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Future Vision

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Numerical Methods

Numerical soln of II order ODE :-

I Runge – Kutta Method of 4^{th} order

$$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0, z_0)$; $l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$; $l_2 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$

$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$; $l_3 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$

$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$; $l_4 = hg(x_0+h, y_0+k_3, z_0+l_3)$

Problems :-

1) Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y(0) = 1$, $y'(0) = 0$.

Evaluate $y(0.1)$ using R.K method of order 4.

Soln :- By data, $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y = 1$, $y' = 0$ at $x = 0$.

Put $\frac{dy}{dx} = z$. $\Rightarrow$ $\frac{d^2y}{dx^2} = \frac{dz}{dx}$.

∴ Given Eqn assumes the form

$$\frac{dz}{dx} - x^2 z - 2xy = 1$$

Thus we have

$\frac{dy}{dx} = z$, $\frac{dz}{dx} = 1 + 2xy + x^2 z$, $y = 1$, $z = 0$ at $x = 0$

Let $f(x, y, z) = z$, $g(x, y, z) = 1 + 2xy + x^2 z$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$. Let $h = 0.1$.

P.T.O.

$k_1 = h f(x_0, y_0, z_0) = (0.1) f(0,1,0) = (0.1)(0) = 0 \Rightarrow \boxed{k_1 = 0}$

$l_1 = h g(x_0, y_0, z_0) = (0.1) g(0,1,0)$

$= (0.1)\left[1 + 2(0)(1) + (0)^2(0)\right]$

$\boxed{l_1 = 0.1}$

$k_2 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= (0.1) f(0.05, 1, 0.05) = (0.1)(0.05) \Rightarrow \boxed{k_2 = 0.005}$

$l_2 = (0.1)\left[1 + 2(0.05)(1) + (0.05)^2(0.05)\right] \Rightarrow \boxed{l_2 = 0.11}$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1) f(0.05, 1.0025, 0.055) \Rightarrow \boxed{k_3 = 0.0055}$

$= (0.1)(0.055)$

$l_3 = (0.1)\left[1 + 2(0.05)(0.055) + (0.05)^2(0.055)\right]$

$\Rightarrow \boxed{l_3 = 0.11}$

$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.1) f(0.1, 1.055, 0.11) \Rightarrow \boxed{k_4 = 0.011}$

$l_4 = (0.1)\left[1 + 2(0.1)(1.055) + (0.1)^2(0.11)\right]$

$\Rightarrow \boxed{l_4 = 0.1202}$

$\therefore \; y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\boxed{y(0.1) = 1.0053.}$

2) Using R.K method, find $y(0.2)$, $y'(0.2)$ given

$\frac{d^2y}{dx^2} = y + x\frac{dy}{dx}$, $y(0) = 1$, $y'(0) = 0$ taking $h = 0.2$

Sol:- Let $\frac{dy}{dx} = z$. so that given Eq becomes

$\frac{dz}{dx} = y + xz$

Let $f(x,y,z) = z$, $g(x,y,z) = y + zx$.

$x_0 = 0, \quad y_0 = 1, \quad z_0 = 0$, let $h = 0.2$

$k_1 = h\, f(x_0, y_0, z_0)$

$= (0.2)\, f(0,1,0)$

$= (0.2)(0)$

$\boxed{k_1 = 0}$

$l_1 = (0.2)\; g(0,1,0)$

$= (0.2)\;[1+0]$

$\boxed{l_1 = 0.2}$

$k_2 = h f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_1}{2},\, z_0 + \frac{l_1}{2}\right)$

$= (0.2)\, f(0.05, 1, 0.1)$

$= (0.2)(0.1)$

$\boxed{k_2 = 0.02}$

$l_2 = (0.2)\; g\,(0.05, 1, 0.1)$

$= (0.2)\,[1 + (0.05)(0.1)]$

$\boxed{l_2 = 0.201}$

$k_3 = h\, f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_2}{2},\, z_0 + \frac{l_2}{2}\right)$

$= (0.2)\, f\,(0.05, 1.01, 0.1005)$

$= (0.2)\,(0.1005)$

$\boxed{k_3 = 0.0201}$

$l_3 = (0.2)\; g\;(0.05, 1.01, 0.1005)$

$= (0.2)\,[1.01 + (0.05)(0.1005)]$

$\boxed{l_3 = 0.2030}$

$k_4 = h\, f\,(x_0 + h,\, y_0 + k_3,\, z_0 + l_3)$

$= (0.2)\, f\,(0.2, 1.0201, 0.203)$

$= (0.2)(0.203)$

$\boxed{k_4 = 0.0406}$

$l_4 = 0.2\, g\,(0.2, 1.0201, 0.203)$

$= 0.2\,[1.0201 + (0.2)(0.203)]$

$\boxed{l_4 = 0.2121}$

$\therefore \quad y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

& $\quad y'(x_0+h) = z(x_0+h) = z_0 + \frac{1}{6}\,(l_1 + 2l_2 + 2l_3 + l_4)$

$\therefore \quad \boxed{y(0.2) = 1.0201}$ and $\boxed{z(0.2) = y'(0.2) = 0.2034}$

P.T.O.

3) Obtain the value of x and $\frac{dx}{dt}$ when $t = 0.1$ given x satisfies the Eqn $\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$; $x = 3$, $\frac{dx}{dt} = 0$ when $t = 0$ using 4th order R.K method.

Soln :- Let $\frac{dx}{dt} = z$; $\frac{dz}{dt} = tz - 4x$.

$x_0 = 3$, $z_0 = 0$ when $t_0 = 0$.

