(x, 0) = f(x)$, $-\infty < x < \infty$, and is initially at rest. Show that

$$y(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)].$$

9. A semi-infinite transmission line has negligible inductance and leakage per unit length. A voltage v is applied at the sending end ($x = 0$) which is given by

$$v(0, t) = \begin{cases} v_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$

Show that the voltage at any point $x > 0$ at any time $t > 0$ is given by

$$v(x, t) = v_0 \operatorname{erfc} \left[\frac{x}{2} \sqrt{RC/t} \right].$$

22.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 22.5

Fill in the blanks or choose the correct answer in each of the following problems :

- Fourier cosine transform of $f(t)$ is
- Fourier sine transform of $1/x$ is
- Convolution theorem for Fourier transforms states that
- If Fourier transform of $f(x)$ is $F(s)$, then the inversion formula is
- $F[x^n f(x)] = \dots\dots\dots$
- If $F\{f(x)\} = F(s)$, then $F\{f(x - a)\} = \dots\dots\dots$
- Fourier sine integral representation of a function $f(x)$ is given by
- If $F_c\{f(ax)\} = k F_c(s/a)$, then $k = \dots\dots\dots$
- Fourier transform of second derivative of $u(x, t)$ is
- If $f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is
- Fourier sine transform of $f'(x)$ in the interval $(0, l)$ is
- If $F(\lambda)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is
- Inverse finite Fourier sine transform of $F_s(p) = \frac{1 - \cos p\pi}{(p\pi)^2}$ for $p = 1, 2, 3, \dots$ and $0 < x < \pi$ is
- If Fourier transform of $f(x) = F(s)$, then Fourier Transform of $f(2x)$ is
- Fourier cosine transform of e^{-x} is
- $f(x) = 1, 0 < x < \infty$ cannot be represented by a Fourier integral. (True or False)
- $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 dx$. (True or False)
- Fourier transform is a linear operation. (True or False)
- $F_s[x f(x)] = -\frac{d}{ds} F_c(s)$. (True or False)
- Kernel of Fourier transform is e^{sx} . (True or False)
- Finite Fourier cosine transform of $f(x) = 1$ in $(0, \pi)$ is zero. (True or False)

Z-Transforms

1. Introduction. 2. Definition. 3. Some standard Z-transforms. 4. Linearity property. 5. Damping rule. 6. Some standard results. 7. Shifting u_n to the right and to the left. 8. Multiplication by n . 9. Two Basic theorems. 10. Some useful Z-transforms. 11. Some useful inverse Z-transforms. 12. Convolution theorems. 13. Convergence of Z-transforms. 14. Two-sided Z-transform. 15. Evaluation of inverse Z-transforms. 16. Application to Difference equations. 17. Objective Type of Questions.

23.1 INTRODUCTION

The development of communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform (§ 21.2). The main difference is that the Z-transform operates not on functions of continuous arguments but on sequences of the discrete integer-valued arguments, *i.e.* $n = 0, \pm 1, \pm 2, \dots$. The analogy of Laplace transform to Z-transform can be carried further. For every operational rule of Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transform, there is a corresponding application of Z-transform. A discrete system is expressible as a difference equation (§ 30.2) and its solutions are found using Z-transforms.

23.2 DEFINITION

If the function u_n is defined for discrete values ($n = 0, 1, 2, \dots$) and $u_n = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ whenever the infinite series converges.} \quad \dots(i)$$

The inverse Z-transform is written as $Z^{-1}[U(z)] = u_n$.

If we insert a particular complex number z into the power series (i), the resulting value of $Z(u_n)$ will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z .

23.3 SOME STANDARD Z-TTRANSFORMS

The direct application of the definition gives the following results :

(1) $Z(a^n) = \frac{z}{z-a}$ (2) $Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})$, p being a +ve integer.

Proof. (1) By definition, $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \dots = \frac{1}{1 - (a/z)} = \frac{z}{z - a} \quad (\text{Kottayam, 2005})$$

$$(2) \quad Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} \cdot n \cdot z^{-(n+1)} \quad \dots(i)$$

$$\text{Changing } p \text{ to } p - 1, \text{ we get } Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} \cdot z^{-n}$$

Differentiating it w.r.t. z ,

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1} \cdot (-n) z^{-(n+1)} \quad \dots(ii)$$

$$\text{Substituting (ii) in (i), we obtain } Z(n^p) = -z \frac{d}{dz}[Z(n^{p-1})]$$

which is the desired recurrence formula.

In particular, we have the following formulae :

$$(3) \quad Z(1) = \frac{z}{z-1} \quad [\text{Taking } a = 1 \text{ in (1)}] \quad (4) \quad Z(n) = \frac{z}{(z-1)^2} \quad [\text{Taking } p = 1 \text{ in (2)}]$$

$$(5) \quad Z(n^2) = \frac{z^2 + z}{(z-1)^3} \quad (\text{V.T.U., 2006}) \quad (6) \quad Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4}$$

$$(7) \quad Z(n^4) = \frac{z^4 + 11z^3 + 11z^2 + z}{(z-1)^5}$$

23.4 LINEARITY PROPERTY

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

$$\text{Proof. By definition, } Z(au_n + bv_n - cw_n) = \sum_{n=0}^{\infty} (au_n + bv_n - cw_n)z^{-n}$$

$$= a \sum_{n=0}^{\infty} u_n z^{-n} + b \sum_{n=0}^{\infty} v_n z^{-n} - c \sum_{n=0}^{\infty} w_n z^{-n}$$

$$= aZ(u_n) + bZ(v_n) - cZ(w_n).$$

23.5 DAMPING RULE

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

$$\text{Proof. By definition, } Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n \cdot z^{-n} = \sum_{n=0}^{\infty} u_n \cdot (az)^{-n} = U(az). \quad (\text{Madras, 2006})$$

Cor. $Z(a^n u_n) = U(z/a)$

Obs. The geometric factor a^{-n} when $|a| < 1$, damps the function u_n , hence the name *damping rule*.

23.6 SOME STANDARD RESULTS

The application of the damping rule leads to the following standard results :

$$(1) \quad Z(na^n) = \frac{az}{(z-a)^2} \quad (2) \quad Z(n^2 a^n) = \frac{az^2 + a^2 z}{(z-a)^3}$$

$$(3) Z(\cos n\theta) = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1} \quad (4) Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$(5) Z(a^n \cos n\theta) = \frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2} \quad (6) Z(a^n \sin n\theta) = \frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2}$$

Proofs. (1) We know that $Z(n) = \frac{z}{(z-1)^2}$. Applying damping rule, we have

$$Z(na^n) = U(a^{-1}z) = \frac{a^{-1}z}{(a^{-1}z-1)^2} = \frac{az}{(z-a)^2} \quad (\text{Madras, 2000 S})$$

(2) We know that $Z(n^2) = \frac{z^2+z}{(z-1)^3}$. Applying damping rule, we have

$$Z(n^2a^n) = U(a^{-1}z) = \frac{(a^{-1}z)^2 + a^{-1}z}{(a^{-1}z-1)^3} = \frac{a(z^2+az)}{(z-a)^3}$$

(3) and (4) We know that $Z(1) = \frac{z}{z-1}$. Applying damping rule, we have

$$\begin{aligned} Z(e^{-in\theta}) &= Z(e^{-i\theta})^n \cdot 1 = \frac{ze^{i\theta}}{ze^{i\theta}-1} = \frac{z}{z-e^{-i\theta}} = \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})} \\ &= \frac{z(z-\cos \theta) - iz \sin \theta}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1} = \frac{z(z-\cos \theta) - iz \sin \theta}{z^2 - 2z \cos \theta + 1} \end{aligned}$$

Equating real and imaginary parts, we get (3) and (4).

