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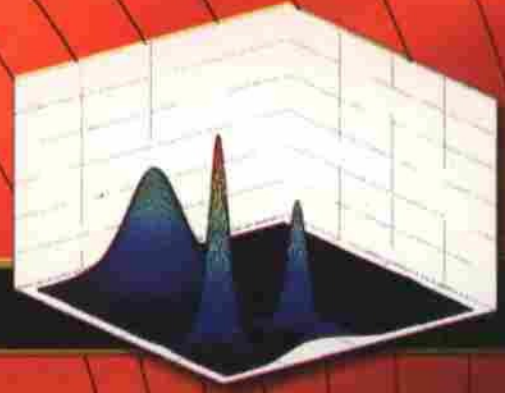
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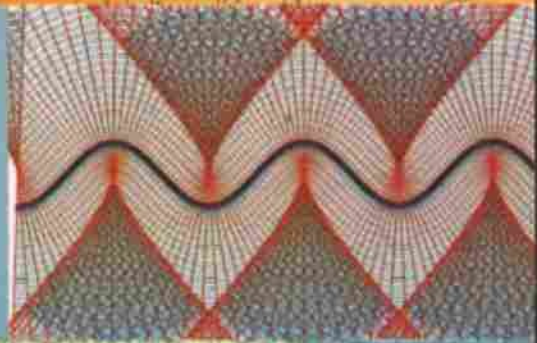
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42nd
Edition



Higher Engineering Mathematics

Dr. B.S. Grewal



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32.12 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

Example 32.25. Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0, y = 1, y' = 0$. (Delhi, 2002)

Solution. Let $dy/dx = z = f(x, y, z)$. Then $dz/dx = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$.

Using $k_1, k_2 \dots$ for $f(x, y, z)$ and l_1, l_2, \dots for $\phi(x, y, z)$, Runge-Kutta formulae become

$k_1 = hf(x_0, y_0, z_0) = 0.2(0) = 0$ $k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1) = 0.2(-0.1) = -0.02$ $k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2) = 0.2(-0.0999) = -0.02$ $k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(-0.1958) = -0.0392$ $\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$	$l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$ $l_2 = h\phi(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1) = 0.2(-0.999) = -0.1998$ $l_3 = h\phi(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2) = 0.2(-0.9791) = -0.1958$ $l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$ $l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + 2l_4) = -0.1970$
--	---

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and

$$y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$$

Example 32.26. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$, obtain y for $x = 0(0.1) 0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$. (Anna, 2004; Madras, 2003 S)

Solution. Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, y' = z \tag{... (i)}$$

We employ Taylor's series method to find y .

Differentiating the given equation n times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

At $x = 0, (y_{n+2})_0 = -(n+1)(y_n)_0$

$\therefore y(0) = 1$, gives $y_2(0) = -1, y_4(0) = 32, y_6(0) = -5 \times 3, \dots$

and $y_1(0) = 0$ yields $y_3(0) = y_5(0) = \dots = 0$.

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots$$

$$\therefore y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!} x^4 - \frac{5 \times 3}{6!} x^6 + \dots \tag{... (ii)}$$

and

$$z(x) = y'(x) = -x + \frac{1}{2} x^3 - \frac{1}{8} x^5 = \dots = -xy \tag{... (iii)}$$

From (ii), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8} (0.1)^4 - \dots = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \dots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48} + \dots = 0.956$$

From (iii), we have

$$z(0.1) = -0.0995, z(0.2) = -0.196, z(0.3) = -0.2863.$$

Also from (i), $z'(x) = -(xz + y) \therefore z'(0.1) = 0.985, z'(0.2) = -0.941, z'(0.3) = -0.87.$

Applying Milne's predictor formula, first to z and then to y , we obtain

$$\begin{aligned} z(0.4) &= z(0) + \frac{4}{3}(0.1)[2z'(0.1) - z'(0.2) + 2z'(0.3)] \\ &= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692 \end{aligned}$$

and

$$\begin{aligned} y(0.4) &= y(0) + \frac{4}{3}(0.1)[2y'(0.1) - y'(0.2) + 2y'(0.3)] \\ &= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231 \end{aligned} \quad [\because y' = z]$$

Also $z'(0.4) = -(x(0.4)z(0.4) + y(0.4)) = (0.4(-0.3692) + 0.9231) = -0.7754.$

Now applying Milne's corrector formula, we get

$$\begin{aligned} z(0.4) &= z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\} \\ &= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692 \end{aligned}$$

and

$$\begin{aligned} y(0.4) &= y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\} \\ &= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232 \end{aligned}$$

Hence $y(0.4) = 0.9232$ and $z(0.4) = -0.3692.$

PROBLEMS 32.6

1. Apply Picard's method to find the third approximation to the values of y and z , given that $dy/dx = z$, $dz/dx = x^3(y + z)$, given $y = 1, z = \frac{1}{2}$ when $x = 0$.
2. Solve the following differential equations using Taylor series method of the 4th order, for $x = 0.1$ and 0.2 , $\frac{dy}{dx} = xz + 1, \frac{dz}{dx} = -xy$; $y(0) = 0$ and $z(0) = 1$.
3. Find $y(0.1), z(0.1), y(0.2)$ and $z(0.2)$ from the system of equations $y' = x + z, z' = x - y^2$ given $y(0) = 0, z(0) = 1$ using Runge-Kutta of 4th order. (J.N.T.U., 2009)
4. Using Picard's method, obtain the second approximation to the solution of

$$\frac{d^2y}{dx^2} = x^3 \frac{dy}{dx} + x^3y \text{ so that } y(0) = 1, y'(0) = \frac{1}{2}.$$

5. Use Picard's method to approximate y when $x = 0.1$, given that $\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + y = 0$ and $y = 0.5, \frac{dy}{dx} = 0.1$, when $x = 0$.
6. Using Runge-Kutta method of order four, solve $y'' = y + xy', y(0) = 1, y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.
7. Consider the second order value problem $y'' - 2y' + 2y = e^{2t} \sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using the fourth order Runge-Kutta method, find $y(0.2)$. (Anna, 2003)

8. The angular displacement θ of a simple pendulum is given by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

where $l = 98$ cm and $g = 980$ cm/sec². If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use Runge-Kutta method to find θ and $d\theta/dt$ when $t = 0.2$ sec.

32.13 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R . Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types :

(i) $\frac{d^2y}{dx^2} + \lambda(x) \frac{dy}{dx} + \mu(x)y = \gamma(x)$ with the conditions $y(x_0) = a, y(x_n) = b$.

(ii) $\frac{d^4y}{dx^4} + \lambda(x)y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$.

While there exist many numerical methods for solving such boundary value problems, the method of finite-differences is most commonly used. We shall explain this method in the next section.

32.14 FINITE-DIFFERENCE METHOD

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite-difference approximations to the various derivatives are derived as under :

If $y(x)$ and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \dots \quad \dots(1)$$

and $y(x - h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \dots \quad \dots(2)$

Equation (1) gives $y'(x) = \frac{1}{h} [y(x + h) - y(x)] - \frac{h}{2}y''(x) - \dots$

i.e., $y'(x) = \frac{1}{h} [y(x + h) - y(x)] + O(h)$

which is the *forward difference approximation* of $y'(x)$ with an error of the order h .

Similarly (2) gives $y'(x) = \frac{1}{h} [y(x) - y(x - h)] + O(h)$

which is the *backward difference approximation* of $y'(x)$ with an error of the order h .

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h} [y(x + h) - y(x - h)] + O(h^2)$$

which is the *central-difference approximation* of $y'(x)$ with an error of the order h^2 . Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2} [y(x + h) - 2y(x) + y(x - h)] + O(h^2)$$

which is the *central difference approximation* of $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence the working expressions for the central difference approximations to the first four derivatives of y_i are as under :

$$y'_i = \frac{1}{2h} (y_{i+1} - y_{i-1}) \quad \dots(3)$$

$$y''_i = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1}) \quad \dots(4)$$

$$y'''_i = \frac{1}{2h^3} (y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \quad \dots(5)$$

$$y^{iv}_i = \frac{1}{h^4} (y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) \quad \dots(6)$$

Obs. The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h , the accuracy improves but the number of equations to be solved also increases.

Example 32.27. Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$. (Calicut, 1999)

Solution. We divide the interval $(0, 1)$ into four sub-intervals so that $h = 1/4$ and the pivot points are $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ and $x_4 = 1$.

The differential equation is approximated as

$$\frac{1}{h^2} [y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or $16y_{i+1} - 33y_i + 16y_{i-1} = x_i, i = 1, 2, 3.$

Using $y_0 = y_4 = 0$, we get the system of equations

$$16y_2 - 33y_1 = \frac{1}{4}$$

$$16y_3 - 33y_2 + 16y_1 = \frac{1}{2}$$

$$-33y_3 + 16y_2 = \frac{3}{4}$$

Their solution gives

$$y_1 = -0.03488, y_2 = -0.05632, y_3 = -0.05003.$$

Obs. The exact solution being $y(x) = \frac{\sinh x}{\sinh 1} - x$, the error at each nodal point is given in the table:

x	Computed value $y(x)$	Exact value $y(x)$	Error
0.25	-0.03488	-0.03505	0.00017
0.5	-0.05632	-0.05659	0.00027
0.75	-0.05003	-0.05028	0.00025

Example 32.28. Determine values of y at the pivotal points of the interval $(0, 1)$, if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2, y(0) = y(1) = y''(0) = y''(1) = 0$. (Take $n = 3$).

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y -values are $y_0 (= 0), y_1, y_2, y_3 (= 0)$.