Let $f(t, x, z) = \frac{dx}{dt} = z$; $g(t, x, z) = tz - 4x$.

Let $h = 0.1$

$k_1 = h f(t_0, x_0, z_0)$

$= (0.1) f(0, 3, 0)$

$= (0.1)(0) \Rightarrow \boxed{k_1 = 0}$

$l_1 = (0.1) g(0, 3, 0)$

$= (0.1)[0 - 12]$

$\boxed{l_1 = -1.2}$

$k_2 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1) f(0.05, 3, -0.6)$

$= (0.1)(-0.6)$

$\boxed{k_2 = -0.06}$

$l_2 = (0.1) g(0.05, 3, -0.6)$

$= (0.1)[(0.05)(-0.6) - 12]$

$\boxed{l_2 = -1.203}$

$k_3 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1) f(0.05, 2.97, -0.6015)$

$= 0.1)(-0.6015)$

$\boxed{k_3 = -0.06015}$

$l_3 = (0.1) g(0.05, 2.97, -0.6015)$

$= (0.1)[(0.05)(-0.6015) - 4(2.97)]$

$\boxed{l_3 = -1.191}$

$k_4 = h f(t_0 + h, x_0 + k_3, z_0 + l_3)$

$= (0.1) f(0.1, 2.9398, -1.191)$

$\boxed{k_4 = -0.1191}$

$l_4 = (0.1)[(0.1)(-1.191) - 4(2.9398)]$

$\boxed{l_4 = -1.18785}$

Thus $x(t_0 + h) = x_0 + \frac{1}{6}(2k_2 + 2k_3 + k_1 + k_4) \Rightarrow x(0.1) = 2.9401$

$z(t_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \Rightarrow z(0.1) = -1.196$.

4) Solve $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$, $y(0)=1$, $y'(0)=0$. (3)

Evaluate $y(0.2)$ correct to 4 dec places using R.K Metho of 4^{th} order.

5) using R.K Method, solve the following D.E at $x=0.1$ under the given condn: $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right)$, $y(0)=1$, $y'(0)=0.5$ taking $h=0.1$

Milne's Method

To solve $y'' = f(x, y, y')$ given $y(x_0) = y_0$, $y'(x_0) = y_0'$

1. Put $y' = z \Rightarrow y'' = \frac{dz}{dx} = z'$, so that given D.E becomes $z' = f(x, y, z)$

2. compute the following table values

x	y	$y' = z$	$y'' = z'$
x_0	y_0	$y_0' = z_0$	$y_0'' = z_0'$
x_1	y_1	$y_1' = z_1$	$y_1'' = z_1'$
x_2	y_2	$y_2' = z_2$	$y_2'' = z_2'$
x_3	y_3	$y_3' = z_3$	$y_3'' = z_3'$

3. Apply Predictor formula to compute $y_4^{(P)}$ and $z_4^{(P)}$

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right) \qquad (\text{since } y' = z)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left(2z_1' - z_2' + 2z_3'\right)$$

4. compute $z_4' = f(x_4, y_4, z_4)$ and then apply corrector formula given by

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\left(z_2' + 4z_3' + z_4'\right)$$

5. Corrector formula can be applied repeatedly for better accuracy.

Problems:

1) Apply Milne's method to compute $y(0.8)$ given that $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ and the foll table of initial values:

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

Apply corrector formula twice in presenting the value of y at $x = 0.8$

Soln:- Put $\frac{dy}{dx} = z \Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx}$

Thus the given Eqn becomes $\frac{dz}{dx} = 1 - 2yz$

i.e. $z' = 1 - 2yz$.

$z_0' = 1 - 2y_0 z_0 = 1 - 2(0)(0) = 1$.

$z_1' = 1 - 2y_1 z_1 = 1 - 2(0.02)(0.1996) = 0.992$

$z_2' = 1 - 2y_2 z_2 = 1 - 2(0.0795)(0.3937) = 0.9374$

$z_3' = 1 - 2y_3 z_3 = 1 - 2(0.1762)(0.5689) = 0.7995$

Thus we have

x	y	$y' = z$	$y'' = z'$
$x_0 = 0$	$y_0 = 0$	$y_0' = 0$	$z_0' = 1$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = 0.1996$	$z_1' = 0.992$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = 0.3937$	$z_2' = 0.9374$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = 0.5689$	$z_3' = 0.7995$

Milne's Predictor formulae

$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3) = 0.3049$ write this as $y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$

$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3') = 0.7055$

Milne's corrector formulae

$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$= 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.7055)$

$\boxed{y_4^{(c)} = 0.3045}$

$z_4' = 1 - 2y_4^{(P)} z_4^{(P)} = 1 - 2(0.3049)(0.7055) = 0.5698$

$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$

$= 0.3935 + \frac{0.2}{3}(0.9374 + 4(0.7995) + 0.5698)$

$\boxed{z_4^{(c)} = 0.7072}$

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}\left(0.3937 + 4(0.5689) + 0.7072\right)$$

$$\boxed{y_4^{(c)} = 0.3046}$$

Thus $y = 0.3046$ at $x = 0.8$

2) Apply Milne's method to compute $y(0.8)$ given that y satisfies the eqn $y'' = 2yy'$ and y and y' are governed by the foll values:

$y(0) = 0$, $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6841$

$y'(0) = 1$, $y'(0.2) = 1.041$, $y'(0.4) = 1.179$, $y'(0.6) = 1.468$.

Apply Corrector formula twice.

Soln:- Let $\frac{dy}{dx} = z$, $\Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that given eqn

becomes $z' = 2yz$.