(V.T.U., 2010 S; Anna, 2009)

(5) We know that $Z(\cos n\theta) = \frac{z(z-\cos \theta)}{z^2-2z \cos \theta+1}$. By damping rule, we have

$$Z(a^n \cos n\theta) = \frac{a^{-1}z(a^{-1}z-\cos \theta)}{(a^{-1}z)^2-2(a^{-1}z)\cos \theta+1} = \frac{z(z-a \cos \theta)}{z^2-2az \cos \theta+a^2} \quad (\text{V.T.U., 2006})$$

Similarly using (4) above, we get (6).

Example 23.1. Find the Z-transform of the following :

(i) $3n - 4 \sin n\pi/4 + 5a$

(ii) $(n+1)^2$

(V.T.U., 2010)

(iii) $\sin(3n+5)$.

(V.T.U., 2009 S; Kottayam, 2005)

Solution. (i) $Z(3n - 4 \sin \frac{n\pi}{4} + 5a) = 3Z(n) - 4Z\left(\sin \frac{n\pi}{4}\right) + 5a Z(1)$ [By Linearty property]

$$= 3 \cdot \frac{z}{(z-1)^2} - 4 \cdot \frac{z \sin n\pi/4}{z^2 - 2z \cos \pi/4 + 1} + 5a \cdot \frac{z}{z-1} \quad [\text{Using formulae for } Z(1), Z(n), Z(\sin n\theta)]$$

$$= \frac{(3-5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1}$$

(ii) $Z(n+1)^2 = Z(n^2 + 2n + 1) = Z(n^2) + 2Z(n) + Z(1)$

$$= \frac{z^2+z}{(z-1)^3} + 2 \frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(2z+1)}{(z-1)^3}$$

(iii) $Z[\sin(3n+5)] = Z(\sin 3n \cos 5 + \cos 3n \sin 5)$

$$= \cos 5 \cdot Z(\sin 3n) + \sin 5 \cdot Z(\cos 3n) \quad (\text{using formulae for } Z(\sin n\theta), Z(\cos n\theta))$$

$$= \cos 5 \cdot \frac{z \sin 3}{z^2 - 2z \cos 3 + 1} + \sin 5 \cdot \frac{z(z - \cos 3)}{z^2 - 2z \cos 3 + 1} = z \cdot \frac{(z \sin 5 - \sin 2)}{z^2 - 2z \cos 3 + 1}$$

Example 23.2. Find the Z-transforms of the following

- (i) e^{an} (ii) ne^{an} (iii) n^2e^{an} .

Solution. (i) Let $u_n = 1$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$. By damping rule $Z(k^{-n} u_n) = U(kz)$,

$$\begin{aligned} \therefore Z(e^{an}) &= Z(k^{-n} \cdot 1) = U(kz) = \frac{kz}{kz - 1} & \left[\because U(z) = Z(1) = \frac{z}{z-1} \right] \\ &= \frac{z}{z - 1/k} = \frac{z}{z - e^a} \end{aligned}$$

(ii) Let $u_n = n$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule, $Z(e^{an} \cdot n) = Z(k^{-n} \cdot n) = U(kz)$ where $U(z) = Z(n) = \frac{z}{(z-1)^2}$

$$\frac{kz}{(kz-1)^2} = \frac{z}{k(z-1/k)^2} = \frac{e^a z}{(z-e^a)^2}$$

(iii) Let $u_n = n^2$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule,

$$\begin{aligned} Z(e^{an} \cdot n^2) &= Z(k^{-n} \cdot n^2) = U(kz) \quad \text{where} \quad U(z) = Z(n^2) = \frac{z^2 + z}{(z-1)^3} \\ &= \frac{(kz)^2 + kz}{(kz-1)^3} = \frac{z(z+1/k)}{(z-1/k)^3} = \frac{ze^a(z+e^a)}{(z-e^a)^3}. \end{aligned}$$

Example 23.3. Find the Z-transform of (i) $\cosh n\theta$ (V.T.U., 2011) (ii) $a^n \cosh n\theta$.

$$\begin{aligned} \text{Solution. (i)} \quad Z(\cosh n\theta) &= Z\left(\frac{e^{n\theta} + e^{-n\theta}}{2}\right) \\ &= \frac{1}{2} \left[Z\{(e^{-\theta})^{-n} \cdot 1\} + Z\{(e^{\theta})^{-n} \cdot 1\} \right] \end{aligned}$$

Apply damping rule to both terms, taking $u_n = 1$.

$$\begin{aligned} Z(\cosh n\theta) &= \frac{1}{2} \left[\frac{ze^{-\theta}}{ze^{-\theta} - 1} + \frac{ze^{\theta}}{ze^{\theta} - 1} \right] & \left[\because z(1) = \frac{z}{z-1} \right] \\ &= \frac{1}{2} \left[\frac{2z^2 - z(e^{\theta} + e^{-\theta})}{z^2 - z(e^{\theta} + e^{-\theta}) + 1} \right] = \frac{z^2 - z \cosh \theta}{z^2 - 2z \cosh \theta + 1} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad Z(a^n \cosh n\theta) &= Z[(a^{-1})^{-n} \cdot \cosh n\theta] & \text{[Apply damping rule using (i)]} \\ &= \frac{(a^{-1}z)^2 - (a^{-1}z) \cosh \theta}{(a^{-1}z)^2 - 2(a^{-1}z) \cosh \theta + 1} = \frac{z(z - a \cosh \theta)}{z^2 - 2az \cosh \theta + a^2}. \end{aligned}$$

Example 23.4. Find the Z-transforms of

- (i) $e^t \sin 2t$ (Madras, 2003) (ii) $c^k \cos k\alpha$, ($k \geq 0$) (U.P.T.U., 2004 S)

$$\text{Solution. (i) We know that } Z(\sin 2t) = \frac{z \sin 2}{z^2 - 2z \cos 2 + 1} \quad \dots(A)$$

$$\begin{aligned} \therefore Z(e^t \sin 2t) &= Z[(e^{-1})^{-t} \cdot \sin 2t] & \text{[Apply damping rule, using (A)]} \\ &= \frac{(e^{-1}z) \sin 2}{(e^{-1}z)^2 - 2(e^{-1}z) \cos 2 + 1} = \frac{ez \sin 2}{z^2 - 2ez \cos 2 + e^2}. \end{aligned}$$

$$\text{(ii) We know that } Z(\cos k\alpha) = \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1} \quad \dots(B)$$

$$\begin{aligned} \therefore Z(c^k \cos k\alpha) &= Z[(c^{-1})^{-k} \cdot \cos k\alpha] & \text{[Apply damping rule, using (B)]} \\ &= \frac{(c^{-1}z)[c^{-1}z - \cos \alpha]}{(c^{-1}z)^2 - 2(c^{-1}z) \cos \alpha + 1} = \frac{z(z - c \cos \alpha)}{z^2 - 2cz \cos \alpha + c^2}. \end{aligned}$$

Example 23.5. Find the Z-transforms of

(i) $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$ (V.T.U., 2011 S) (ii) $\cosh\left(\frac{n\pi}{2} + \theta\right)$ (U.P.T.U., 2008)

Solution. (i) $Z\left[\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)\right] = Z\left[\cos\frac{n\pi}{2} \cos\frac{\pi}{4} - \sin\frac{n\pi}{2} \sin\frac{\pi}{4}\right]$
 $= \cos\frac{\pi}{4} \cdot Z\left(\cos\frac{n\pi}{2}\right) - \sin\frac{\pi}{4} \cdot Z\left(\sin\frac{n\pi}{2}\right)$ [Using formulae for $Z(\sin n\alpha)$ and $Z(\cos n\alpha)$]
 $= \frac{1}{\sqrt{2}} \left\{ \frac{z(z - \cos\pi/2)}{z^2 - 2z \cos\pi/2 + 1} - \frac{z \sin\pi/2}{z^2 - 2z \cos\pi/2 + 1} \right\} = \frac{1}{\sqrt{2}} \left(\frac{z^2}{z^2 + 1} - \frac{z}{z^2 + 1} \right) = \frac{z(z - 1)}{\sqrt{2}(z^2 + 1)}$