Replacing y^{iv} by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4} (y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = 81x_i^2$$

or $y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2, i = 1, 2$

At $i = 1,$ $y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1/9$

At $i = 2,$ $y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$

Using $y_0 = y_3 = 0$, we get $-4y_2 + 7y_2 + y_{-1} = 1/9$...(i)

$y_4 + 7y_1 - 4y_1 = 4/9$...(ii)

Regarding the conditions $y''_0 = y''_3 = 0$, we know that

$$x_i'' = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1})$$

or At $i = 0$, $y''_0 = 9(y_1 - 2y_0 + y_{-1})$

$[\because y_0 = y_0'' = 0] \dots(iii)$

At $i = 3$, $y''_3 = 9(y_4 - 2y_3 + y_2)$
 $y_4 = -y_2$

$[\because y_3 = y_3'' = 0] \dots(iv)$

Using (iii), the equation (i) becomes

$$-4y_2 + 6y_1 = 1/9 \dots(v)$$

Using (iv), the equation (ii) reduces to

$$6y_2 - 4y_1 = 4/9 \dots(vi)$$

Solving (v) and (vi), we obtain

$$y_1 = 11/90 \text{ and } y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

Example 32.29. The deflection of a beam is governed by the equation

$$\frac{d^4 y}{dx^4} + 81y = \phi(x)$$

where $\phi(x)$ is given by the table

x	$1/3$	$2/3$	1
$\phi(x)$	81	162	243

and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y -values are $y_0 (= 0), y_1, y_2, y_3$.

The given differential equation is approximated to

$$\frac{1}{h^4} (y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

At $i = 1$, $y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1 \dots(i)$

At $i = 2$, $y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 2 \dots(ii)$

At $i = 3$, $y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3 \dots(iii)$

We have $y_0 = 0 \dots(iv)$

Since $y'_i = \frac{1}{2h} (y_{i+1} - y_{i-1})$

\therefore for $i = 0$, $0 = y'_0 = \frac{1}{2h} (y_1 - y_{-1})$ i.e. $y_{-1} = y_1 \dots(v)$

Since $y''_i = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1})$

\therefore for $i = 3$, $0 = y''_3 = \frac{1}{h^2} (y_4 - 2y_3 + y_2)$, i.e. $y_4 = 2y_3 - y_2 \dots(vi)$

Also $y'''_i = \frac{1}{2h^3} (y_{i+2} - 2y_{i+1} - 2y_{i-1} - y_{i-2})$

\therefore for $i = 3$, $0 = y'''_3 = \frac{1}{2h^3} (y_5 - 2y_4 + 2y_2 - y_1)$
 $y_5 = 2y_4 - 2y_2 + y_1 \dots(vii)$

Using (iv) and (v), the equation (i) reduces to

$$y_3 - 4y_2 + 8y_1 = 1 \dots(viii)$$

Using (iv) and (vi), the equation (ii) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \dots(ix)$$

Using (vi) and (vii), the equation (iii) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3$$

...(x)

Solving (viii), (ix) and (x), we get

$$y_1 = 8/13, y_2 = 22/13, y_3 = 37/13.$$

Hence

$$y(1/3) = 0.6154, y(2/3) = 1.6923, y(1) = 2.8462.$$

PROBLEMS 32.7

1. Solve the boundary value problem for $x = 0.5$:

$$\frac{d^2y}{dx^2} + y + 1 = 0, y(0) = y(1) = 0. \quad (\text{Take } n = 4)$$

2. Find an approximate solution of the boundary value problem :

$$y'' + 8(\sin^2 \pi y)y = 0, 0 \leq x \leq 1, y(0) = y(1) = 1. \quad (\text{Take } n = 4)$$

3. Solve the boundary value problem

$$xy'' + y = 0, y(1) = 1, y(2) = 2. \quad (\text{Take } n = 4)$$

4. Solve the equation

$$y'' - 4y' + 4y = e^{3x}, \text{ with the conditions } y(0) = 0, y(1) = -2, \text{ taking } n = 4.$$

5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.

6. Solve the boundary value problem

$$y'' + xy' + y = 3x^2 + 2, y(0) = 0, y(1) = 1.$$

7. The boundary value problem governing the deflection of a beam of length 3 metres is given by

$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4, y(0) = y'(0) = y(3) = y'(3) = 0.$$

The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$). Determine y at the pivotal points $x = 1$ and $x = 2$.

8. Solve the boundary value problem,

$$\frac{d^4y}{dx^4} + 81y = 729x^2, y(0) = y'(0) = y''(1) = y'''(1) = 0. \quad (\text{Use } n = 3)$$

9. Solve the equation $y^{(4)} - y''' + y = x^2$ subject to the boundary conditions

$$y(0) = y'(0) = 0 \text{ and } y(1) = 2, y'(1) = 0. \quad (\text{Take } n = 5)$$

32.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 32.8

Select the correct answer or fill up the blanks in the following questions :

- Which of the following is a step by step method :
(a) Taylor's s (b) Adams-Bashforth (c) Picard's (d) None.
- The finite difference scheme for the equation $2y'' + y = 5$ is
- If $y'' = x + y, y(0) = 1$ and $y^{(1)}(x) = 1 + x + x^2/2$, then by Picard's method, the value of $y^{(2)}(x)$ is
- The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is
- Taylor's series for solution of first order ordinary differential equations is
- Using Runge-Kutta method of order four, the value of $y(0.1)$ for $y' = x - 2y, y(0) = 1$ taking $h = 0.1$ is
(a) 0.813 (b) 0.825 (c) 0.0825 (d) none.
- Given y_0, y_1, y_2, y_3 , Milne's corrector formula to find y_4 for $dy/dx = f(x, y)$, is
- The second order Runge-Kutta formula is
- Adams-Bashforth predictor formula to solve $y' = f(x, y)$ given $y_0 = y(x_0)$ is
- The multi-step methods available for solving ordinary differential equations are
- To predict Adam's method atleast values of y , prior to the desired value, are required.
- Taylor's series solution of $y' = -xy, y(0) = 1$ upto x^4 is

13. Using modified Euler's method, the value of $y(0.1)$ for $\frac{dy}{dx} = x - y, y(0) = 1$ is
(a) 0.809 (b) 0.909 (c) 0.0809 (d) none.
14. Milne's Predictor formula is
15. Adam's corrector formula is
16. Using Euler's method, $dy/dx = (y - 2x)/y, y(0) = 1$; gives $y(0.1) =$
17. $\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations and
18. The formula for the 4th order Runge-Kutta method is
19. Taylor's series method will be useful to give some of Milne's method.
20. The name of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are
21. In the derivation of fourth order Runge-Kutta formula, it is called fourth order because
22. If $y' = x, y(0) = 1$ then by Picard's method, the value of $y(1)$ is
(a) 0.915 (b) 0.905 (c) 0.981 (d) none.
23. The finite difference scheme of the differential equation $y'' + 2y = 0$ is
24. If $y' = -y, y(0) = 1$, the Euler's method, the value of $y(1)$ is
(a) 0.99 (b) 0.999 (c) 0.981 (d) none.
25. In Euler's method if h is small the method is too slow, if h is large, it gives inaccurate value. (True or False)
26. Runge-Kutta method is a self-starting method. (True or False)
27. Predictor-corrector methods are self-starting methods. (True or False)

Calculus of Variations

1. Introduction. 2. Functionals. 3. Euler's equation. 4. Solutions of Euler's equation. 5. Geodesics. 6. Isoperimetric problems. 7. Several dependent variables. 8. Functionals involving higher order derivatives. 9. Approximate solution of boundary value problems—Rayleigh–Ritz method. 10. Weighted residual method—Galerkin's method. 11. Hamilton's principle. 12. Lagrange's equations.

35.1 INTRODUCTION

The calculus of variations is a powerful technique for the solution of problems in dynamics of rigid bodies, optimization of orbits and vibration problems. The subject primarily concerns with finding maximum or minimum value of a definite integral involving a certain function. It is something beyond finding stationary values of a given function. Only an elementary exposition of the subject is given here with the sole aim of introducing the student to a topic whose importance is fast growing in science and engineering.

Before proceeding further, the student should revise § 5.12 concerning maxima and minima of functions of several variables.

35.2 FUNCTIONALS

Consider the problem of finding a curve through two points (x_1, y_1) and (x_2, y_2) whose length is a minimum (Fig. 35.1). It is same as determining the curve $y = y(x)$ for which $y(x_1) = y_1$, $y(x_2) = y_2$ such that $\int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$ is a minimum.

In general terms, we wish to find the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function $f(x, y, y')$,

$$\int_{x_1}^{x_2} f(x, y, y') dx \text{ is a stationery value or an extremum.} \quad \dots(1)$$

An integral such as (1), which assumes a definite value for functions of the type $y = y(x)$ is called a **functional**.

In differential calculus, we deal with the problems of maxima and minima of functions. The calculus of variations is however, concerned with maximizing or minimizing functionals.

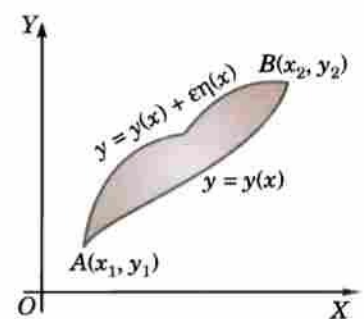


Fig. 35.1

35.3 EULER'S EQUATION

A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y, y') dx \text{ to be an extremum is that}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

This is called *Euler's equation*.

Proof. Let $y = y(x)$ be the curve joining points $A(x_1, y_1), B(x_2, y_2)$ which makes I an extremum. Let

$$y = y(x) + \epsilon \eta(x) \quad \dots(1)$$

be a neighbouring curve joining these points so that at $A, \eta(x_1) = 0$ and at $B, \eta(x_2) = 0$ (2)

The value of I along (1) is $I = \int_{x_1}^{x_2} f[x, y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x)] dx$

This being a function of ϵ , is a maximum or minimum for $\epsilon = 0$, when

$$\frac{dI}{d\epsilon} = 0 \text{ at } \epsilon = 0 \quad \dots(3)$$

\therefore Differentiating I under the integral sign by Leibnitz's rule (p. 139), we have

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \epsilon} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \epsilon} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \epsilon} \right) dx \quad \dots(4)$$

But ϵ being independent of $x, \frac{\partial x}{\partial \epsilon} = 0$. Also from (1), $\frac{\partial y}{\partial \epsilon} = \eta(x)$ and $\frac{\partial y'}{\partial \epsilon} = \eta'(x)$.