$\therefore \quad z'(0) = 0$

$z'(0.2) = 2(0.2027)(1.041) = 0.422$

$z'(0.4) = 2(0.4228)(1.179) = 0.997$

$z'(0.6) = 2(0.6841)(1.468) = 2.009$

Thus we have

x	y	$y' = z$	$y'' = z'$
$x_0 = 0$	$y_0 = 0$	$y_0' = 1$	$z_0' = 0$
$x_1 = 0.2$	$y_1 = 0.2027$	$y_1' = 1.041$	$z_1' = 0.422$
$x_2 = 0.4$	$y_2 = 0.4228$	$y_2' = 1.179$	$z_2' = 0.997$
$x_3 = 0.6$	$y_3 = 0.6841$	$y_3' = 1.468$	$z_3' = 2.009$

Milne's Predictor formula:

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3) = 1.0237$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3') = 2.0307.$$

Milne's corrector formulae:

$$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

we have $z_4' = 2y_4^{(P)} z_4^{(P)} = 4.1577$

$\therefore \boxed{y_4^{(c)} = 1.0282}$, $\boxed{z_4^{(c)} = 2.0584}$

Applying corrector formula again,

$$y_4^{(c)} = 0.4228 + \frac{0.2}{3}(1.179 + 4(1.468) + 2.0584)$$

$$\boxed{y_4^{(c)} = 1.0301}$$

Thus $y(0.8) = 1.0301$.

3) Obtain the soln of the eqn $y'' + xy' + y = 0$; $y(0) = 1$, $y'(0) = 0$. ~~$y'(0)$~~ compute $y(0.4)$ given

$y(0.1) = 0.995$, $y(0.2) = 0.9801$, $y(0.3) = 0.956$

$z(0.1) = -0.0995$, $z(0.2) = -0.196$, $z(0.3) = -0.2867$

soln :- Let $\frac{dy}{dx} = z \Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx}$

$\therefore$ given eqn becomes $z' + xz + y = 0$.

$$z' = -(y + xz)$$

$z'(0) = -(1 + 0) = -1$

$z'(0.1) = -(0.995 + (0.1)(-0.0995)) = -0.985$

$z'(0.2) = -(0.9801 + (0.2)(-0.196)) = -0.941$

$z'(0.3) = -(0.956 + (0.3)(-0.2867)) = -0.87$

Thus we have the following table:

P.T.O.

x	y	$y' = z$	$y'' = z'$
$x_0 = 0$	$y_0 = 1$	$z_0 = 0$	$z_0' = -1$
$x_1 = 0.1$	$y_1 = 0.995$	$z_1 = -0.0995$	$z_1' = -0.985$
$x_2 = 0.2$	$y_2 = 0.9801$	$z_2 = -0.196$	$z_2' = -0.941$
$x_3 = 0.3$	$y_3 = 0.956$	$z_3 = -0.2867$	$z_3' = -0.87$

Milne's Predictor formula :

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$\boxed{y_4^{(P)} = 0.9231}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$\boxed{z_4^{(P)} = -0.3692}$$

Now, $z_4' = -\left(y_4^{(P)} + x_4^{(P)} z_4^{(P)}\right) = -0.7754$

Milne's corrector formula :-

$$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4) = 0.923.$$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4') = -0.3692.$$

Thus $\underline{y(0.1) = 0.923}$

4) Obtain the soln of the eqn $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ at the point $x = 1.4$ by applying Milne's method given that $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$, $y'(1) = 2$, $y'(1.1) = 2.3178$, $y'(1.2) = 2.6725$, $y'(1.3) = 3.0657$.

$\rightarrow$ $y_4^{(P)} = 3.0793$, $z_4^{(P)} = 3.4996$, $y_4^{(c)} = 3.0794$.

P.T.O.

5) Using Milne's method, obtain an approximate sol^n at the point $x = 0.4$ of the problem $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$

$y(0) = 1$, $y'(0) = 0.1$. Given that $y(0.1) = 1.03995$,

$y(0.2) = 1.138036$, $y(0.3) = 1.29865$, $y'(0.1) = 0.6955$,

$y'(0.2) = 1.258$, $y'(0.3) = 1.873$.

Ans :- 1.5139

Special Functions

Series solution of Bessel's Differential Equation leading to $J_n(x)$

Bessel's function of first kind:

The Bessel differential equation of order n is of the form,

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$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \text{——} \quad (1)$$

where n is a non negative real constant. (parameter)

We employ Frobenius method to solve this equation,

By (1) coefficient of y'' is $x^2 = P_0(x)$ and $P_0(x) = 0$ at $x = 0$.

We assume the series solution of (1) in the form,

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \text{——} \quad (2)$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

Now (1) becomes,

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r} +$$

$$\sum_{r=0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

ie $$\sum_{r=0}^{\infty} a_r x^{k+r} \left[(k+r)(k+r-1) + (k+r) - n^2 \right] + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

ie $$\sum_{r=0}^{\infty} a_r x^{k+r} \left[(k+r)[k+r-\cancel{1}+\cancel{1}] - n^2 \right] + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

i.e. $$\sum_{r=0}^{\infty} a_r x^{k+r} \left[(k+r)^2 - n^2 \right] + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0 \quad \text{——} \quad (3)$$

The coefficient of lowest degree term in x is x^k

Its coefficient is $a_0(k^2-n^2)$

Let $a_0(k^2-n^2)=0$

Since, $a_0 \neq 0$, $k^2-n^2=0. \Rightarrow k=\pm n.$ — (4)

Coefficient of x^{k+1} is $a_1[(k+1)^2-n^2]$

Let $a_1[(k+1)^2-n^2]=0 \Rightarrow (k+1)^2=n^2 \Rightarrow k+1=\pm n$

— not possible, since $k=\pm n$ by (4). $\therefore \boxed{a_1=0}$.