(ii) $Z\left[\cosh\left(\frac{n\pi}{2} + \theta\right)\right] = Z\left[\frac{e^{n\pi/2 + \theta} + e^{-(n\pi/2 + \theta)}}{2}\right] = \frac{1}{2} [e^\theta Z(e^{n\pi/2}) + e^{-\theta} Z(e^{-n\pi/2})]$

Since, $Z(a^n) = \frac{z}{z - a}$, $\therefore Z(e^{n\pi/2}) = Z(e^{\pi/2})^n = \frac{z}{z - e^{\pi/2}}$, $Z(e^{-n\pi/2}) = \frac{z}{z - e^{-\pi/2}}$

Thus $Z\left[\cosh\left(\frac{n\pi}{2} + \theta\right)\right] = \frac{1}{2} \left\{ e^\theta \cdot \frac{z}{z - e^{\pi/2}} + e^{-\theta} \cdot \frac{z}{z - e^{-\pi/2}} \right\}$
 $= \frac{z}{2} \left\{ \frac{z(e^\theta + e^{-\theta}) - [e^{(\pi/2 - \theta)} + e^{-(\pi/2 - \theta)}]}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1} \right\} = \frac{z^2 \cosh\theta - z \cosh\left(\frac{\pi}{2} - \theta\right)}{z^2 - 2z \cosh\left(\frac{\pi}{2}\right) + 1}$

Example 23.6. Find the Z-transform of

(i) ${}^n C_p$ ($0 \leq p \leq n$) (ii) ${}^{n+p} C_p$

Solution. (i) $Z({}^n C_p) = \sum_{p=0}^n ({}^n C_p z^{-p}) = 1 + {}^n C_1 z^{-1} + {}^n C_2 z^{-2} + \dots + {}^n C_n z^{-n} = (1 + z^{-1})^n$

(ii) $Z({}^{n+p} C_n) = \sum_{p=0}^n {}^{n+p} C_p z^{-p}$
 $= 1 + {}^{n+1} C_1 z^{-1} + {}^{n+2} C_2 z^{-2} + {}^{n+3} C_3 z^{-3} + \dots \infty$
 $= 1 + (n+1)z^{-1} + \frac{(n+2)(n+1)}{2!} z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!} z^{-3} + \dots \infty$
 $= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2!} (-z^{-1})^2$
 $+ \frac{(-n-1)(-n-2)(-n-3)}{3!} (-z^{-1})^3 + \dots \infty$
 $= (1 - z^{-1})^{-n-1}$

Example 23.7. Find the Z-transform of

(i) unit impulse sequence $\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$ (ii) unit step sequence $u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$

Solution. (i) $Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n)z^{-n} = 1 + 0 + 0 + \dots = 1$

(ii) $Z[u(n)] = \sum_{n=0}^{\infty} u(n)z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$

23.7 (1) SHIFTING U_N TO THE RIGHT

If $Z(u_n) = U(z)$, then $Z(u_{n-k}) = z^{-k} U(z)$ ($k > 0$)

Proof. By definition,

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \dots = z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z)$$

Obs. This rule will be very useful in applications to difference equations.

(2) **Shifting u_n to the left.** If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$$

Proof.
$$Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$$

$$= z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right]$$

Hence $Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$

(J.N.T.U., 2002)

In particular, we have the following standard results :

- (1) $Z(u_{n+1}) = z[U(z) - u_0]$; (2) $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$
 (3) $Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$.

Example 23.8. Show that $Z\left(\frac{1}{n!}\right) = e^{1/z}$.

Hence evaluate $Z[1/(n+1)!]$ and $Z[1/(n+2)!]$.

(Madras, 2006)

Solution. We have
$$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots = e^{1/z}$$

Shifting $(1/n!)$ one unit to the left gives

$$Z\left[\frac{1}{(n+1)!}\right] = z \left[Z\left(\frac{1}{n!}\right) - 1 \right] = z(e^{1/z} - 1)$$

Similarly shifting $(1/n!)$ two units to the left gives

$$Z\left[\frac{1}{(n+2)!}\right] = z^2(e^{1/z} - 1 - z^{-1}).$$

23.8 MULTIPLICATION BY n

If $Z(u_n) = u(z)$, then $Z(nu_n) = -z \frac{dU(z)}{dz}$

Proof.
$$Z(nu_n) = \sum_{n=0}^{\infty} n \cdot u_n z^{-n} = -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1} = -z \sum_{n=0}^{\infty} u_n \frac{d}{dz} (z^{-n}).$$

$$= -z \sum_{n=0}^{\infty} \frac{d}{dz} (u_n z^{-n}) = -z \frac{d}{dz} \left(\sum_{n=0}^{\infty} u_n z^{-n} \right) = -z \frac{d}{dz} U(z).$$

Obs. We have, $Z(n^2 u_n) = \left(-z \frac{d}{dz}\right)^2 u(z)$

(Madras, 2006)

In general, $Z(n^m u_n) = \left(-z \frac{d}{dz}\right)^m u(z)$.

Example 23.9. Find the Z-transform of (i) $n \sin n\theta$ (ii) $n^2 e^{n\theta}$.

Solution. (i) We know that $Z(nu_n) = -z \frac{dU(z)}{dz}$ and $Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$

$$\begin{aligned} \therefore Z(n \sin n\theta) &= -z \frac{d}{dz} [Z(\sin n\theta)] = -z \frac{d}{dz} \left(\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \right) \\ &= -z \frac{\sin \theta - z^2 \sin \theta}{(z^2 - 2z \cos \theta + 1)^2} = \frac{z(z^2 - 1) \sin \theta}{(z^2 - 2z \cos \theta + 1)^2} \end{aligned}$$

(ii) We know that $Z(e^{n\theta}) = \frac{z}{z - e^\theta}$

$$\begin{aligned} \therefore Z(n^2 e^{n\theta}) &= \left(-z \frac{d}{dz} \right)^2 (Z e^{n\theta}) = \left(-z \frac{d}{dz} \right) \left[-z \frac{d}{dz} \left(\frac{z}{z - e^\theta} \right) \right] \\ &= \left(-z \frac{d}{dz} \right) \left\{ -z \frac{(z - e^\theta)(1) - z(1)}{(z - e^\theta)^2} \right\} = -z \frac{d}{dz} \left\{ \frac{ze^\theta}{(z - e^\theta)^2} \right\} \\ &= -ze^\theta \left\{ \frac{(z - e^\theta)^2(1) - z[2(z - e^\theta)]}{(z - e^\theta)^4} \right\} = -ze^\theta \frac{z - e^\theta - 2z}{(z - e^\theta)^3} = \frac{z(z + e^\theta)e^\theta}{(z - e^\theta)^3} \end{aligned}$$

23.9 TWO BASIC THEOREMS

In applications, we often need the values of u_n for $n = 0$ or as $n \rightarrow \infty$ without requiring complete knowledge of u_n . We can find this as the behaviour of u_n for small values of n is related to the behaviour of $U(z)$ as $z \rightarrow \infty$ and vice-versa. The precise relationship is given by the following *initial and final value theorems* :

(1) Initial value theorem. If $Z(u_n) = U(z)$, then $u_0 = \lim_{z \rightarrow \infty} U(z)$

Proof. We know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots$

Taking limits as $z \rightarrow \infty$, we get $\lim_{z \rightarrow \infty} [U(z)] = u_0$, as required.