Substituting these values in (4), we get $\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx$

Integrating the second term on the right by parts, we have

$$\begin{aligned} \frac{dI}{d\epsilon} &= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) dx + \left[\frac{\partial f}{\partial y'} \eta(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta(x) dx \\ &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \eta(x) dx \end{aligned} \quad \text{[By (2)]}$$

Since this has to be zero by (3),

$$\therefore \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad \dots(I)$$

which is the desired *Euler's equation*.

Obs. 1. Other forms of Euler's equation.

(a) Since f is a function of x, y, y' , we have $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx}$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad \dots(5)$$

and

$$\frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) = y' \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{\partial f}{\partial y'} y'' \quad \dots(6)$$

Subtracting (6) from (5), we get $\frac{df}{dx} - \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' - y' \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)$

or

$$\frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial x} = y' \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right\} = 0 \quad \text{[By (I)]}$$

Hence

$$\frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial x} = 0 \quad \dots(II)$$

which is another form of (I).

(b) Again since $\frac{\partial f}{\partial y'}$ is also a function of x, y, y' , say : $\psi(x, y, y')$.

$$\therefore \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{d\psi}{dx} = \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} + \frac{\partial \psi}{\partial y'} \frac{dy'}{dx}$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y'} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y'} \right) y' + \frac{\partial}{\partial y'} \left(\frac{\partial f}{\partial y'} \right) y'' = \frac{\partial^2 f}{\partial x \partial y'} + y' \frac{\partial^2 f}{\partial y \partial y'} + y'' \frac{\partial^2 f}{\partial y'^2}$$

$$\text{Substituting this in (I), we get } \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y' \frac{\partial^2 f}{\partial y \partial y'} - y'' \frac{\partial^2 f}{\partial y'^2} = 0 \quad (\text{V.T.U., 2001}) \dots(\text{III})$$

which is an extended form of (I).

Obs. 2. The above problem can easily be extended to the integral

$$= \int_{x_1}^{x_2} f(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n') dx$$

involving n functions y_1, y_2, \dots, y_n of x . Then the necessary condition for this integral to be stationary is

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) = 0, \quad i = 1, 2, \dots, n \quad \dots(\text{IV})$$

These are Euler's equations for the n functions.

35.4 SOLUTIONS OF EULER'S EQUATION

Every solution of the Euler's equation which satisfies the boundary conditions, is called an *extremal* or a *stationary function* of the problem. The extremal can easily be obtained in the following cases :

(1) When f is independent of x

$$\text{We have } \partial f / \partial x = 0 \text{ and Euler's equation (II) above becomes } \frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) = 0$$

Integrating, we get $f - y' \frac{\partial f}{\partial y'} = \text{constant}$. This directly gives a solution of Euler's equation.

(2) When f is independent of y

$$\text{We have } \partial f / \partial y = 0 \text{ and Euler's equation (I) reduces to } \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Integrating $\frac{\partial f}{\partial y} = \text{constant}$ which gives a solution directly.

(3) When f is independent of y' .

$$\text{We have } \partial f / \partial y' = 0 \text{ and the equation (I) becomes } \frac{\partial f}{\partial y} = 0 \text{ which gives the desired solution.}$$

(4) When f is independent of x and y

$$\text{We have } \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial^2 f}{\partial x \partial y'} = 0, \frac{\partial^2 f}{\partial y \partial y'} = 0.$$

$$\text{Then the equation (III) above becomes } y'' \frac{\partial^2 f}{\partial y'^2} = 0.$$

If $\frac{\partial^2 f}{\partial y'^2} \neq 0$, it reduces to $y'' = 0$ which gives a solution of the form $y = ax + b$.

Example 35.1. Find the extremals of the functional $\int_{x_0}^{x_1} (y'^2/x^3) dx$. (V.T.U., 2003)

Solution. We have $f = y'^2/x^3$ which is independent of y i.e., $\partial f / \partial y = 0$.

$$\text{Also } \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{d}{dx} \left(\frac{2y'}{x^3} \right) = 2 \frac{x^3 y'' - y' \cdot 3x^2}{x^6} = \frac{2}{x^4} (xy'' - 3y')$$

$$\therefore \text{ Euler's equation reduces to } \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

$$\text{i.e., } \frac{2}{x^4} (xy'' - 3y') = 0 \text{ or } y''/y' = 3/x$$

Integrating both sides, $\int \frac{y''}{y'} dy = 3 \int \frac{dx}{x} + \log c$

i.e., $\log y' = 3 \log x + \log c$ or $y' = cx^3$
 Hence $y = cx^4/4 + c'$ or $y = c_1x^4 + c_2$
 This is the required extremal.

Example 35.2. Prove that the shortest distance between two points in a plane is a straight line.
 (V.T.U., 2003 S ; Bhopal, 2003)

Solution. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points and s the arc length of a curve connecting them (Fig. 35.2). Then

$$s = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$$

Now s will be minimum if it satisfies Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Here $f = \sqrt{1 + y'^2}$ which is independent of y i.e., $\partial f / \partial y = 0$

$$\therefore \frac{d}{dx} \left\{ \frac{\partial}{\partial y'} \sqrt{1 + y'^2} \right\} = 0 \quad \text{or} \quad \frac{d}{dx} \left\{ \frac{y'}{\sqrt{1 + y'^2}} \right\} = 0.$$

\therefore On integration, we have $y' / \sqrt{1 + y'^2} = \text{constant}$ $y' = \text{constant}$, m say.

Integrating, we get $y = mx + c$, which is a straight line, the constants m and c are determined from the fact that the straight line passes through A and B .

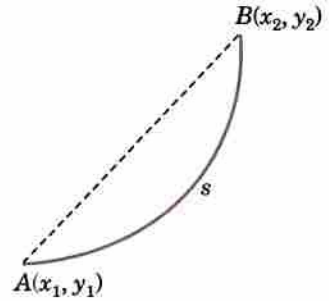


Fig. 35.2

Example 35.3. Find the curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about the x -axis gives a minimum surface area.
 (V.T.U., 2009)

Solution. In Fig. 35.3, the surface area = $\int_{x_1}^{x_2} 2\pi y ds$
 $= 2\pi \int_{x_1}^{x_2} y \sqrt{1 + y'^2} dx$. This has to be minimum.

Since $f = y \sqrt{1 + y'^2}$ is independent of x , therefore, Euler's equation reduces to

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}, c : \text{say} \quad [\text{By } \S 35.4 (1)]$$

$$\therefore y \sqrt{1 + y'^2} - y' \frac{\partial}{\partial y'} \left\{ y \sqrt{1 + y'^2} \right\} = c$$

$$\text{i.e.,} \quad y \sqrt{1 + y'^2} - y' \left\{ \frac{y}{2} (1 + y'^2)^{-1/2} \cdot 2y' \right\} = c$$

$$\text{or} \quad y / \sqrt{1 + y'^2} = c \quad \text{or} \quad y' = \frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c}$$

Separating the variables and integrating, we have

$$\int \frac{dy}{\sqrt{y^2 - c^2}} = \int \frac{dx}{c} + c' \quad \text{or} \quad \cosh^{-1} \left(\frac{y}{c} \right) = \frac{x + a}{c}$$

$$\text{i.e.,} \quad y = c \cosh \left(\frac{x + a}{c} \right)$$

which is a *catenary*. The constants a and c are determined from the points (x_1, y_1) and (x_2, y_2) .

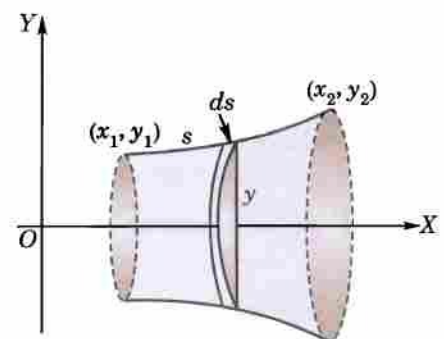


Fig. 35.3

Obs. This problem is also important in connection with *soap films* which are known to have shapes with minimum surface areas.

Example 35.4. Find the path on which a particle in the absence of friction, will slide from one point to another in the shortest time under the action of gravity. (V.T.U., 2004)

Solution. Let the particle start sliding on the curve OP_1 from O with zero velocity (Fig. 35.4). At time t , let the particle be $P(x, y)$ such that arc $OP = s$.

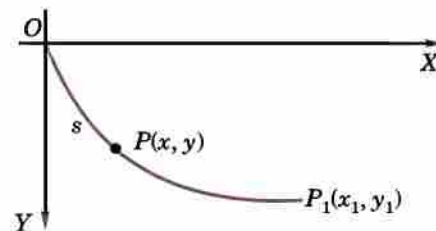


Fig. 35.4

By the principle of work and energy, we have

K.E. at P – K.E. at O = Work done in moving the particle from O to P .

$$\frac{1}{2} m \left(\frac{ds}{dt} \right)^2 - 0 = mgy$$

or
$$ds/dt = \sqrt{(2gy)}$$
 ... (i)

Thus the time taken by the particle to move from O to P_1 is

$$T = \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{(2gy)}} = \frac{1}{\sqrt{(2g)}} \int_0^{x_1} \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} dx$$

Here $f = \sqrt{(1+y'^2)/y}$ is independent of x .