For $r \geq 2$, coefficient of x^{k+r} is,

$$a_r[(k+r)^2-n^2]+a_{r-2}=0.$$

or $$a_r = \frac{-a_{r-2}}{[(k+r)^2-n^2]} \quad (r \geq 2) \quad \text{—— (5)}$$

when, $k=\cancel{\pm n}$ $+n$, (5) becomes,

$$a_r = \frac{-a_{r-2}}{(n+r)^2-n^2} = \frac{-a_{r-2}}{2nr+r^2}$$

Putting $r=2,3,4\ldots$ we get,

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}$$

$$a_3 = \frac{-a_1}{6n+9} = 0 \quad \text{since } a_1=0.$$

IIIly, $a_5, a_7 \ldots = 0$.

$$a_4 = \frac{-a_2}{8n+16} = \frac{-a_2}{8(n+2)} = \frac{a_0}{32(n+1)(n+2)} \ldots\ldots$$

by (2) $y = x^k(a_0+a_1x+a_2x^2+\ldots)$

Substituting values of $a_0, a_2, a_3 \ldots$ also considering (2)

the solution for $k = +n$ as y_1 we get,

$$y_1 = x^n \left[a_0 - \frac{a_0}{4(n+1)} x^2 + \frac{a_0}{32(n+1)(n+2)} x^4 - \ldots \right]$$

ie $$y_1 = a_0 x^n \left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \ldots \right] \quad \text{— (6)}$$

Let y_2 be the solution when $k = -n$,

Then $$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5[-n+1](-n+2)} - \ldots \right] \quad \text{— (7)}$$

The complete (general) solution of (1) is given by

$$y = Ay_1 + By_2$$

where A and B are arbitrary constants.

We shall standardise the solution as in (6) by choosing,

$a_0 = \dfrac{1}{2^n \, \Gamma(n+1)}$ and is denoted by Y_1

$$\therefore Y_1 = \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot 2} - \ldots \right\}$$

$$= \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1)\cdot 2} \right\}$$

WKT, $\Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(n+2) = (n+1)\Gamma(n+1)$

$\Gamma(n+3) = (n+2)\Gamma(n+2) = (n+2)(n+1)\Gamma(n+1)$

$$\therefore Y_1 = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(n+3)\cdot 2} - \ldots \right\}$$

$$= \left(\frac{x}{2}\right)^n \left\{ \frac{(-1)^0}{\Gamma(n+1)\cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\, 2!}\left(\frac{x}{2}\right)^4 + \cdots \right.$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\, r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$$

This function is called Bessel's function of I kind order n denoted by $J_n(x)$.

$$\therefore \quad J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!} \quad \text{—— (8)}$$

Further the solution for $k=-n$ (in respect of y_2) be denoted by $J_{-n}(x)$.

Hence general solution of Bessel's equation is given by,

$$y = a J_n(x) + b J_{-n}(x).$$

where a and b are arbitrary constants and n is not an integer.

Properties of Bessel's function:

Property 1:

$J_{-n}(x) = (-1)^n J_n(x)$ where n is a positive integer.

Proof: WKT, $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \quad \text{--- (1)}.$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!} \quad \text{--- (2)}$$

$\Gamma(-n+r+1) = \Gamma(r-(n-1))$

For $r = 1, 2, 3 \dots n-2$, $r-(n-1) < 0$. Let $r-(n-1) = -k$. $k>0$.

and for $r = n-1$, $\Gamma(r-(n-1)) = \Gamma(0)$.

WKT, $\Gamma(-k) = \Gamma(0) = \infty$ (not defined)

$\therefore \frac{1}{\Gamma(-k)} = 0$. for $r = 0, 1, 2, 3 \dots (n-1)$. ie $\sum_{r=0}^{n-1} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-k)\, r!} = 0$.

$\therefore$ by (2), $$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!} \quad \text{--- (3)}$$

Let $r-n = s$ or $r = s+n$.

$\therefore$ when $r = n$, $s = 0$.

$\therefore$ by (3) $$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{-n+2s+2n} \frac{1}{\Gamma(s+1)\,(s+n)!}$$

$$= \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{\Gamma(s+1)\,(s+n)!}$$

Using the properties of gamma function, we can write,

$\Gamma(s+1) = s!$ and $(s+n)! = \Gamma(s+n+1)$

$$\therefore \quad J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{s!\,\Gamma(s+n+1)}$$

$$= (-1)^{n} \sum_{s=0}^{\infty} (-1)^{s} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{\Gamma(n+s+1)\,s!}$$

Comparing with (1),

$$J_{-n}(x) = (-1)^{n} J_{n}(x)$$

Remark: $J_{-n}(x)$ and $J_{n}(x)$ are linearly dependent where n is an integer. When n is not an integer then $J_{-n}(x)$ and $J_{n}(x)$ are linearly independent.

Property 2: $J_{n}(-x) = (-1)^{n} J_{n}(x) = J_{-n}(x)$ where n is a positive integer.

Proof: We have $$J_{n}(x) = \sum_{r=0}^{\infty} (-1)^{r} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_{n}(-x) = \sum_{r=0}^{\infty} (-1)^{r} \left(-\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= \sum_{r=0}^{\infty} (-1)^{r} (-1)^{n+2r} \frac{x^{n+2r}}{2^{n+2r}} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= (-1)^{n} \sum_{r=0}^{\infty} (-1)^{3r} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} \{(-1)^{r}\}^{r} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$ (4)

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$\therefore \quad J_n(-x) = (-1)^n J_n(x).$

Since $(-1)^n J_n(x) = J_{-n}(x)$ we have,

$$J_n(-x) = (-1)^n J_n(x) = J_{-n}(x).$$

Orthogonal property of Bessel Function: Dec - 2015.