Similarly additional initial values can be found successively, giving :

$$u_1 = \lim_{z \rightarrow \infty} \{z [U(z) - u_0]\}; u_2 = \lim_{z \rightarrow \infty} \{z^2 [U(z) - u_0 - u_1 z^{-1}]\} \text{ and so on.}$$

(2) Final value theorem. If $Z(u_n) = U(z)$, then

$$\lim_{n \rightarrow \infty} (u_n) = \lim_{z \rightarrow 1} (z - 1) U(z)$$

Proof. By definition, $Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

$$\text{or } Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

$$\text{or } z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

$$\text{or } U(z)(z - 1) - u_0 z = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

Taking limits of both sides as $z \rightarrow 1$, we get

$$\lim_{z \rightarrow 1} [(z - 1) U(z)] - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n) = \lim_{n \rightarrow \infty} [(u_1 - u_0) + (u_2 - u_1) + \dots + (u_{n+1} - u_n)]$$

$$= \lim_{n \rightarrow \infty} [u_{n+1}] - u_0 = u_\infty - u_0$$

Hence

$$u_\infty = \lim_{z \rightarrow 1} [(z-1)U(z)].$$

(Anna, 2005 S)

Example 23.10. If $U(z) = \frac{2z^2 + 5z + 14}{(z-1)^4}$, evaluate u_2 and u_3 .

Solution. Writing $U(z) = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1-z^{-1})^4}$

By initial value theorem, $u_0 = \lim_{z \rightarrow \infty} U(z) = 0$

Similarly, $u_1 = \lim_{z \rightarrow \infty} [z[U(z) - u_0]] = 0$

Now $u_2 = \lim_{z \rightarrow \infty} [z^2[U(z) - u_0 - u_1z^{-1}]] = 2 - 0 - 0 = 2$

and

$$u_3 = \lim_{z \rightarrow \infty} z^3[U(z) - u_0 - u_1z^{-1} - u_2z^{-2}] = \lim_{z \rightarrow \infty} z^3[U(z) - 0 - 0 - 2z^{-2}]$$

$$= \lim_{z \rightarrow \infty} z^3 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2} \right] = \lim_{z \rightarrow \infty} z^3 \left\{ \frac{13z^3 + 2z^2 + 8z - 2}{z^2(z-1)^4} \right\} = 13.$$

PROBLEMS 23.1

1. Find the Z-transforms of the following sequences :

(i) $\frac{a^n}{n!}$ ($n \geq 0$) (S.V.T.U., 2009) (ii) $\frac{1}{(n+1)!}$ (iii) $(\cos \theta + i \sin \theta)^n$.

2. Using the linearity property, find the Z-transforms of the following functions :

(i) $2n + 5 \sin n\pi/4 - 3a^4$ (ii) $\frac{1}{2}(n-1)(n+2)$ (S.V.T.U., 2007)

(iii) $(n+1)(n+2)$ (Anna, 2008) (iv) $(2n-1)^2$ (V.T.U., 2011 S)

3. Show that (i) $Z(\sinh n\theta) = \frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1}$ (V.T.U., 2011) (ii) $Z(a^n \sinh n\theta) = \frac{az \sinh \theta}{z^2 - 2az \cosh \theta + a^2}$.

4. Show that (i) $Z(e^{-in} \cos n\theta) = \frac{ze^{\alpha}(ze^{\alpha} - \cos \theta)}{z^2 e^{2\alpha} - 2ze^{\alpha} \cos \theta + 1}$; (ii) $Z(e^{-in} \sin n\theta) = \frac{ze^{\alpha} \sin \theta}{z^2 e^{2\alpha} - 2ze^{\alpha} \cos \theta + 1}$

Also evaluate $Z(e^{3n} \sin 2n)$. (S.V.T.U., 2007)

5. Using $Z(n^2) = \frac{z^2 + z}{(z-1)^3}$, show that $Z(n+1)^2 = \frac{z^3 + z^2}{(z-1)^3}$.

6. Find the Z-transforms of (i) $\sin(n+1)\theta$, (ii) $\cos\left(\frac{k\pi}{8} + \alpha\right)$. (Marathwada, 2008)

7. Find the Z-transform of $\cos n\theta$ and hence find $Z(n \cos n\theta)$. (Anna, 2009)

8. Find the Z-transform of $\cos(n\pi/2)$ and $a^n \cos(n\pi/2)$. (Anna, 2008 S)

9. Find the Z-transforms of the following

(i) e^{-an} (ii) e^{-2n} (V.T.U., 2010 S) (iii) $e^{-an} n^2$.

10. Show that (i) $Z[\delta(n+1)] = 1/z$ (ii) $(1/2)^n u(n) = \frac{2z}{2z-1}$.

11. Show that $Z(n^{\rho} C_p) = (1-1/z)^{-(\rho+1)}$. Using the damping rule, deduce that $Z(n^{\rho} C_p a^n) = (1+a/z)^{-(\rho+1)}$.

12. If $Z(u_n) = \frac{z}{z-1} + \frac{z}{z^2+1}$, find the Z-transform of u_{n+2} . (S.V.T.U., 2009)

13. If $U(z) = \frac{2z^2 + 3z + 12}{(z-1)^4}$, find the value of u_2 and u_3 .

14. Given that $Z(u_n) = \frac{2z^2 + 3z + 4}{(z-3)^3}$, $|z| > 3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

15. Show that (i) $Z\left(\frac{1}{n}\right) = z \log \frac{z}{z-1}$. (Madras, 2003 S) (ii) $Z\left\{\frac{1}{n(n+1)}\right\}$. (Anna, 2005 S)

16. Using $Z(n) = \frac{z}{(z-1)^2}$, show that $Z(n \cos n\theta) = \frac{(z^3 + z) \cos \theta - 2z^2}{(z^2 - 2z \cos \theta + 1)^2}$.

23.10 SOME USEFUL Z-TRANSFORMS

Sr. No.	Sequence u_n ($n \geq 0$)	Z-transform $U(z) = Z(u_n)$
1.	k	$kz/(z-1)$
2.	$-k$	$kz/(z+1)$
3.	n	$z/(z-1)^2$
4.	n^2	$(z^2 + z)/(z-1)^3$
5.	n^p	$-z d/dz [Z(n^{p-1})]$, $p + ve$ integer.
6.	$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$	1
7.	$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$	$z/(z-1)$
8.	a^n	$z/(z-a)$
9.	na^n	$az/(z-a)^2$
10.	n^2a^n	$(az^2 + a^2z)/(z-a)^3$
11.	$\sin n\theta$	$\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$
12.	$\cos n\theta$	$\frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$
13.	$a^n \sin n\theta$	$\frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2}$
14.	$a^n \cos n\theta$	$\frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2}$
15.	$\sinh n\theta$	$\frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1}$
16.	$\cosh n\theta$	$\frac{z(z - \cosh \theta)}{z^2 - 2z \cosh \theta + 1}$
17.	$a^n \sinh n\theta$	$\frac{az \sinh \theta}{z^2 - 2az \cosh \theta + a^2}$
18.	$a^n \cosh n\theta$	$\frac{z(z - a \cosh \theta)}{z^2 - 2az \cosh \theta + a^2}$
19.	$a^n u_n$	$U(z/a)$
20.	u_{n+1}	$z[U(z) - u_0]$
	u_{n+2}	$z^2[U(z) - u_0 - u_1z^{-1}]$
	u_{n+3}	$z^3[U(z) - u_0 - u_1z^{-1} - u_2z^{-2}]$
21.	u_{n-k}	$z^{-k}U(z)$
22.	nu_n	$-z d/dz [U(z)]$
23.	u_0	$\text{Lt}_{z \rightarrow \infty} U(z)$
24.	$\text{Lt}_{n \rightarrow \infty} (u_n)$	$\text{Lt}_{z \rightarrow 1} [(z-1)U(z)]$

23.11 SOME USEFUL INVERSE Z-TRANSFORMS

Sr. No.	$U(z)$	Inverse Z-transform $u_n = z^{-1}[U(z)]$
1.	$\frac{1}{z-a}$	a^{n-1}
2.	$\frac{1}{z+a}$	$(-a)^{n-1}$
3.	$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
4.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
5.	$\frac{z}{z-a}$	a^n
6.	$\frac{z}{z+a}$	$(-a)^n$
7.	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$
8.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(n+1)(n+2)a^n u(n)$

23.12 CONVOLUTION THEOREM

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$, then

$$Z^{-1}[U(z) \cdot V(z)] = \sum_{m=0}^n u_m \cdot v_{n-m} = u_n * v_n$$

where the symbol * denotes the convolution operation.