∴ Euler's equation reduces to $f - y' \partial f / \partial y' = \text{constant}, c$: say

i.e.,
$$\frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y' \frac{\partial}{\partial y'} \left\{ \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} \right\} = c \quad \text{or} \quad \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y' \left\{ \frac{y'}{\sqrt{(1+y'^2)}\sqrt{y}} \right\} = c$$

or
$$\sqrt{[y(1+y'^2)]} = 1/c = \sqrt{a}, \text{ say.}$$

Solving for y' , we have
$$y' = \frac{dy}{dx} = \sqrt{\left(\frac{a-y}{y} \right)}$$

Separating the variables and integrating, we get

$$\int_0^x dx = \int_0^y \sqrt{\left(\frac{y}{a-y} \right)} dy \quad \text{[Put } y = a \sin^2 \theta \text{] ... (i)}$$

or
$$x = \int_0^\theta \sqrt{\left(\frac{a \sin^2 \theta}{a - a \sin^2 \theta} \right)} 2a \sin \theta \cos \theta d\theta$$

$$= a \int_0^\theta 2 \sin^2 \theta d\theta = a \int_0^\theta (1 - \cos 2\theta) d\theta = \frac{a}{2} (2\theta - \sin 2\theta) \quad \dots (ii)$$

Writing $a/2 = b$ and $2\theta = \phi$, equations (ii) and (i) become $x = b (\phi - \sin \phi)$, $y = b (1 - \cos \phi)$ which is a *cyloid*. The constant b is found from the fact that the curve goes through (x_1, y_1) .

Obs. This is the well-known **brachistochrone problem** which derives its name from the Greek words 'brachistos' meaning shortest and 'chronos' meaning time. It was proposed by John Bernoulli in 1696 and its solution formed the basis for the study of the 'Calculus of Variations'. (V.T.U., 2006)

Example 35.5. Find the curves on which the functional $\int_0^1 [(y')^2 + 12xy] dx$ with $y(0) = 0$ and $y(1) = 1$ can be extremised. (V.T.U., 2010)

Solution. We have $f = y'^2 + 12xy$

∴
$$\partial f / \partial y = 12x ; \quad \frac{\partial f}{\partial y'} = 2y' ; \quad \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 2y''$$

Hence the Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$ becomes

$$12x - 2y'' = 0 \quad \text{i.e., } y'' = 6x \quad \dots (i)$$

Integrating (i), $y' = 3x^2 + C$... (ii) and $y = x^3 + Cx + C'$... (iii)

Using the boundary conditions, when $x = 0, y = 0$ (iii) gives $C' = 0$.

When $x = 1, y = 1$, (iii) again gives $C = 0$.

Hence (iii) reduces to $y = x^3$ which is the only curve on which extremum can be attained.

Example 35.6. On which curve the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy) dy$ with $y(0) = 0$ and $y(\pi/2) = 0$, be extremized? (V.T.U., 2006)

Solution. Let $f = y'^2 - y^2 + 2xy$ so that $\frac{\partial f}{\partial y} = 0 - 2y + 2x$

and
$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{d}{dx} (2y') = 2y''$$

\therefore Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$ becomes

$$-2y + 2x - 2y'' = 0 \quad \text{or} \quad y'' + y = x \quad \text{or} \quad (D^2 + 1)y = x$$

Its A.E. $D^2 + 1 = 0$ gives $D = \pm i$.

\therefore C.F. = $c_1 \cos x + c_2 \sin x$

and P.I. = $\frac{1}{D^2 + 1} x = (1 + D^2)^{-1} x = (1 - D^2) x = x$

Thus $y = c_1 \cos x + c_2 \sin x + x$... (i)

Using boundary conditions : when $x = 0, y = 0$, (i) gives $c_1 = 0$;

when $x = \pi/2, y = 0$, (i) gives $0 = c_2 + \pi/2$, i.e., $c_2 = -\pi/2$.

Hence (i) reduces to $y = x - \frac{\pi}{2} \sin x$, which is the only curve on which the given functional can be extremized.

Example 35.7. Solve the variational problem

$$\delta \int_1^2 [x^2 (y')^2 + 2y(x+y)] dx = 0, \text{ given } y(1) = y(2) = 0. \quad (\text{V.T.U., 2006})$$

Solution. Let $f = x^2 (y')^2 + 2xy + 2y^2$ so that $\frac{\partial f}{\partial y} = 2x + 4y$, $\frac{\partial f}{\partial y'} = 2x^2 y'$

\therefore Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$ becomes

$$2x + 4y - \frac{d}{dx} (2x^2 y') = 0 \quad \text{or} \quad 2x + 4y - (2x^2 y'' + 4xy') = 0$$

or $x^2 y'' + 2xy' - 2y = x$. This is Cauchy's homogeneous linear (§ 13.9)

Putting $x = e^t$, it reduces to $(D^2 + D - 2)y = e^t$.

Its solution is $y = c_1 e^t + c_2 e^{-2t} + \frac{1}{3} t e^t$ or $y = c_1 x + \frac{c_2}{x^2} + \frac{1}{3} x \log x$... (i)

Since $y(1) = 0$, we have $c_1 + c_2 = 0$

and $y(2) = 0$ gives $0 = 2c_1 + \frac{1}{4} c_2 + \frac{2}{3} \log 2$

Solving these equations, we get $c_1 = -c_2 = \frac{-8}{21} \log 2$.

Putting the values of c_1 and c_2 in (i), we get

$$y = \frac{1}{21} \{ 8 \log 2 (x^{-2} - x) + 7x \log x \}$$

which is the required solution.

35.5 GEODESICS

A geodesic on a surface is a curve along which the distance between any two points of the surface is a minimum. To find the geodesics on a surface is a variational problem involving the conditional extremum. This problem was first studied by Jacob Bernoulli in 1698 and its general method of solution was given by Euler.

Example 35.8. Show that the geodesics on a plane are straight lines.

(V.T.U., 2009)

Solution. Let $y = y(x)$ be a curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the xy -plane. Then the length of a curve joining A and B is given by

$$s = \int_A^B \frac{ds}{dx} dx = \int_{x_1}^{x_2} \sqrt{[1 + (dy/dx)^2]} dx \quad \text{i.e.,} \quad s = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$$

The geodesic on the xy -plane is the curve $y = y(x)$ for which s is minimum.

We have $f(x, y, y') = \sqrt{1 + y'^2}$ which depends on y' only. Hence the Euler's equation.

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad \text{yields}$$

$$\frac{d}{dx} \left\{ \frac{2y'}{2\sqrt{1+y'^2}} \right\} = 0 \quad \text{i.e.,} \quad y'' \sqrt{1+y'^2} - \frac{y' y'' y'}{2\sqrt{1+y'^2}} = 0$$

$$\text{i.e.} \quad y''(1+y'^2) - y' y'' = 0 \quad \text{i.e.,} \quad \frac{d^2 y}{dx^2} = 0$$

Integrating twice, we get $y = c_1 x + c_2$

which is a straight line.

Hence the geodesics on a plane are straight lines.

Example 35.9. Find the geodesics on a right circular cylinder of radius a .

Solution. In cylindrical coordinates (ρ, ϕ, z) we have

$$x = r \cos \phi, y = r \sin \phi, z = z. \quad (\text{p. 357})$$

\therefore The element of arc on a right circular cylinder of radius a , is given by

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + (\rho d\phi)^2 + dz^2 = a^2 d\phi^2 + dz^2 \quad [\because \rho = a \text{ and } d\rho = 0]$$

$$\text{or} \quad ds = \sqrt{a^2 + (dz/d\phi)^2} \cdot d\phi \quad \text{or} \quad s = \int_{\phi_1}^{\phi_2} \sqrt{a^2 + z'^2} d\phi \quad \dots(i)$$

Now the geodesic for the given cylinder is the curve for which s is minimum. Here $f = \sqrt{a^2 + z'^2}$, which is a function of ϕ and z' while z is absent.

\therefore Euler's equation for the functional (i) reduces to

$$\frac{\partial f}{\partial z'} = \text{constant} \quad \text{or} \quad \frac{z'}{\sqrt{a^2 + z'^2}} = c, \text{ say.}$$

This simplifies to $(z')^2 = \text{constant}$ or $dz/d\phi = c_1$, say.

Integrating, $z = c_1 \phi + c_2$

This is the desired geodesics on a circular cylinder which is a circular helix. (Example 8.3, p. 318)

Example 35.10. Show that the geodesics on a sphere of radius a are its great circles.

Solution. In spherical coordinates (r, θ, ϕ) , we have

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta. \quad (\text{p. 359}) \quad \dots(i)$$

\therefore The arc element on a sphere of radius a , is given by

$$ds^2 = dr^2 + (r d\theta)^2 + (r \sin \theta d\phi)^2 = a^2 d\theta^2 + (a \sin \theta)^2 d\phi^2 \quad [\because r = a, dr = 0]$$

$$\text{or} \quad ds = a \sqrt{[1 + \sin^2 \theta (d\phi/d\theta)^2]} d\theta, \quad \text{or} \quad s = a \int_{\theta_1}^{\theta_2} \sqrt{[1 + \sin^2 \theta \cdot (\phi')^2]} d\theta$$

Now the geodesic on the sphere $r = a$ is the curve for which s is minimum. Here $f = a \sqrt{(1 + \sin^2 \theta \cdot \phi'^2)}$ which is a function of θ and ϕ' while ϕ is absent.

\therefore Euler's equation reduces to $\partial f / \partial \phi' = \text{constant}$.

$$\frac{\partial f}{\partial \phi'} = \frac{a \sin^2 \theta \cdot \phi'}{\sqrt{(1 + \sin^2 \theta \cdot \phi'^2)}} = \text{constant.}$$

or
$$\frac{\sin^2 \theta \cdot \phi'}{\sqrt{(1 + \sin^2 \theta \cdot \phi'^2)}} = c \text{ (say) or } \sin^2 \theta (\sin^2 \theta - c^2) \phi'^2 = c^2$$

or
$$\frac{d\phi}{d\theta} = \frac{c}{\sin \theta \sqrt{(\sin^2 \theta - c^2)}} = \frac{c \operatorname{cosec}^2 \theta}{\sqrt{(1 - c^2 \operatorname{cosec}^2 \theta)}}$$

Integrating
$$\phi = \int \frac{c \operatorname{cosec}^2 \theta \cdot d\theta}{\sqrt{[(1 - c^2) - (c \cot \theta)^2]}} + c' = -\sin^{-1} \left\{ \frac{c \cot \theta}{\sqrt{(1 - c^2)}} \right\} + c'$$

or
$$\cot \theta = A \cos \phi + B \sin \phi \text{ or } a \cos \theta = Aa \sin \theta \cos \phi + Ba \sin \theta \sin \phi$$

or
$$z = Ax + By \quad \text{[By (i) when } r = a]$$

This is a plane through the centre $(0, 0, 0)$ of the sphere which cuts the sphere along a great circle. Hence the required *geodesics are the arcs of the great circles*.