If α and β are 2 distinct roots of $J_n(x) = 0$ then

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha \neq \beta. \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2}[J_{n+1}(\alpha)]^2 & \text{if } \alpha = \beta. \end{cases}$$

Proof: WKT $y = J_n(\lambda x)$ is a soln of the eqn,

$$x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0.$$

If $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ the associated D.E are,

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \quad \text{—— (1)}$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \quad \text{—— (2)}$$

$\times$ (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$ we get,

$$x v u'' + v u' + \alpha^2 u v x - \frac{n^2 u v}{x} = 0.$$

$$x u v'' + u v' + \beta^2 u v x - \frac{n^2 u v}{x} = 0.$$

On subtraction we get,

$$x[v u'' - u v''] + (v u' - u v') + u v x(\alpha^2 - \beta^2) = 0$$

$$\Rightarrow \frac{d}{dx}[x\{v u' - u v'\}] = (\beta^2 - \alpha^2) u v x$$

Integrating w.r.t x on both sides between 0 to 1, we have

$$\Big[x[v u' - u v']\Big]_{x=0}^{1} = (\beta^2 - \alpha^2) \int_0^1 x u v\,dx$$

i.e $(v u' - u v')_{x=1} - 0 = (\beta^2 - \alpha^2) \int_0^1 x u v\,dx$ —— (3)

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$ we have,

$$u' = \alpha J_n'(\alpha x), \quad v' = \beta J_n'(\beta x)$$

(3) becomes,

$$\Big[J_n(\beta x) \cdot \alpha \cdot J_n'(\alpha x) - J_n(\alpha x)\, \beta J_n'(\beta x)\Big]_{x=1}$$

$$= (\beta^2 - \alpha^2) \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx.$$

$$\therefore \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{\beta^2-\alpha^2}\left[\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right] \quad \text{—— (4)}$$

Given α and β are distinct roots of $J_n(x)=0$

$\Rightarrow J_n(\alpha)=0$, $J_n(\beta)=0$.

$\therefore$ by (4), $\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$ provided $\beta^2-\alpha^2 \neq 0$

or $\beta \neq \alpha$.

Thus we have proved that,

$$\boxed{\text{if } \alpha \neq \beta \quad \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0} \quad \text{—— (5)}$$

Now, to discuss the case when $\alpha=\beta$.

when $\alpha=\beta$, RHS of (4) becomes indeterminate form of the type $\frac{0}{0}$.

Taking limits as $\beta\to\alpha$ keeping α fixed, by L'Hospital's rule,

$$(4) \Rightarrow \lim_{\beta\to\alpha} \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

$$= \lim_{\beta\to\alpha} \frac{1}{\beta^2-\alpha^2}\left\{\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right\}$$

Since α is fixed, we must have $J_n(\alpha)=0$ as α is a root of $J_n(x)=0$.

$$\therefore \lim_{\beta\to\alpha} \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

$$= \lim_{\beta\to\alpha} \frac{1}{\beta^2-\alpha^2}\left[\alpha J_n(\beta) J_n'(\alpha)\right]$$

by L'Hospital's rule,

$$= \lim_{\beta\to\alpha} \frac{1}{2\beta}\left[\alpha J_n'(\beta) J_n'(\alpha)\right]$$

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\alpha x)\,dx = \frac{1}{2\alpha}\,\alpha J_n'(\alpha) J_n'(\alpha)$$

$$\Rightarrow \int_0^1 x J_n^2(\alpha x)\,dx = \frac{1}{2}\left[J_n'(\alpha)\right]^2 \quad \text{—— (6)}$$

Further we have the recurrence relation,

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$\text{i.e. } J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha)$$

but $J_n(\alpha) = 0$

$$J_n'(\alpha) = -J_{n+1}(\alpha)$$

$$(6) \Rightarrow \int_0^1 x J_n^2(\alpha x)\,dx = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2 \quad \text{—— (7)}$$

This result is known as Lommel integral formula.

Thus combining (5), (6), (7),

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha \neq \beta. \\ \frac{1}{2}\left[J_n'(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}^2(\alpha)\right] & \text{if } \alpha = \beta. \end{cases}$$

<u>Note:</u> Orthogonal property is also represented in the form,

$$\int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha = \beta. \\ \frac{a^2}{2}\left[J_{n+1}(a\alpha)\right]^2 & \text{if } \alpha = \beta. \end{cases}$$

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* <u>Series solution of Legendre's D.E:</u> June - 2014, 2012.

We have Legendre's D.E,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{—— (1)}$$

The coefficient of $y'' = (1-x^2) = P_0(x)$ and $P_0(x) \neq 0$ at $x = 0$

We employ power series method to solve this eqn,

We assume the series solution of (1) in the form,

$$y = \sum_{r=0}^{\infty} a_r x^r. \quad \text{—— (2)}$$

$$\therefore \frac{dy}{dx} = \sum_0^\infty a_r r x^{r-1}$$

$$\frac{d^2y}{dx^2} = \sum_0^\infty a_r r(r-1) x^{r-2}$$

Now (1) becomes,

$$(1-x^2)\sum_0^\infty a_r r(r-1)x^{r-2} - 2x\sum_0^\infty a_r r x^{r-1} + n(n+1)\sum a_r x^r = 0$$

ie. $$\sum_0^\infty a_r r(r-1)x^{r-2} - \sum_0^\infty a_r r(r-1)x^r - \sum_0^\infty 2a_r r x^r + n(n+1)\sum_0^\infty a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