Proof. We have $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, $V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$

$$\begin{aligned} \therefore U(z) V(z) &= (u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_n z^{-n} + \dots \infty) \times (v_0 + v_1 z^{-1} + v_2 z^{-2} + \dots + v_n z^{-n} + \dots \infty) \\ &= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \dots + u_n v_0) z^{-n} = Z(u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0) \end{aligned}$$

whence follows the desired result.

Obs. The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Example 23.11. Use convolution theorem to evaluate $Z^{-1} \left\{ \frac{z^2}{(z-a)(z-b)} \right\}$.

Solution. We know that $Z^{-1} \left\{ \frac{z}{z-a} \right\} = a^n$ and $Z^{-1} \left\{ \frac{z}{z-b} \right\} = b^n$

$$\begin{aligned} \therefore Z^{-1} \left\{ \frac{z^2}{(z-a)(z-b)} \right\} &= Z^{-1} \left\{ \frac{z}{z-a} \cdot \frac{z}{z-b} \right\} = a^n * b^n \\ &= \sum_{m=0}^n a^m \cdot b^{n-m} = b^n \cdot \sum_{m=0}^n \left(\frac{a}{b} \right)^m \text{ which is a G.P.} \\ &= b^n \cdot \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b} \end{aligned}$$

23.13 CONVERGENCE OF Z-TRANSFORMS

Z-transform operation is performed on a sequence u_n which may exist in the range of integers $-\infty < n < \infty$, and we write

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \quad \dots(1)$$

where u_n represents a number in the sequence for $n = \text{an integer}$. The region of the z -plane in which (1) converges absolutely is known as the region of convergence (ROC) of $U(z)$.

We have so far discussed one-sided Z-transform only for which $n \geq 0$. Here the sequence is always right-sided and the ROC is always outside a prescribed circle say $|z| > |a|$ [Fig. 23.2 (i)]. For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$. [Fig. 23.2 (ii)].

23.14 TWO-SIDED Z-TRANSFORM OF u_n IS DEFINED BY

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \quad \dots(2)$$

In this case, the sequence is two-sided and the region of convergence for (2) is the annular region $|b| < |z| < |c|$ [Fig. 23.2 (iii)]. The inner circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z . The shaded annulus of convergence is necessary for the two sided sequence and its Z-transform to exist.

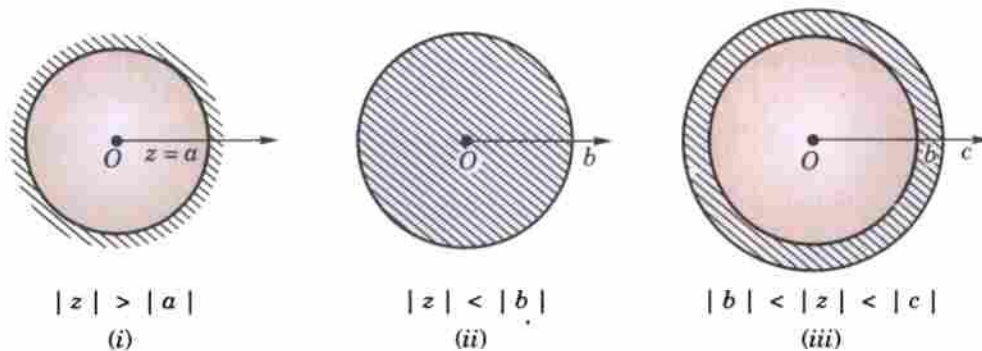


Fig. 23.1

Example 23.12. Find the Z-transform and region of convergence of

$$(a) u(n) = \begin{cases} 4^n & \text{for } n < 0 \\ 2^n & \text{for } n \geq 0 \end{cases}$$

$$(b) u(n) = n^k c^n, n \geq k.$$

Solution. By definition $Z[u(n)] = \sum_{-\infty}^{\infty} u(n)Z^{-n} = \sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$

Putting $-n = m$ in the first series, we get

$$\begin{aligned} Z[u(n)] &= \sum_1^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n} \\ &= \left\{ \frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \dots \right\} + \left\{ 1 + \frac{2}{z} + \frac{2^2}{z^2} + \dots \right\} \\ &= \frac{z}{4} \left\{ 1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \dots \right\} + \left\{ 1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \dots \right\} \quad \dots(i) \\ &= \frac{z}{4} \cdot \frac{1}{1 - (z/4)} + \frac{1}{1 - (2/z)} = \frac{z}{4 - z} + \frac{z}{z - 2} = \frac{2z}{(4 - z)(z - 2)} \end{aligned}$$

Now the two series in (i) being G.P. will be convergent if $|z/4| < 1$ and $|2/z| < 1$ i.e., if $|z| < 4$ and $2 < |z|$ i.e. $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in Fig. 23.3. Hence ROC is $2 < z < 4$.

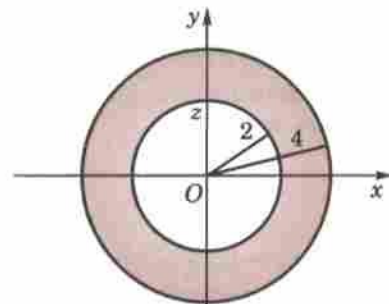


Fig. 23.2

$$[\because {}^k C_r = {}^k C_{k-r}]$$

$$(b) \text{ By definition, } Z[u(n)] = \sum_{n=-\infty}^{\infty} {}^n C_k z^{-n} = \sum_{n=k}^{\infty} {}^n C_k 2^n z^{-n}$$

To find the sum of this series, we replace n by $k+r$

$$\begin{aligned} \therefore Z[u(n)] &= \sum_{r=0}^{\infty} {}^{k+r} C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r} C_r z^{-r} \\ &= z^{-k} [1 + {}^{k+1} C_1 z^{-1} + {}^{k+1} C_2 z^{-2} + \dots] \\ &= z^{-k} (1 - 1/z)^{-(k+1)} \end{aligned}$$

This series is convergence for $|1/z| < 1$ i.e., for $|z| > 1$.

Hence ROC is $|z| > 1$.

Example 23.13. Find the Z-transform and the radius of convergence of

(a) $f(n) = 2^n, n < 0$

(b) $f(n) = 5^n/n!, n \geq 0$.

(Mumbai, 2009)

Solution. (a) Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$\begin{aligned} Z[f(n)] &= \sum_{n=-\infty}^{\infty} f(n) z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where } m = -n \\ &= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots = \frac{z}{2} [1 + (z/2) + (z/2)^2 + \dots \infty] \\ &= \frac{z}{2} \cdot \frac{1}{1 - (z/2)} = \frac{z}{2 - z} \end{aligned}$$

This series being a G.P. is convergent if $|z/2| < 1$ i.e., $|z| < 2$.