PROBLEMS 35.1

1. Solve the Euler's equation for the following functionals :

(i) $\int_{x_0}^{x_1} (x + y') y' dx$

(ii) $\int_{x_0}^{x_1} (1 + x^2 y') y' dx.$

(V.T.U., 2004)

2. Show that the general solution of the Euler's equation for the integral

$$\int_a^b \frac{1}{y} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ is } (x - h)^2 + y^2 = h^2.$$

Find the extremals of the following functionals :

3. $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x) dx.$

(V.T.U., 2004)

4. $\int_0^{\pi/2} (y^2 + y'^2 - 2y \sin x) dx, y(0) = y(\pi/2) = 0.$

(V.T.U., 2008)

5. $\int_0^{\pi} (y'^2 - y^2 + 4y \cos x) dx; y(0) = 0, y(\pi) = 0.$

(V.T.U., 2008 S)

6. $\int_{x_0}^{x_1} \frac{1 + y^2}{y^3} dx.$

7. $\int_1^2 \frac{x^3}{y^2} dx; y(1) = 0, y(2) = 3.$

8. $\int_1^2 \frac{\sqrt{(1 + y'^2)}}{x} dx; y(1) = 0, y(2) = 1.$

(Madurai, M.E., 2000 S)

9. Solve the variational problem $\delta \int_0^{\pi/2} [y^2 - (y')^2] dx$ under the conditions $y(0) = 0, y(\pi/2) = 2.$

(V.T.U., 2010)

10. A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary.
11. A particle is moving with a force perpendicular to and proportional to its distance from the line of zero velocity. Show that the path of quickest descent (brachistochrone) is a circle.
12. Find the geodesics on a *right circular cone* of semi-vertical angle α .

35.6 ISOPERIMETRIC PROBLEMS

In certain problems, it is necessary to make a given integral

$$I = \int_{x_1}^{x_2} f(x, y, y') dx \quad \dots(1)$$

maximum or minimum while keeping another integral

$$J = \int_{x_1}^{x_2} g(x, y, y') dx \quad \dots(2)$$

constant. Such problems involve one or more constraint conditions, just as $J = a$ constant. A typical example of this type is that of finding a closed curve of a given perimeter and maximum area. This being one of the earliest problems to engage attention, we often refer to problems of this type as *isoperimetric problems*.

Such problems are generally solved by the method of Lagrange multipliers. To extremize (1), we multiply (2) by λ and add to (1) where λ is the Lagrange multiplier. Then the necessary condition for the integral $\int_{x_1}^{x_2} H dx$ to be an extremum is $\frac{\partial H}{\partial y} - \frac{d}{dx} \left(\frac{\partial H}{\partial y'} \right) = 0$ where $H = f + \lambda g$. The values of the two constants of integration and the parameter λ are determined from the three conditions namely: the two boundary conditions and the integral J having given constant value.

Example 35.11. Find the plane curve of fixed perimeter and maximum area. (V.T.U., 2000 S)

Solution. Let l be the fixed perimeter of a plane curve between the points with abscissae x_1 and x_2 (Fig. 35.5). Then

$$l = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx \quad \dots(i)$$

Also the area between the curve and the x -axis is

$$A = \int_{x_1}^{x_2} y dx \quad \dots(ii)$$

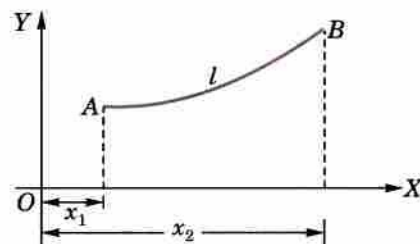


Fig. 35.5

We have to maximize (ii) subject to constraint (i).

\therefore Taking $f = y$ and $g = \sqrt{1 + y'^2}$, we write $H = f + \lambda g = y + \lambda \sqrt{1 + y'^2}$

Now H must satisfy the Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx} \left(\frac{\partial H}{\partial y'} \right) = 0 \quad \therefore \quad 1 - \frac{d}{dx} \left[\frac{\lambda y'}{\sqrt{1 + y'^2}} \right] = 0$$

Integrating w.r.t. x , we have $x - \lambda y' / \sqrt{1 + y'^2} = 0$

Solving for y' , we get $y' = \frac{x - a}{\sqrt{\lambda^2 - (x - a)^2}}$

Integrating again, $y = \sqrt{\lambda^2 - (x - a)^2} + b$ i.e. $(x - a)^2 + (y - b)^2 = \lambda^2$ which is a circle.

Example 35.12. Prove that the sphere is the solid figure of revolution which, for a given surface area, has maximum volume. (V.T.U., 2006 ; Madras, 2000 S)

Solution. Consider the arc OPA of the curve which rotates about the x -axis as shown in Fig. 35.6.

Then surface area
$$S = \int_{x=0}^a 2\pi y ds = \int_0^a 2\pi y \sqrt{1 + y'^2} dx$$

and volume of the solid so formed
$$V = \int_0^a \pi y^2 dx.$$

Here we have to maximize V subject to fixed S . Taking $f = \pi y^2$ and $g = 2\pi y \sqrt{1 + y'^2}$, we write $H = f + \lambda g = \pi y^2 + 2\pi\lambda y \sqrt{1 + y'^2}$.

Now H has to satisfy Euler's equation. But it does not contain x explicitly.

$$\therefore H - y' \frac{\partial H}{\partial y'} = \text{constant}, c : \text{say}$$

$$\text{i.e., } \pi y^2 + 2\pi\lambda y \sqrt{1 + y'^2} - y' \cdot 2\pi\lambda y \frac{y'}{\sqrt{1 + y'^2}} = c$$

$$\text{or } \pi y^2 + \frac{2\pi\lambda y}{\sqrt{1 + y'^2}} = c \quad \dots(i)$$

Since the curve passes through O and A for which $y = 0$, (i) gives $c = 0$.

$$\therefore y + 2\lambda \sqrt{1 + y'^2} = 0$$

$$\text{Solving for } y', \quad y' \left(= \frac{dy}{dx} \right) = \frac{\sqrt{4\lambda^2 - y^2}}{y}$$

Separating the variables and integrating, we get

$$\int dx = \int \frac{y dy}{\sqrt{4\lambda^2 - y^2}} + k \quad \text{or} \quad x = k - \sqrt{4\lambda^2 - y^2} \quad \dots(ii)$$

When $x = 0, y = 0 \quad \therefore k = 2\lambda$

\therefore (ii) becomes $(x - 2\lambda)^2 + y^2 = (2\lambda)^2$ which is a circle with centre $(2\lambda, 0)$ and radius 2λ .

Hence the figure formed by the revolution of given arc is a sphere.

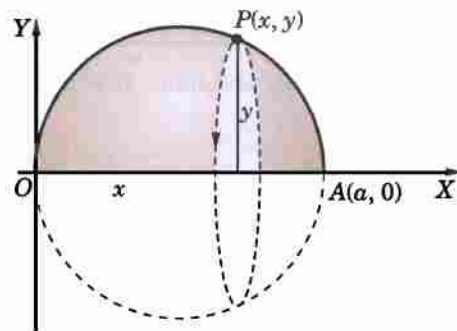


Fig. 35.6

PROBLEMS 35.2

- Find a function $y(x)$ for which $\int_0^1 (x^2 + y^2) dx$ is stationary, given that $\int_0^1 y^2 dx = 2; y(0) = 0, y(1) = 0$.
(Madras, 2000 S)
- Find the extremals of the isoperimetric problem $v[y(x)] = \int_{x_0}^{x_1} y'^2 dx$ given that $\int_{x_0}^{x_1} y dx = c$, a constant.
- Show that the curve c of given length l which minimizes the curved surface area of the solid generated by the revolution of c about the x -axis is a catenary.
(V.T.U., 2000 S)
- Find the extremal of the functional $I = \int_0^\pi [(y')^2 - y^2] dx$ under the conditions $y(0) = 0, y(\pi) = 1$ and subject to the constraint $\int_0^\pi y dx = 1$.
- Prove that the extremal of the isoperimetric problem $v[y(x)] = \int_1^4 y_2^2 dx, y(1) = 3, y(4) = 24$, subject to the condition $\int_1^4 y dx = 36$ is a parabola.
(Madras M.E., 2000)

35.7 SEVERAL DEPENDENT VARIABLES

We now extend the variational problem of § 35.3 to a problem with several variables as functions of a single independent variable i.e., A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx \quad \dots(1)$$

to be an extremum is that
$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) = 0, i = 1, 2, \dots, n \quad \dots(2)$$

Let y_1, y_2, \dots, y_n satisfy the boundary conditions $y_i(x_1) = y_{i1}, y_i(x_2) = y_{i2}$

Consider arbitrary functions $\eta_1(x), \eta_2(x), \dots, \eta_n(x)$ which are all zero on the boundary i.e.,

$$\eta_i(x_1) = 0 = \eta_i(x_2)$$

Replacing y_1, y_2, \dots by $y_1 + \varepsilon_1 \eta_1, y_2 + \varepsilon_2 \eta_2, \dots$ in (1), we get

$$I(\varepsilon) = \int_{x_1}^{x_2} f(x, y_1 + \varepsilon_1 \eta_1, y_2 + \varepsilon_2 \eta_2, \dots, y_1' + \varepsilon_1 \eta_1', y_2' + \varepsilon_2 \eta_2', \dots) dx \quad \dots(3)$$

This is a function of the parameters $\varepsilon_1, \varepsilon_2, \dots$ and reduces to (1) for $\varepsilon_1 = \varepsilon_2 = \dots = 0$.