~~We~~ coefft of x^{-2} : $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$.

coefft of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero,

ie $a_{r+2}(r+2)(r+1) - a_r r(r-1) - 2a_r r + n(n+1)a_r = 0$

ie $a_{r+2}(r+2)(r+1) = a_r[r(r-1) + 2r - n(n+1)]$.

or $$a_{r+2} = \frac{-[n(n+1) - r^2 - r]}{(r+2)(r+1)} a_r \quad \text{———} (3)$$

putting $r = 0, 1, 2, 3 \ldots$ in (3) we get,

$$a_2 = \frac{-n(n+1)}{2} a_0 \qquad a_3 = \frac{-(n^2+n-2)}{6} a_1 = \frac{-(n-1)(n+2)}{6} a_1$$

$$a_4 = \frac{-(n^2+n-6)}{12} \cdot a_2 = \frac{-(n-2)(n+3)}{12} \cdot \frac{-n(n+1)}{2} a_0 .$$

ie $$a_4 = \frac{n(n+1)(n-2)(n+3)}{24} a_0 .$$

$$a_5 = \frac{-(n^2+n-12)}{20} \cdot a_3 = \frac{-(n-3)(n+4)}{20} \cdot \frac{-(n-1)(n+2)}{6} a_1 .$$

ie $$a_5 = \frac{(n-1)(n+2)(n-3)(n+4)}{120} a_1$$ and so on.

We substitute the values in the expanded form of (11)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^2 + a_4 x^4 + a_5 x^5 + \dots$$

$$= (a_0 + a_2 x + a_4 x^2 +) \dots) + (a_1 x + a_3 x^3 + a_5 x^5 + \dots)$$

$$= a_0 \left[1 - \frac{n(n+1)}{2} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 - \dots \right]$$

$$+ a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} x^5 - \dots \right] \quad \text{——} (4)$$

$$= a_0 u(x) + a_1 v(x) \quad \text{——} (5).$$

This is series soln of Legendre's D.E.

Legendre Polynomials:

If n is a positive even integer $a_0 u(x)$ reduces to a polynomial of degree n and if n is a positive odd integer $a_1 v(x)$ reduces to a polynomial of degree n. Otherwise these will give infinite series called Legendre functions of second kind.

$u(x)$ and $v(x)$ contain alternate powers of x and general form of the polynomial that represents either of them in descending powers of x can be represented in the form

$$y = f(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \dots + F(x) \quad \text{——} (5)$$

where $F(x) = \begin{cases} a_0 & \text{is } n \text{ is even} \\ a_1 x & \text{if } n \text{ is odd} \end{cases}$

we have by (3) $$a_{r+2} = \frac{-[n(n+1) - r(r+1)]}{(r+2)(r+1)} a_r \quad \text{——} (6)$$

We plan to express $a_{n-2}, a_{n-4}, \ldots$ present in (i) in terms of a_n. Replacing r by $(n-2)$ in (ii) we obtain,

$$a_n = \frac{-[n(n+1)-(n-2)(n-1)]}{n(n-1)} a_{n-2}$$

i.e $$a_n = \frac{-(4n-2)}{n(n-1)} a_{n-2}$$

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

by (2), replacing r by $n-4$ we get,

$$a_{n-2} = \frac{-[n(n+1)-(n-4)(n-3)]}{(n-2)(n-3)} a_{n-4}.$$

$$= \frac{-(8n-12)}{(n-2)(n-3)} a_{n-4}.$$

$$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$= \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} a_n$$

by using the value of a_{n-2} and so on.

using these values in (5), we have

$$y = f(x) = a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} - \cdots + G(x) \right]$$

where $$G(x) = \begin{cases} \dfrac{a_0}{a_n} & \text{if } n \text{ is even} \\ \dfrac{a_1 x}{a_n} & \text{if } n \text{ is odd.} \end{cases}$$

If the constant a_n is chosen such that $y = f(x)$ becomes 1 (4)
when $x=1$, the polynomials so obtained are called Legendre polynomials denoted by $P_n(x)$.

Let us choose $a_n = \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{n!}$ to meet the said requirement for Legendre polynomial.

We write, $$P_n(x) = \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \ldots \ldots + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \ldots\right] \quad \text{—(7)}$$

We obtain first few Legendre polynomials by putting $n = 0, 1, 2, 3, 4$,

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{1!}[x] = x$$

$$P_2(x) = \frac{1\cdot 3}{2!}\left[x^2 - \frac{2(2-1)}{2\cdot 3}x^0\right] = \frac{3}{2}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1\cdot 3\cdot 5}{3!}\left[x^3 - \frac{3(2)}{2\cdot 5}x\right] = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{4!}\left[x^4 - \frac{4(3)}{2(7)}x^2 + \frac{4(3)(2)(1)}{2\cdot 4\cdot 7\cdot 5}\right]$$

$$P_4(x) = \frac{35}{8}\left[x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right] = \frac{1}{8}[35x^4 - 30x^2 + 3] \text{ etc.}$$

when $x = 1$, $P_0, P_1, P_2 \ldots = 1$.

4. Rodrigue's formula:

We derive a formula for Legendre' polynomials $P_n(x)$ in the form $$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$ Dec-2012, 2008, 14 July-2008, 2013.

Proof: Let $u = (x^2-1)^n$.