Hence ROC is $|z| < 2$.

$$\begin{aligned} (b) \text{ By definition, } Z[u(n)] &= \sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot z^{-n} = \sum_{n=0}^{\infty} \frac{(5/z)^n}{n!} = 1 + \left(\frac{5}{z}\right) + \frac{1}{2!} \left(\frac{5}{z}\right)^2 + \frac{1}{3!} \left(\frac{5}{z}\right)^3 + \dots \infty \\ &= e^{5/z} \end{aligned}$$

The above series is convergent for all values of z .

Hence ROC is the entire z -plane.

PROBLEMS 23.2

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 4^n, n \geq 0$.
2. $u(n) = 2^n, n < 0$.
3. $u(n) = 4^n$, for $n < 0$ and $= 3^n$ for $n \geq 0$.
4. $u(n) = n5^n, n \geq 0$.
5. $u(n) = 2^n/n, n > 1$.
6. $u(n) = 3^n/n!, n \geq 0$.
7. $u(n) = e^{an}, n \geq 0$.

23.15 EVALUATION OF INVERSE Z-TRANSFORMS

We can obtain the inverse Z-transforms using any of the following three methods :

I. Power series method. This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example 23.14. Find the inverse Z-transform of $\log(z/z+1)$ by power series method.

Solution. Putting $z = 1/t$, $U(z) = \log\left(\frac{1/y}{1/y+1}\right) = -\log(1+y) = -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \dots$
 $= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \dots$

Thus $u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}$

Example 23.15. Find the inverse Z-transform of $z/(z+1)^2$ by division method.

Solution. $U(z) = \frac{z}{z^2 + 2z + 1} = z^{-1} - \frac{2+z^{-1}}{z^2 + 2z + 1}$, by actual division
 $= z^{-1} - 2z^{-2} + \frac{3z^{-1} + 2z^{-2}}{z^2 + 2z + 1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2} + 3z^{-3}}{z^2 + 2z + 1}$

Continuing this process of division, we get an infinite series i.e.,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1} n z^{-n}$$

Thus $u_n = (-1)^{n-1} n$.

II. Partial fractions method. This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example 23.16. Find the inverse Z-transforms of

$$(i) \frac{2z^2 + 3z}{(z+2)(z-4)} \quad (\text{V.T.U., 2008 S; S.V.T.U., 2007}) \quad (ii) \frac{z^3 - 20z}{(z-2)^3(z-4)} \quad (\text{V.T.U., 2011})$$

Solution. (i) We write $U(z) = \frac{2z^2 + 3z}{(z+2)(z-4)}$ as $\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$
 where $A = 1/6$ and $B = 11/6$

$$\therefore U(z) = \frac{1}{6} \frac{z}{z+2} + \frac{11}{6} \frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6} (-2)^n + \frac{11}{6} (4)^n \quad [\text{Using } \S 23.10 (9)]$$

$$(ii) \text{ We write } U(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)}$$

$$\text{as } \frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{A + Bz + Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Readily we get $D = 1/2$. Multiplying throughout by $(z-2)^3(z-4)$, we get

$$z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3.$$

Putting $z = 0, 1, -1$ successively and solving the resulting simultaneous equations, we get $A = 6, B = 0, C = 1/2$.

$$\text{Thus } U(z) = \frac{1}{2} \cdot \frac{12z + z^3}{(z-2)^3} - \frac{z}{z-4} = \frac{1}{2} \frac{z(z-2)^2 + 4z^2 + 8z}{(z-2)^3} - \frac{z}{z-4}$$

$$= \frac{1}{2} \left[\frac{z}{z-2} + 2 \frac{2z^2 + 4z}{(z-2)^3} \right] - \frac{z}{z-4}$$

On inversion, we get
$$u_n = \frac{1}{2} (2^n + 2 \cdot n^2 2^n) - 4^n \quad [\text{Using } \S 23.10 (9) \text{ \& (11)}]$$

$$= 2^{n-1} + n^2 2^n - 4^n.$$

Example 23.17. Find the inverse Z-transform of $2(z^2 - 5z + 6.5) / [(z-2)(z-3)^2]$, for $2 < |z| < 3$.

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \quad \text{where } A = B = C = 1$$

$$\begin{aligned} \therefore U(z) &= \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2} \\ &= \frac{1}{z} \left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3} \left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9} \left(1 - \frac{z}{3}\right)^{-2} \quad \text{so that } 2/z < 1 \text{ and } z/3 < 1 \\ &= \frac{1}{z} \left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots\right) - \frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \dots\right) + \frac{1}{9} \left(1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \dots\right) \\ &\quad \text{where } 2 < |z| < 3. \\ &= \left(\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \dots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \dots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \dots\right) \\ &= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{3}\right)^{n+2} z^n \end{aligned}$$

On inversion, we get $u_n = 2^{n-1}$, $n \geq 1$ and $u_n = -(n+2)3^{n-2}$, $n \leq 0$.

III. Inversion integral method. The inverse Z-transform of $U(z)$ is given by the formula

$$u_n = \frac{1}{2\pi i} \int_C U(z) z^{n-1} dz$$

= sum of residues of $U(z) z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given.

The following examples will illustrate the application of this formula :

Example 23.18. Using the inversion integral method, find the inverse Z-transform of

$$\frac{z}{(z-1)(z-2)} \quad (\text{V.T.U., 2010 S})$$

Solution. Let $U(z) = \frac{z}{(z-1)(z-2)}$. Its poles are at $z = 1$ and $z = 2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i} \int_C U(z) z^{n-1} dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

= sum of residues of $U(z) z^{n-1}$ at $z = 1$ and $z = 2$.

Now
$$\text{Res} [U(z) z^{n-1}]_{z=1} = \text{Lt}_{z \rightarrow 1} \left\{ (z-1) \cdot \frac{z^n}{(z-1)(z-2)} \right\} = -1$$

and
$$\text{Res} [U(z) z^{n-1}]_{z=2} = \text{Lt}_{z \rightarrow 2} \left\{ (z-2) \cdot \frac{z^n}{(z-1)(z-2)} \right\} = 2^n$$

Thus the required inverse Z-transform $u_n = 2^n - 1$, $n = 0, 1, 2, \dots$

Example 23.19. Find the inverse Z-transform of $2z / [(z-1)(z^2+1)]$.

(Madras, 2000 S)

Solution. Let $U(z) = \frac{2z}{(z-1)(z+i)(z-i)}$. It has three poles at $z = 1, z = \pm i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i} \int_C U(z) \cdot z^{n-1} dz, \text{ where } C \text{ is a circle large enough to enclose the poles of } U(z). \\ = \text{sum of residues of } U(z) \cdot z^{n-1} \text{ at } z = 1, z = \pm i.$$

$$\text{Now } \text{Res} [U(z) z^{n-1}]_{z=1} = \text{Lt}_{z \rightarrow 1} \left\{ (z-1) \frac{2z^n}{(z-1)(z^2+1)} \right\} = 1$$

$$\text{Res} [U(z) z^{n-1}]_{z=i} = \text{Lt}_{z \rightarrow i} \left\{ (z-i) \frac{2z^n}{(z-1)(z+i)(z-i)} \right\} = \frac{-(i)^n}{1+i}$$

$$\text{Res} [U(z) z^{n-1}]_{z=-i} = \text{Lt}_{z \rightarrow -i} \left\{ (z+i) \frac{2z^n}{(z-1)(z+i)(z-i)} \right\} = \frac{(-i)^n}{i-1}$$

$$\text{Hence } u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}.$$

PROBLEMS 23.3

Using convolution theorem, evaluate the inverse Z-transforms of the following :

$$1. \frac{z^2}{(z-1)(z-3)}, \quad 2. \left(\frac{z}{z-a} \right)^2 \quad (\text{Madras, 2003}) \quad 3. \left(\frac{z}{z-1} \right)^3.$$

$$4. \text{ Show that (a) } \frac{1}{n!} * \frac{1}{n!} = \frac{2^n}{n!} \quad (\text{b) } Z^{-1} \left(\frac{z^2}{(z+a)(z+b)} \right) = \frac{(-1)}{b-a} (b^{n+1} - a^{n+1}). \quad (\text{Anna, 2009})$$