To find the stationary value of (1), we find the stationary value of $I(\varepsilon)$ for $\varepsilon_1 = \varepsilon_2 = \dots = 0$, $I(\varepsilon)$ will have a stationary value when

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0, \frac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0, \dots$$

Writing

$$f = f(x, y_1, y_2, \dots, y_1', y_2', \dots)$$

and

$$F = f(x, y_1 + \varepsilon_1 \eta_1, y_2 + \varepsilon_2 \eta_2, \dots, y_1' + \varepsilon_1 \eta_1', y_2' + \varepsilon_2 \eta_2', \dots).$$

$$(3) \text{ becomes } I(\varepsilon) = \int_{x_1}^{x_2} F dx.$$

x_1, x_2 being constants independent of ε_1 , differentiating under the integral sign, we get

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varepsilon_1} dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_1} \eta_1 + \frac{\partial F}{\partial y_1'} \eta_1' \right) dx$$

$$\therefore \frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0, \text{ when } \varepsilon_1 = \varepsilon_2 = \dots = 0 \text{ gives}$$

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y_1} \eta_1 + \frac{\partial f}{\partial y_1'} \eta_1' \right) = 0$$

Integrating by parts the second term, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_1} \eta_1 + \left[\frac{\partial f}{\partial y_1'} \eta_1 \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y_1'} \right) \eta_1(x) dx = 0$$

i.e.,

$$\int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y_1} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_1'} \right) \right\} \eta_1(x) dx = 0 \quad [\because \eta_1(x_1) = 0 = \eta_1(x_2)]$$

Since this equation must hold good for all values of $\eta_1(x)$, we get

$$\frac{\partial f}{\partial y_1} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_1'} \right) = 0$$

Similarly $\frac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0$ when $\varepsilon_1 = \varepsilon_2 = \dots = 0$, will give

$$\frac{\partial f}{\partial y_2} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_2'} \right) = 0 \text{ and so on.}$$

All these conditions give a system of Euler's equation (2). A solution of these equations leads to the desired curves.

Example 35.13. Show that the functional $\int_0^{\pi/2} \left\{ 2xy + \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} dt$ such that $x(0) = 0, x(\pi/2) = -1, y(0) = 0, y(\pi/2) = 1$ is stationary for $x = -\sin t, y = \sin t$.

Solution. Euler's equations are $\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial x'} \right) = 0$... (i)

$$\frac{\partial f}{\partial y} - \frac{d}{dt} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad \dots (ii)$$

Here $f = 2xy + x'^2 + y'^2$. $\therefore \frac{\partial f}{\partial x} = 2y, \frac{\partial f}{\partial x'} = 2x', \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial y'} = 2y'$

(i) becomes $2y - \frac{d}{dt}(2x') = 0$ i.e., $2y - 2 \frac{d^2x}{dt^2} = 0$ or $\frac{d^2x}{dt^2} = y$... (iii)

(ii) becomes $2x - \frac{d}{dt}(2y') = 0$ i.e., $2x - 2 \frac{d^2y}{dt^2} = 0$ or $\frac{d^2y}{dt^2} = x$... (iv)

Now to solve these simultaneous differential equations, we differentiate (iii) twice,

$$\frac{d^4x}{dt^4} = \frac{d^2y}{dt^2} = x \quad \text{[By (iv)]}$$

or $(D^4 - 1)x = 0$ which is a linear equation with constant coefficients.

Its solution is $x = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$... (v)

From (iii), $y = x'' = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$... (vi)

Since $x = 0$ when $t = 0$ $\therefore 0 = c_1 + c_2 + c_3$... (vii)

$y = 0$ when $t = 0$ $\therefore 0 = c_1 + c_2 - c_3$... (viii)

Also $x = -1$ when $t = \pi/2$ $\therefore -1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4$... (ix)

$y = 1$ when $t = \pi/2$ $\therefore 1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} - c_4$... (x)

Adding (vii) and (viii), $c_1 + c_2 = 0$

Adding (ix) and (x), $c_1 e^{\pi/2} + c_2 e^{-\pi/2} = 0$

Solving these equations, we get $c_1 = c_2 = 0$.

From (viii), $c_3 = c_1 + c_2 = 0$. From (ix), $c_4 = -1$.

Hence from (v), $x = -\sin x$ and from (vi), $y = \sin x$.

35.8 FUNCTIONALS INVOLVING HIGHER ORDER DERIVATIVES

A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y, y', y'') dx \quad \dots (1)$$

to be extremum is $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0$.

Let $y(x)$ be the function which makes (1) stationary and satisfies the boundary conditions

$$y(x_1) = y_1, y(x_2) = y_2, y'(x_1) = y_1' \text{ and } y'(x_2) = y_2'$$

Consider the differentiable function $\eta(x)$ such that

$$\eta(x_1) = 0 = \eta(x_2) \text{ and } \eta'(x_1) = 0 = \eta'(x_2) \quad \dots (2)$$

Replacing y by $y + \epsilon\eta$ in (1), we get

$$I(\epsilon) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta'') dx \quad \dots (3)$$

This is a function of the parameter ϵ and reduces to (1) for $\epsilon = 0$.

To find the stationary value of (1), we find the stationary value of $I(\epsilon)$ for $\epsilon = 0$. But $I(\epsilon)$ will have a stationary value when $dI(\epsilon)/d\epsilon = 0$.

Writing $f = f(x, y, y', y'')$ and $F = f(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta'')$.

(3) becomes $I(\epsilon) = \int_{x_1}^{x_2} F dx$

x_1, x_2 being constants independent of ϵ , differentiating under the integral sign, we get

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{x_1}^{x_2} \frac{dF}{d\epsilon} dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' + \frac{\partial F}{\partial y''} \eta'' \right) dx$$

$$\therefore \frac{dI(\epsilon)}{d\epsilon} = 0 \text{ when } \epsilon = 0 \text{ gives } \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' + \frac{\partial f}{\partial y''} \eta'' \right) dx = 0$$

Integrating by parts once the second term and twice the third term, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta dx + \left[\frac{\partial f}{\partial y'} \eta \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \cdot \eta dx + \left[\frac{\partial f}{\partial y''} \eta' - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \cdot \eta \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) \cdot \eta dx = 0$$

or

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) \right] \eta(x) dx = 0 \tag{By (2)}$$

Since this equation must hold good for all values of $\eta(x)$, we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0$$

In general, a necessary condition for the functional $I = \int_{x_1}^{x_2} f(x, y, y', y'', \dots, y^{(n)}) dx$ to be stationary will be

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial f}{\partial y^{(n)}} \right) = 0$$

which is called the *Euler-Poisson equation* and its solutions are called *extremals*.

Example 35.14. Show that the curve which extremizes the functional $I = \int_0^{\pi/4} (y''^2 - y'^2 + x^2) dx$ under the conditions $y(0) = 0, y'(\pi/4) = 1, y(0) = 0, y'(\pi/4) = 1/\sqrt{2}$ is $y = \sin x$. (Madras M.E., 2000 S)

Solution. The Euler-Poisson equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0 \tag{... (i)}$$

Here $f = (y'')^2 - y'^2 + x^2, \frac{\partial f}{\partial y} = -2y, \frac{\partial f}{\partial y'} = 0, \frac{\partial f}{\partial y''} = 2y''$

\therefore (i) becomes $-2y + \frac{d^2}{dx^2} (2y'') = 0$ or $y^{iv} - y = 0$ or $(D^4 - 1)y = 0$

Its solution is $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$... (ii)

$\therefore y'(x) = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x$... (iii)

Applying the given boundary conditions to (ii) and (iii), we get

$$0 = y(0) = c_1 + c_2 + c_3, 1 = y'(\pi/4) = c_1 - c_2 + c_4$$

$$\frac{1}{\sqrt{2}} = y(\pi/4) = c_1 e^{\pi/4} + c_2 e^{-\pi/4} + \frac{1}{\sqrt{2}} c_3 + \frac{1}{\sqrt{2}} c_4 \tag{... (iv)}$$

$$\frac{1}{\sqrt{2}} = y'(\pi/4) = c_1 e^{\pi/4} - c_2 e^{-\pi/4} - \frac{1}{\sqrt{2}} c_3 + \frac{1}{\sqrt{2}} c_4$$

Solving the equation (iv), we get $c_1 = c_2 = c_3 = 0, c_4 = 1$. Hence the required curve is $y = \sin x$.

PROBLEMS 35.3

1. Show that the functional $\int_0^1 \left\{ 2x + \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} dt$, such that $x(0) = 1, y(0) = 1; x(1) = 1.5, y(1) = 1$ is stationary for $x = 1 + t^2/2, y = t$.
2. Find the extremals of the functional $v(y, z) = \int_{x_0}^{x_1} (2yz - 2y^2 + y'^2 - z'^2) dx$ (V.T.U., M.E., 2006)
3. Find a function $y(x)$ such that $\int_0^\pi y^2 dx = 1$ which makes $\int_0^\pi y'^2 dx$ a minimum if $y(0) = 0 = y(\pi), y''(0) = 0 = y''(\pi)$.
4. Find the extremals of the functional $\int_0^{\pi/2} (y'^2 - y^2 + x^2) dx$ that satisfies the conditions $y(0) = 1, y'(0) = 0, y(\pi/2) = 0, y'(\pi/2) = -1$. (Madras, M.E. 2000 S)
5. Find the extremals of the functional $\int_{-a}^a (\lambda y + \frac{1}{2} \mu y'^2) dx$ which satisfies the boundary conditions $y(-a) = 0, y'(-a) = 0, y(a) = 0, y'(a) = 0$.
6. Find the extremals of the following functionals :
 (i) $v[y(x)] = \int_{x_0}^{x_1} (16y^2 - y'^2 + x^2) dx$ (Nagpur, 1997) (ii) $v[y(x)] = \int_{x_0}^{x_1} (2xy + y'^2) dx$.

35.9 APPROXIMATE SOLUTION OF BOUNDARY VALUE PROBLEMS — Rayleigh-Ritz Method

In § 35.3, we have seen that the solution of Euler’s differential equation alongwith boundary conditions amounts to extremising a certain definite integral. This fact provides a technique of solving a boundary value problem approximately by assuming a trial solution satisfying the given boundary conditions and then extremising the integral whose integrand is found from the given differential equation.