Let us find nth derivative of u i.e. u_n is a soln of Legendre D.E. $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ——— (1)

Differentiating u w.r.t. x,

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1}\, 2x.$$

$$(x^2-1)u_1 = 2nx(x^2-1)^n$$

$$\therefore (x^2-1)u_1 = 2nxu.$$

Differentiating w.r.t. x again,

$$(x^2-1)u_2 + u_1 2x = 2n(xu_1+u)$$

$$\therefore D^n[(x^2-1)u_2] + D^n[2xu_1] = 2n\, D^n[xu_1+u].$$

By Leibnitz theorem, $\left(D^n(uv) = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} \cdots \right)$

$$\left[(x^2-1)u_{n+2} + n\cdot 2x u_{n+1} + \frac{n(n-1)}{\cancel{2}}\cdot \cancel{2}\cdot u_n \right]$$

$$+ 2\{ xu_{n+1} + n u_n \} = 2n\{ xu_{n+1} + n\cdot u_n + u_n \}.$$

ie $(x^2-1)u_{n+2} + \cancel{2nxu_{n+1}} + 2xu_{n+1} + (n^2-n)u_n + \cancel{2nu_n}$

$$= \cancel{2nxu_{n+1}} + 2n^2u_n + \cancel{2nu_n}$$

ie $(x^2-1)u_{n+2} + 2xu_{n+1} + (n^2+\cancel{2n}-n)u_n = \cancel{2n(n^2+2n)u_n}\ 2n^2u_n.$

ie $(x^2-1)u_{n+2} + 2xu_{n+1} + (n^2-n)u_n = 0.$

ie $(x^2-1)u_{n+2} + 2xu_{n+1} - (n^2+n)u_n = 0$

ie $(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0.$

ie $(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0.$ ——— (2)

Comparing ① & ② u_n is a soln of Legendre D.E.

where u_n is a polynomial of degree n and

u is a polynomial of degree $2n$.

Also $P_n(x)$ which satisfies Legendre D.E is also a polynomial of deg n and hence u_n must be same as $P_n(x)$ but for some constant factor k.

ie $P_n(x) = k\,u_n = k\left[(x^2-1)^n\right]_n$

$= k\left[(x-1)^n(x+1)^n\right]_n$.

Apply Leibnitz theorem for RHS,

$$P_n(x) = k\left[(x-1)^n\{x+1\}_n + n.\,n(x-1)^{n-1}(x+1)_{n-1}\right.$$

$$+\frac{n(n-1)}{2}n(n-1)(x-1)^{n-2}\{(x+1)^n\}_{n-2}$$

$$\left.\cdots + \{(x-1)^n\}_n(x+1)^n\right] \quad \text{——— (3)}$$

let $z = (x-1)^n$ then

$z_1 = n(x-1)^{n-1} \quad z_2 = n(n-1)(x-1)^{n-2}$ etc.

$z_n = n(n-1)(n-2)\cdots 2.1.\,(x-1)^{n-n}$

ie $z_n = n!\,(x-1)^0 = n!$

$\therefore \{(x-1)^n\}_n = n!$

We proceed to find k choosing suitable value for x,

putting $x=1$ in (3), we get,

$P_n(1) = k.\,n!\,2^n$

but $P_n(1) = 1$ by def of Legendre polynomials.

$\therefore 1 = k\,n!\,2^n$ or $k = \dfrac{1}{2^n n!}$

Since $P_n(x) = k u_n$,

$$P_n = \frac{1}{2^n n!}\{(x^2-1)^n\}_n = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Using Rodrigue's formula obtain expressions for $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$, $P_5(x)$. Hence express x^2, x^3, x^4, x^5 in terms of Legendre polynomials.

By Rodrigue's formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n. \quad \text{--- (1)}$$

put $n=0$ in (1), $P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2-1)^0$

$\therefore \boxed{P_0(x) = 1}$.

put $n=1$ in (1), $P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx} (x^2-1)$

$= \frac{1}{\cancel{2}} \cdot \cancel{2}x = x \quad \therefore \boxed{P_1(x) = x}$

put $n=2$ in (1), $P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2-1)^2$

$= \frac{1}{4\times 2} \frac{d}{dx} \left[2(x^2-1) 2x \right]$

$= \frac{1}{8} \frac{d}{dx} (4x^3 - 4x)$

$= \frac{1}{8} \; 4 \cdot 3x^2 - 4. = \frac{\cancel{4}(3x^2-1)}{\cancel{8}\,2}$

$\therefore \boxed{P_2(x) = \frac{3x^2-1}{2}}$.

put $n=3$ in (1), $P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3.$

$= \frac{1}{8\times 6} \frac{d^2}{dx^2} \left(3(x^2-1)^2 \cdot 2x \right)$

$= \frac{1}{48} \frac{d^2}{dx^2} \left(3(x^4 - 2x^2 + 1) 2x \right)$

$= \frac{1}{\cancel{48}\,8} \frac{d^2}{dx^2} \cancel{6}\left[x^5 - 2x^3 + x\right]$

$$= \frac{1}{8}\frac{d}{dx}\left[5x^4 - 6x^2 + 1\right]$$

$$= \frac{1}{8}\left(20x^3 - 12x\right)$$

$$= \frac{\cancel{4}(5x^3 - 3x)}{\cancel{8}\,2}$$

$$\therefore \boxed{P_3(x) = \frac{5x^3 - 3x}{2}}$$

put $n = 4$, $P_4(x) = \frac{1}{2^4\,4!}\frac{d^4}{dx^4}(x^2-1)^4$.