Find the inverse Z-transforms of the following :

$$5. \frac{4z}{z-a}, \quad |z| > |a|. \quad (\text{Kottayam, 2005}) \quad 6. \frac{5z}{(2-z)(3z-1)}. \quad (\text{Madras, 1999})$$

$$7. \frac{z}{(z-1)^2}, \quad 8. \frac{18z^2}{(2z-1)(4z+1)}. \quad (\text{S.V.T.U., 2009})$$

$$9. \frac{8z-z^3}{(4-z)^3}, \quad 10. \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}. \quad (\text{Anna, 2005 S})$$

$$11. \frac{4z^2-2z}{z^3-5z^2+8z-4}. \quad (\text{V.T.U., 2011 S}) \quad 12. \frac{z^3+3z}{(z-1)^2(z^2+1)}. \quad (\text{Anna, 2009})$$

$$13. \frac{(1-e^{at})z}{(z-1)(z-e^{-at})}.$$

$$14. \text{ Obtain } Z^{-1}\{1/[(z-2)(z-3)]\} \text{ for (i) } |z| < 2; \text{ (ii) } 2 < |z| < 3; \text{ (iii) } |z| > 3. \quad (\text{Marathwada, 2008})$$

$$15. \text{ Evaluate } Z^{-1}\{(z-5)^{-3}\} \text{ for } |z| > 5. \quad (\text{Mumbai, 2009})$$

Using inversion integral, find the inverse Z-transform of the following functions :

$$16. \frac{z+3}{(z+1)(z-2)}, \quad 17. \frac{(2z-1)z}{2(z-1)(z+0.5)}.$$

$$18. \frac{1}{z(z-1)(z+0.5)}. \quad (\text{S.V.T.U., 2008}) \quad 19. \frac{z^2+z}{(z-1)(z^2+1)}. \quad (\text{Madras, 2003})$$

$$20. \frac{2z(z^2-1)}{(z^2+1)^2}.$$

23.16 (1) APPLICATION TO DIFFERENCE EQUATIONS

Just as the Laplace transforms method is quite effective for solving linear differential equations (§ 21.15), the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems. To determine the frequency response of such systems, the solution of difference equations is required for which Z-transform method proves useful.

(2) Working procedure to solve a linear difference equation with constant coefficients by Z-transforms :

1. Take the Z-transform of both sides of the difference equations using the formulae of § 26.16 and the given conditions.
2. Transpose all terms without $U(z)$ to the right.
3. Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z .
4. Express this function in terms of the Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.

Example 23.20. Using the Z-transform, solve

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0, u_1 = 1.$$

(U.P.T.U., 2003)

Solution. If $Z(u_n) = U(z)$, then $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1z^{-1}]$$

Also

$$Z(2^n) = z/(z-2)$$

\therefore taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1z^{-1}] + 4z[U(z) - u_0] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)} = \frac{3}{8} \frac{1}{z+1} + \frac{1}{24} \frac{1}{z-3} - \frac{5}{12} \frac{1}{z+3},$$

on breaking into partial fractions.

$$U(z) = \frac{3}{8} \frac{z}{z+1} + \frac{1}{24} \frac{z}{z-3} - \frac{5}{12} \frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8} Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24} Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12} Z^{-1}\left(\frac{z}{z+3}\right) = \frac{3}{8} (-1)^n + \frac{1}{24} 3^n - \frac{5}{12} (-3)^n.$$

Example 23.21. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z-transforms.

(V.T.U., 2011 ; Anna, 2009 ; S.V.T.U., 2009)

Solution. If $Z(y_n) = Y(z)$, then $Z(y_{n+1}) = z[Y(z) - y_0]$, $Z(y_{n+2}) = z^2[Y(z) - y_0 - y_1z^{-1}]$

Also $Z(2^n) = z/(z-2)$.

Taking Z-transforms of both sides, we get

$$z^2[Y(z) - y_0 - y_1z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = z/(z-2)$$

Since $y_0 = 0$, and $y_1 = 0$, we have $Y(z)(z^2 + 6z + 9) = z/(z-2)$

or
$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1}{25} \left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2} \right], \text{ on splitting into partial fractions.}$$

or
$$Y(z) = \frac{1}{25} \left[\frac{z}{z-2} - \frac{z}{z+3} - 5 \frac{z}{(z+3)^2} \right]$$

On taking inverse Z-transform of both sides, we obtain

$$y_n = \frac{1}{25} \left[Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z+3}\right) + \frac{5}{3} Z^{-1}\left(-\frac{3z}{(z+3)^2}\right) \right]$$

$$= \frac{1}{25} \{2^n - (-3)^n + \frac{5}{3} n(-3)^n\}$$

$$\left[\because Z^{-1}\left\{\frac{az}{(z-a)^2}\right\} = na^n \right]$$

Example 23.22. Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0$, $y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \dots$ by Z-transform method. (V.T.U., 2010)

Solution. Taking Z-transform of both sides of the given equation, we get

$$z^2(Y(z) - y_0 - y_1z^{-1}) - 5z(Y(z) - y_0) + 6Y(z) = \frac{z}{z-1}$$

Substituting the values $y_0 = 0$, $y_1 = 1$, it reduces to

$$(z^2 - 5z + 6)Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

or

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$\text{where } A = \frac{1}{2}, B = -2, C = \frac{3}{2}$$

so that

$$Y(z) = \frac{1}{2} \frac{z}{z-1} - 2 \frac{z}{z-2} + \frac{3}{2} \frac{z}{z-3}$$

On inversion, we obtain

$$y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$$

Obs. The initial values given in the problem automatically appear in the generated sequence.

Example 23.23. Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.

Solution. Taking Z-transform of both sides of the given equation, we get

$$Y(z) + \frac{1}{4}z^{-1}Y(z) = 1 + \frac{1}{3}z^{-1}$$

or

$$Y(z) = \left(1 + \frac{1}{3}z^{-1}\right) / \left(1 + \frac{1}{4}z^{-1}\right) = \left(z + \frac{1}{3}\right) / \left(z + \frac{1}{4}\right)$$

There being only one simple pole at $z = -1/4$, consider the contour $|z| > 1/4$.

$$\begin{aligned} \therefore \text{Res } [Y(z)z^{n-1}]_{z=-1/4} &= \text{Lt}_{z \rightarrow -1/4} \left\{ \left(z + \frac{1}{4}\right) \cdot \left(z + \frac{1}{3}\right) z^{n-1} / \left(z + \frac{1}{4}\right) \right\} \\ &= \text{Lt}_{z \rightarrow -1/4} \left(z + \frac{1}{3} \right) z^{n-1} = \left(-\frac{1}{4} + \frac{1}{3}\right) \left(-\frac{1}{4}\right)^{n-1} = \frac{1}{12} \cdot \left(-\frac{1}{4}\right)^{n-1} \end{aligned}$$

Hence by inversion integral method, we have

$$y_n = \frac{1}{12} \left(-\frac{1}{4}\right)^{n-1}$$

Example 23.24. Using the Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

(S.V.T.U., 2007)

Solution. Given equation is $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

Taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 \cdot \frac{z}{(z-1)^2} + 5 \cdot \frac{z}{z-1}$$

or

$$U(z)(z^2 - 2z + 1) = \frac{5z^2 - 2z}{(z-1)^2} + u_0(z^2 - 2z) + u_1z$$

or

$$U(z) = \frac{5z^2 - 2z}{(z-1)^4} + u_0 \frac{z^2 - 2z}{(z-1)^2} + u_1 \frac{z}{(z-1)^2}$$