To solve a boundary value problem of Rayleigh-Ritz method, we try to write the given differential equation as the Euler’s equation of some variational problem. Then we reduce this variational problem to a minimizing problem assuming an approximate solution in the form

$$\bar{y}(x) = y_0(x) + \sum c_i \phi_i(x) \tag{1}$$

where the trial functions $\phi_i(x)$ satisfy the boundary conditions and $\phi_i(x) = 0$ on the boundary C of its region R .

Let the integral to be extremised be $I = \int_a^b f(y, y', x) dx$... (2)

such that $y(a) = A$ and $y(b) = B$.

Substituting (1) in (2) by replacing y in \bar{y} in I , giving \bar{I} as a function of the unknowns c_i . Then c ’s become parameters which are so determined as to extremise \bar{I} . This requires

$$\frac{\partial \bar{I}}{\partial c_i} = 0, \quad i = 1, 2, \dots$$

Solving these equations, we get the values of c_i , which when substituted in (1) give the desired solution.

Example 35.15. Solve the boundary value problem $y'' - y + x = 0$ ($0 \leq x \leq 1$), $y(0) = y(1) = 0$ by Rayleigh-Ritz method.

Solution. Given differential equation is $y'' - y + x = 0$... (i)

Its solution is equivalent to extremising the integral $I = \int_0^1 F(x, y, y') dx$

where $F(x, y, y') = 2xy - y^2 - y'^2$, ... (ii)

since the Euler’s equation $\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$ gives (i).

Assume that the trial function is $\bar{y}(x) = c_0 + c_1x + c_2x^2$... (iii)

To satisfy $y(0) = 0, y(1) = 0$, we require $c_0 = 0, c_2 = -c_1$.

\therefore (iii) becomes $\bar{y}(x) = c_1x(1-x)$... (iv)

Substituting \bar{y} and \bar{y}' in I , we have

$$\begin{aligned} \bar{I} &= \int_0^1 [2x\bar{y} - \bar{y}^2 - (\bar{y}')^2] dx = \int_0^1 [2c_1(x^2 - x^3) - c_1^2(x - x^2)^2 - c_1^2(1-x)^2] dx \\ &= \frac{1}{6}c_1 - \frac{11}{30}c_1^2 \end{aligned}$$

Its stationary values are given by $d\bar{I}/dc_1 = 0$. $\therefore \frac{1}{6} - \frac{11}{15}c_1 = 0$ i.e., $c_1 = \frac{5}{22}$.

Thus the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$... (v)

35.10 WEIGHTED RESIDUAL METHOD—Galerkin's Method

The starting point of this method is to guess a solution to the differential equation which satisfies the boundary conditions. This trial solution will contain certain parameters which can be adjusted to minimize the errors so that the trial solution is as close to the exact solution as possible.

Consider the boundary value problem

$$y'' = f(y, y', x) \text{ with } y(a) = A \text{ and } y(b) = B \quad \dots(1)$$

We write the differential equation as $R = \bar{y}'' - f(\bar{y}, \bar{y}', x)$... (2)

where R is the residual of the equation ($R = 0$ for the exact solution $y(x)$ only which will satisfy the boundary conditions).

Consider the trial solution as $\bar{y}(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots$

where $\bar{y}(a) = A$ and $\bar{y}(b) = B$. The trial solution is differentiated twice and is substituted in (2).

To find c_1, c_2, \dots , we weight the residual by trial functions $\phi_1(x), \phi_2(x), \dots$ and set the integrals to zero. Thus we have $\int_c R\phi_1(x) dx = 0, \int_c R\phi_2(x) dx = 0, \dots$

These lead to simultaneous equations in the unknowns.

Having found c_1, c_2, \dots , the approximate solution $\bar{y}(x)$ is obtained.

Example 35.16. Use Galerkin's method to solve the boundary value problem of Example 35.14. Compare your approximate solution with the exact solution.

Solution. The residual is $R = \bar{y}'' - \bar{y} + x$... (i)

To find the trial solution which satisfies the boundary conditions, we derive a Lagrangian polynomial (§ 28.8) which passes through the points :

$$\begin{array}{lcl} x & : & 0 \quad 1/2 \quad 1 \\ y & : & 0 \quad c \quad 0 \end{array}$$

The resulting polynomial is $\bar{y}(x) = 4cx(1-x)$, so that $\phi(x) = x(1-x)$.

Substituting $\bar{y}(x), \bar{y}'(x)$ in (i), we get $R = 4cx^2 + (1-4c)x - 8c$

Thus $\int R\phi(x) dx = 0$ gives $\int_0^1 [4cx^2 + (1-4c)x - 8c] x(1-x) dx = 0$ whence $c = 5/88$.

Hence the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$ which is same as found in Example 34.14.

Exact solution. Rewriting the given equation as $(D^2 - 1)y = -x$, we find its solution as $y = c_1e^x + c_2e^{-x} + x$

Since $y(0) = 0$ and $y(1) = 0$, therefore $c_2 = -c_1 = 1/(e - e^{-1})$.

Hence the exact solution is $y = x - \frac{e^x - e^{-x}}{e - e^{-1}}$

The approximate and the exact solutions for some values of x are given below for comparison :

x	Approx. Sol.	Exact Sol.
0.25	0.043	0.035
0.50	0.057	0.057
0.75	0.043	0.05

Obs. To obtain a trial solution containing two unknown parameters, we derive a Lagrangian polynomial which passes through the points :

x	:	0	1/3	2/3	1
y	:	0	c_1	c_2	0

More the undetermined parameters, the more accurate is the solution, but it involves more labour to find their values.

Example 35.17. Find the approximate deflection of a simply supported beam under a uniformly distributed load w Fig. 35.7, using Galerkin's method.

Solution. The differential equation governing the deflection $y(x)$ of the

beam is $EI \frac{d^4 y}{dx^4} - w = 0, 0 < x < l$ (i) [§ 14.8]

The boundary conditions to be satisfied are

$$y(x=0) = y(x=l) = 0 \quad (\text{deflection is zero at ends}) \quad \dots(ii)$$

$$\frac{d^2 y}{dx^2}(x=0) = \frac{d^2 y}{dx^2}(x=l) = 0 \quad (\text{bending moment zero at ends})$$

...(iii)

We assume the trial solution $\bar{y}(x) = c_1 \sin(\pi x/l) + c_2 \sin(3\pi x/l)$, which satisfies the boundary conditions (ii) and (iii).

Substituting the trial solution in (i), we obtain the residual

$$R = EIC_1 (\pi/l)^4 \sin(\pi x/l) + EIC_2 (3\pi/l)^4 \sin(3\pi x/l) - w$$

Thus $\int_0^l R \cdot \sin(\pi x/l) dx = 0$ and $\int_0^l R \cdot \sin(3\pi x/l) dx = 0$

Computing these integrals, we get

$$EIC_1 (\pi/l)^4 l/2 - w \cdot 2l/\pi = 0, EIC_2 (3\pi/l)^4 l/2 - w \cdot 2l/3\pi = 0$$

Solving these, we obtain $c_1 = \frac{4wl^4}{\pi^5 EI}$ and $c_2 = \frac{4wl^4}{243\pi^5 EI}$.

Hence the deflection of the beam is given by

$$\bar{y}(x) = \frac{4wl^4}{\pi^5 EI} \left\{ \sin\left(\frac{\pi x}{l}\right) + \frac{1}{243} \sin\left(\frac{3\pi x}{l}\right) \right\}.$$

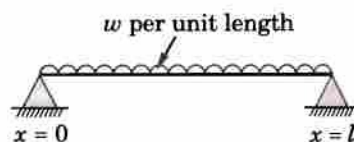


Fig. 35.7

PROBLEMS 35.4

1. Solve the boundary value problem :

$$y'' + y + x = 0 \quad (0 \leq x \leq 1), y(0) = y(1) = 0 \text{ by}$$

(i) Rayleigh-Ritz method, (ii) Galerkin's method. Compare your solution with the exact solution.

2. Using Galerkin's method, solve the boundary value problem $y'' = 3x + 4y$; $y(0) = 0, y(1) = 1$.

3. Apply Galerkin's method to the boundary value problem $y'' + y + x = 0$ ($0 \leq x \leq 1$); $y(0) = y(1) = 0$, to find the coefficients of the approximate solution $\bar{y}(x) = c_1 x(1-x) + c_2 x^2(1-x)$.

[Hint. Substituting $\bar{y}(x)$, $\bar{y}'(x)$ in the given equation replacing y, y'' by \bar{y}, \bar{y}'' , we get the residual $R = -2c_1 + c_2(2 - 6x) + x(1 - x)(c_1 + c_2x) + x$

Thus $\int_0^1 R \cdot x(1-x) dx = 0$ and $\int_0^1 R \cdot x^2(1-x) dx = 0$.

Computing these integrals, we get

$$\frac{3}{10}c_1 + \frac{3}{20}c_2 = \frac{1}{12}, \quad \frac{3}{20}c_1 + \frac{13}{305}c_2 = \frac{1}{20}.$$

Solving these, we obtain $c_1 = 71/369$, $c_2 = 7/41$.

- Using Ritz method, find an approximate solution of the problem $y'' - y + 4xe^x = 0$, $y'(0) = 1$, $y'(1) + y(1) = -e$.
- Solve the boundary value problem : $y'' + (1 + x^2)y + 1 = 0$, $y(-1) = y(1) = 0$, by taking the approximate solution $\bar{y}(x) = c_1(1 - x^2) + c_2x^2(1 - x^2)$ and using (i) Ritz method, (ii) Galerkin's method.
- Given the boundary value problem : $y'' + \pi^2y = x$, $y(0) = 1$, $y(1) = -0.9$.

Use Galerkin's method to estimate $y(0.5)$, taking the trial solution :

$$y = 1 - 1.9x + c_1x(1-x) + c_2x^2(1-x).$$

- Using Galerkin's method, obtain an approximate solution of the boundary value problem :

$$\frac{d}{dx} \left(x \frac{dy}{dx} \right) + y = x, \quad y(0) = 0, \quad y(1) = 1,$$

in the form $\bar{y}(x) = x + x(1-x)(c_1 + c_2x)$.