$$= \frac{1}{16\times 24}\frac{d^4}{dx^4}\left\{(x^2)^4 - {}^4c_1(x^2)^3 + {}^4c_2(x^2)^2 - {}^4c_1 x^2 + {}^4c_0\right\}$$

$$= \frac{1}{16\times 24}\frac{d^4}{dx^4}\left\{x^8 - 4x^6 + 6x^4 - 4x^2 + 1\right\}$$

$$= \frac{1}{16\times 24}\left\{\frac{d^4}{dx^4}(x^8) - 4\frac{d^4}{dx^4}(x^6) + 6\frac{d^4}{dx^4}(x^4) - 4\frac{d^4}{dx^4}(x^2) + \frac{d^4}{dx^4}(1)\right\}$$

$$= \frac{1}{16\times 24}\left\{\frac{8!}{4!}x^4 - 4\frac{6!}{2!}x^2 + 6\times 4! + 0 + 0\right\}$$

$$= \frac{1}{16\times 24}\left\{5\times 6\times 7\times 8\,x^4 - 4\times 3\times 4\times 5\times 6\,x^2 + 6\times 24\right\}$$

$$= \frac{24}{16\times 24}\left\{5\times 2\times 7\,x^4 - 3\times 4\times 5\cdot x^2 + 6\right\}$$

$$= \frac{1}{16}\left\{70x^4 - 60x^2 + 6\right\}$$

$$= \frac{2}{16}\left\{35x^4 - 30x^2 + 3\right\}$$

$$\boxed{P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)}$$

w. $n=5$ in (1),

$$P_5(x) = \frac{1}{2^5 \cdot 5!} \frac{d^5}{dx^5} (x^2-1)^5$$

$$= \frac{1}{32 \times 120} \frac{d^5}{dx^5} \left\{ (x^2)^5 - {}^5C_1 (x^2)^4 + {}^5C_2 (x^2)^3 - {}^5C_3 (x^2)^2 + {}^5C_4 (x^2) - {}^5C_5 \right\}$$

$$= \frac{1}{32 \times 120} \frac{d^5}{dx^5} \left(x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1 \right\}$$

$$= \frac{1}{32 \times 120} \left\{ \frac{d^5}{dx^5}(x^{10}) - 5 \cdot \frac{d^5}{dx^5}(x^8) + 10 \frac{d^5}{dx^5}(x^6) - 10 \cdot \frac{d^5}{dx^5}(x^4 + 5x^2 - 1) \right\}$$

$$= \frac{1}{32 \times 120} \left\{ \frac{10!}{(10-5)!} x^{10-5} - 5 \cdot \frac{8!}{(8-5)!} x^{8-5} + 10 \cdot \frac{6!}{(6-5)!} x^{6-5} - 0 \right\}$$

$$= \frac{1}{32 \times 120} \left\{ 6 \times 7 \times 8 \times 9 \times 10 \cdot x^5 - 5 \cdot 4 \times 5 \times 6 \times 7 \times 8\, x^3 + 10 \cdot 6!\, x \right\}$$

$$= \frac{1}{32} \left\{ 252x^5 - 280x^3 + 60x \right\}$$

$$= \frac{1}{8} \left\{ 63x^5 - 70x^3 + 15x \right\}.$$

$$\therefore \boxed{P_5(x) = \frac{63x^5 - 70x^3 + 15x}{8}}$$

Now we express x^2, x^3, x^4, x^5 in terms of Legendre's polynomials.

WKT $P_2(x) = \frac{3x^2-1}{2} \Rightarrow 2P_2(x) = 3x^2 - 1$

$$\Rightarrow 3x^2 = 2P_2(x) + 1$$

$$\Rightarrow x^2 = \frac{2P_2(x)+1}{3}.$$

bw. $1 = P_0(x)$.

$$\therefore \quad x^2 = \frac{2P_2(x) + P_0(x)}{3}.$$

$$\boxed{x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)}.$$

WKT, $P_3(x) = \frac{5x^3 - 3x}{2} \Rightarrow 2P_3(x) = 5x^3 - 3x$.

$$\Rightarrow 5x^3 = 2P_3(x) + 3x.$$

$$= 2P_3(x) + 3 \cdot P_1(x) \qquad \because P_1(x) = x.$$

$$\Rightarrow \boxed{x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)}.$$

WKT, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$\therefore \quad 8P_4(x) = 35x^4 - 30x^2 + 3.$$

$$\Rightarrow 35x^4 = 8P_4(x) + 30x^2 - 3$$

$$= 8P_4(x) + 30\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - 3P_0(x)$$

$$= 8P_4(x) + 20P_2(x) + 10P_0(x) - 3P_0(x)$$

$$= 8P_4(x) + 20P_2(x) + 7P_0(x)$$

$$x^4 = \frac{8}{35} P_4(x) + \frac{20}{35} P_2(x) + \frac{7}{35} P_0(x)$$

$$\boxed{x^4 = \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x)}$$

WKT, $P_5(x) = \frac{1}{8}\left[63x^5 - 70x^3 + 15x\right]$

$$\therefore \quad 8P_5(x) = 63x^5 - 70x^3 + 15x.$$

$$\cancel{8P_5(x)}\ 63x^5 = 8P_5(x) + 70x^3 - 15x.$$

$$63x^5 = 8P_5(x) + 70\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] - 15P_1(x)$$

$$= 8P_5(x) + 14\times 2P_3(x) + 14\times 3P_1(x) - 15P_1(x)$$

$$= 8P_5(x) + 28P_3(x) + 27P_1(x)$$

$$\therefore x^5 = \frac{1}{63}\left[8P_5(x) + 28P_3(x) + 27P_1(x)\right]$$

$$\boxed{x^5 = \frac{8}{63}P_5(x) + \frac{4}{9}P_3(x) + \frac{27}{63}P_1(x).}$$