On inversion, we obtain

$$u_n = Z^{-1} \left\{ \frac{5z^2 - 2z}{(z-1)^4} \right\} + u_0 Z^{-1} \left\{ \frac{z^2 - 2z}{(z-1)^2} \right\} + u_1 Z^{-1} \left\{ \frac{z}{(z-1)^2} \right\} \quad \dots(i)$$

Noting that $Z(1) = \frac{z}{z-1}$, $Z(n) = \frac{z}{(z-1)^2}$

$$Z(n^2) = \frac{z^2 + z}{(z-1)^3}, \quad Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4}$$

We write $\frac{5z^2 - 2z}{(z-1)^4} \equiv A \frac{z^3 + 4z^2 + z}{(z-1)^4} + B \frac{z^2 + z}{(z-1)^3} + C \frac{z}{(z-1)^2} + D \frac{z}{z-1}$

Equating coefficients of like powers of z , we find

$$A = \frac{1}{2}, B = 1, C = -\frac{3}{2}, D = 0$$

$$\therefore Z^{-1} \left\{ \frac{5z^2 - 2z}{(z-1)^4} \right\} = \frac{1}{2} n^3 + n^2 - \frac{3}{2} n = \frac{1}{2} n(n-1)(n+3)$$

Also $Z^{-1} \left\{ \frac{z^2 - 2z}{(z-1)^2} \right\} = Z^{-1} \left\{ \frac{z}{z-1} \right\} - Z^{-1} \left\{ \frac{z}{(z-1)^2} \right\} = 1 - n$

and

$$Z^{-1} \left\{ \frac{z}{(z-1)^2} \right\} = n.$$

Substituting these values in (i) above, we get

$$\begin{aligned} u_n &= \frac{1}{2} n(n-1)(n+3) + u_0(1-n) + u_1 n \\ &= \frac{1}{2} n(n-1)(n+3) + c_0 + c_1 n. \end{aligned}$$

where $c_0 = u_0$, $c_1 = u_1 - u_0$

Example 23.25. Using residue method, solve $y_k + \frac{1}{9} y_{k-2} = \frac{1}{3^k} \cos \frac{k\pi}{2}$, $k \geq 0$.

Solution. Taking Z-transform of both sides of the given equation, we get

$$Z \left\{ y_k + \frac{1}{9} y_{k-2} \right\} = Z \left\{ \frac{1}{3^k} \cos \frac{k\pi}{2} \right\}$$

or

$$Y(z) + \frac{1}{9} z^{-2} Y(z) = \frac{z^2}{z^2 + 1/9} \quad \text{or} \quad \left(1 + \frac{1}{9} z^{-2} \right) Y(z) = \frac{z^2}{z^2 + \frac{1}{9}}$$

or

$$Y(z) = \frac{z^2}{\left(1 + \frac{1}{9} z^{-2} \right) \left(z^2 + \frac{1}{9} \right)} = \frac{z^4}{\left(z^2 + \frac{1}{9} \right)^2}$$

There are two poles of second order at $z = i/3$ and $z = -i/3$.

$$\begin{aligned} \therefore \text{Residue at } (z = i/3) &= \left[\frac{d}{dz} \left\{ \left(\frac{z-i}{3} \right)^2 \frac{z^{k-1} z^4}{(z^2 + 1/9)^2} \right\} \right] \\ &= \left[\frac{d}{dz} \left\{ \frac{z^{k+3}}{(z+i/3)^2} \right\} \right]_{z=i/3} = \left[\frac{(z+i/3)^2 (k+3) z^{k+2} - z^{(k+3)} \cdot 2(z+i/3)}{(z+i/3)^4} \right]_{z=i/3} \\ &= \left[\frac{(z+i/3)(k+3)z^{k+2} - 2z^{k+3}}{(z+i/3)^3} \right]_{z=i/3} = \left(\frac{3}{2i} \right)^3 \left[(2k+6) \left(\frac{i}{3} \right)^{k+3} - 2 \left(\frac{i}{3} \right)^{k+3} \right] \end{aligned}$$

$$= \frac{1}{8} (2k+4) \left(\frac{i}{3}\right)^k = \frac{1}{4} (k+2) \left(\frac{1}{3}\right)^k \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^k = \frac{1}{4} (k+2) \left(\frac{1}{3}\right)^k \left(\cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2}\right) \quad \dots(i)$$

Changing i to $-i$ in (i), we have

$$\text{Residue at } (z = -i/3) = \frac{1}{4} (k+2) \left(\frac{1}{3}\right)^k \left(\cos \frac{k\pi}{2} - i \sin \frac{k\pi}{2}\right) \quad \dots(ii)$$

$$\text{Adding (i) and (ii), we obtain } y_k = \frac{1}{2} (k+2) \left(\frac{1}{3}\right)^k \cos \frac{k\pi}{2}.$$

PROBLEMS 23.4

Solve the following difference equations using Z-transforms (1–8):

- $6y_{k+2} - y_{k+1} - y_k = 0$, given that $y(0) = y(1) = 1$. (Kottayam, 2005)
- $y(n+2) + 2y(n+1) + y(n) = 0$, given that $y(0) = y(1) = 0$. (V.T.U., 2008 S)
- $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$. (U.P.T.U., 2008)
- $f(n) + 3f(n-1) - 4f(n-2) = 0, n \geq 2$, given that $f(0) = 3, f(1) = -2$. (Madras, 2003 S)
- $y_{(n+3)} - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4, y(1) = 0$ and $y(2) = 8$. (Anna, 2005 S)
- $y_{n+2} - 5y_{n+1} + 6y_n = 36$, given that $y(0) = y(1) = 0$. (Anna, 2009)
- $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.
- $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.
- $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n$ ($n \geq 0$), $y_0 = 0$.
- $u_{x+2} + u_x = 5(2^x)$ given that $u_0 = 1, u_1 = 0$. (Marathwada, 2008)
- $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0, y_1 = 1$. (Madras, 2006)
- $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2, y_1 = 1$.
- $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
- $y_k + \frac{1}{25}y_{k-2} = \left(\frac{1}{5}\right)^k \cos \frac{k\pi}{2}$, ($k \geq 0$).
- Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y_{(-1)} = 1$.
- Find the impulse response of a system described by $y_{(n+1)} + 2y_{(n)} = \delta_n$; $y_0 = 0$.

23.1 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 23.5

Choose the correct answer or fill up the blanks in each of the following problems:

- $Z(1) = \dots$
- If u_n is defined for $n = 0, 1, 2, \dots$ only, then $Z(u_n) = \dots$
- Z-transform of $n = \dots$ (Anna, 2009)
- $Z(na^n) = \dots$
- $Z(\sin n\theta) = \dots$
- Z-transform of $(1/n!)$ is
- $Z(n^2) = \dots$
- Linear property of Z-transform states that...
- $Z^{-1}\left(\frac{1}{z-2}\right) = \dots$
- $Z^{-1}\left\{\frac{z}{(z+1)^2}\right\} = \dots$
- Initial value theorem on Z-transform states that
- Z-transform is linear. (True or False)
- If $Z(u_n) = u(z)$, then $\lim_{n \rightarrow \infty} (u_n) = \lim_{z \rightarrow \infty} (z-1)u(z)$. (True or False)
- Z-transform of the sequence $\{2^k\}, k \geq 0$ is $z/(z-2)$. (True or False)
- Z-transform of $\{a^k/k!\}, k \geq 0 = e^{az}$. (True or False)
- Z-transform of $\{{}^nC_r\}, (0 \leq r \leq n)$ is $(1+z)^n$. (True or False)
- Z-transform of unit impulse sequence $\delta(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$ is $z/z-1$. (True or False)
- Z-transform of unit step sequence $u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$ is 1. (True or False)