- Of all the parabolas which pass through $(0, 0)$ and $(1, 1)$, determine the one, which when rotated about the x -axis, generates a solid of revolution with least possible volume between $x = 0$ and $x = 1$.

[Hint. Take the parabola as $y = x + cx(1-x)$.]

- Using Rayleigh-Ritz method, find the potential at any point due to a charged sphere of radius a .

[Hint. Potential at a distance r from the centre of the sphere is $\phi = \phi_0(r/a)^k$, where ϕ_0 is the value of ϕ for $r = a$ and $k < 0$ so that $\phi \rightarrow 0$ as $r \rightarrow \infty$.

Electrostatic field due to charged sphere being conservative, electrostatic intensity $E = \nabla\phi$.

Also potential energy for unit volume = $\frac{1}{8\pi} E^2$

\therefore Total potential energy of the field in the entire region R exterior to the given sphere is

$$\begin{aligned} V &= \frac{1}{8\pi} \iiint_R E^2 dx dy dz = \frac{1}{8\pi} \iiint_R \left[\left(\frac{\partial\phi}{\partial x} \right)^2 + \left(\frac{\partial\phi}{\partial y} \right)^2 + \left(\frac{\partial\phi}{\partial z} \right)^2 \right] dx dy dz \\ &= \frac{1}{8\pi} \int_a^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{\partial\phi}{\partial r} \right)^2 r^2 \sin\theta d\phi d\theta dr. \end{aligned}$$

Electrostatic field will be in stable equilibrium if V is minimum, i.e., $dV/dp = 0$ and $d^2V/dp^2 > 0$.

This gives $k = -1$. Hence $\phi = \phi_0 a/r$.]

35.11 HAMILTON'S PRINCIPLE*

An important concept of mathematical physics is *Hamilton's principle* which states that *the time integral of the difference between the kinetic and potential energies of a dynamical system is stationary*

Consider a particle of mass m moving from a fixed origin O under the effect of a force F (Fig. 35.8). At any time t , let its position vector be \mathbf{R} . Then by Newton's second law,

$$\frac{m d^2 \mathbf{R}}{dt^2} = \mathbf{F} \quad \dots(1)$$

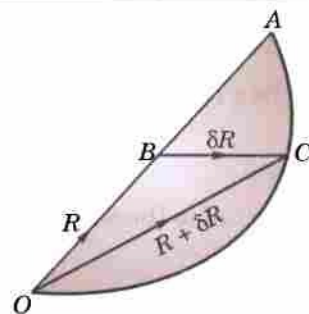


Fig. 35.8

* Named after the Irish mathematician *William Rowan Hamilton* (1805–1865) who is known for his contributions to dynamics.

Let the natural path OBA of the particle be changed to another path OCA , end points remaining the same. Let this variation in path, often called *virtual displacement*, be $\delta\mathbf{R}$. Then the work done during this displacement is

$$\delta W = \mathbf{F} \cdot \delta\mathbf{R} = \frac{m d^2 \mathbf{R}}{dt^2} \cdot \delta\mathbf{R} \quad [\text{By (1)}]$$

Also the kinetic energy of the particle is

$$T = \frac{1}{2} m \left(\frac{d\mathbf{R}}{dt} \right)^2$$

$$\therefore \delta T = m \frac{d\mathbf{R}}{dt} \cdot \delta \left(\frac{d\mathbf{R}}{dt} \right) = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt} (\delta\mathbf{R})$$

$$\text{Thus} \quad \delta T + \delta W = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt} (\delta\mathbf{R}) + m \frac{d^2 \mathbf{R}}{dt^2} \cdot \delta\mathbf{R} = m \frac{d}{dt} \left(\frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R} \right)$$

Integrating both sides w.r.t. t from t_0 to t_1 , we get

$$\int (\delta T + \delta W) dt = m \left[\frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R} \right]_{t_0}^{t_1} = 0 \quad \dots(2) \quad [\because \delta\mathbf{R} = 0 \text{ at } t_0 \text{ and } t_1]$$

If the force field is conservative, there exists a potential V such that $W = -V$. Then (2) takes the form

$$\int (\delta T - \delta V) dt = 0 \quad \text{or} \quad \delta \int (T - V) dt = 0 \quad \text{i.e.,} \quad \int (T - V) dt \quad \dots(3)$$

is stationary. This proves the Hamilton's principle for a particle. Its derivation can be extended to a system of particles by summation and to a rigid body by integration. Hence the principle is true for any dynamical system.

Obs. It can be easily shown that the integral (3) is a minimum along the natural path as compared to that along any other path joining O to A .

Def. The energy difference $T - V = L$ is called the **kinetic potential** or the **Lagrangian function**.

35.12 LAGRANGE'S EQUATION

In a dynamical system, the position of a body can be specified by the quantities q_1, q_2, \dots, q_n which are called the *generalised coordinates*.

The potential energy V , being a function of position only depends on these generalised coordinates q_i . The kinetic energy T , however, depends upon q_i and the velocities dq_i/dt (i.e., \dot{q}_i) $i = 1, 2, \dots, n$. Therefore, Lagrangian function $L = T - V$ is also a function of q_i and \dot{q}_i , $i = 1, 2, \dots, n$.

Thus by Hamilton's principle, the system moves so that $\int_{t_0}^{t_1} L dt$ is stationary.

$$\therefore \text{Euler's equation must hold good, i.e.,} \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad i = 1, 2, \dots, n.$$

These are called *Lagrange's equations* which determine the motion of the system.

Example 35.18. A mass, suspended at the end of a light spring having spring constant k , is set into vertical motion. Use Lagrange's equation, to find the equation of motion of the mass.

Solution. At any time t , let the displacement of m from the equilibrium position O be x (Fig. 35.9). Then the kinetic energy of P is

$$T = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2$$

Also the work done during its fall from O to P is

$$W = \int_0^x (mg - kx) dx = mgx - \frac{1}{2} kx^2.$$

If V is the potential energy of the mass at P , then

$$V = -W = \frac{1}{2} kx^2 - mgx$$

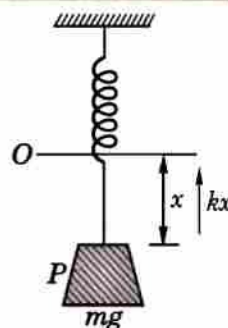


Fig. 35.9

∴ Lagrangian $L = T - V = \frac{1}{2} m \dot{x}^2 + mgx - \frac{1}{2} kx^2$.

Thus the Lagrange's equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0$$

becomes $(mg - kx) - \frac{d}{dt} (m\dot{x}) = 0$ or $m \frac{d^2x}{dt^2} = mg - kx$

which is the required equation of motion.

Example 35.19. Apply Lagrange's equations, to show that the equations of motion of the double pendulum of Fig. 35.10 are given by

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 + (m_1 + m_2) g \theta_1 = 0$$

and $l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 + g \theta_2 = 0$

for small angles θ_1, θ_2

(Punjab, M.E., 1997)

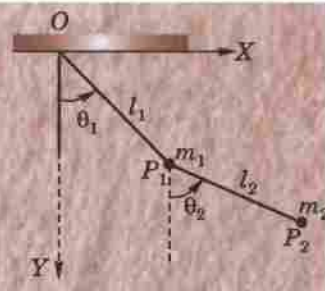


Fig. 35.10

Solution. At any time t , let the masses m_1, m_2 be at $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (Fig. 35.10) so that

$$\left. \begin{aligned} x_1 &= l_1 \sin \theta_1, y_1 = l_1 \cos \theta_1 \\ x_2 &= l_1 \sin \theta_1 + l_2 \sin \theta_2, y_2 = l_1 \cos \theta_1 + l_2 \cos \theta_2 \end{aligned} \right\} \dots(i)$$

Then total kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \end{aligned} \quad \text{[Using (i)]}$$

Also total potential energy is

$$V = m_1 g l_1 (1 - \cos \theta_1) + m_2 g (l_1 + l_2 - l_1 \cos \theta_1 - l_2 \cos \theta_2)$$

∴ Lagrangian $L = T - V = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 (1 - \cos \theta_1) - m_2 g l_2 (1 - \cos \theta_2) \dots(ii)$

Thus the Lagrange's equation corresponding to θ_1 , is

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = 0, \text{ becomes}$$

$$-m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin \theta_1 - \frac{d}{dt} [(m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 + \theta_2)] = 0$$

or $(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin \theta_1 = 0 \dots(iii)$

Similarly from (ii), Lagrange's equation corresponding to θ_2 , i.e.,

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) = 0$$

becomes $m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin \theta_2 - \frac{d}{dt} [m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)] = 0$

or $l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin \theta_2 = 0 \dots(iv)$

Now $\dot{\theta}_1$ and $\dot{\theta}_2$ being small, retaining first order terms only, (iii) and (iv) reduce to

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 + (m_1 + m_2) g \theta_1 = 0 \quad \text{and} \quad l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 + g \theta_2 = 0.$$

PROBLEMS 35.5

1. In a single pendulum, a mass m is suspended by a light rod of length l and the system vibrates in a plane. Using Lagrange's equation, show that $\ddot{\theta} + (g/l) \sin \theta = 0$.

Show that if θ is small, the period of oscillation is $2\pi \sqrt{l/g}$.

2. Two masses m_1 and m_2 are connected by an inextensible string which passes over a fixed pulley. Using Lagrange's equations, show that the acceleration of either mass is numerically

$$= (m_1 - m_2)g / (m_1 + m_2).$$

3. A perfectly flexible rope of uniform density per unit length is suspended with its end points fixed. Show that it assumes the shape of a catenary.
4. A bead of mass m from rest slides without friction under gravity along a wire inclined at an angle α to the vertical and rotating with constant angular velocity ω . Show that in times t , the bead has slid through a distance

$$\frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \cosh (\omega t \sin \alpha - 1).$$