MODULE-2 NUMERICAL METHODS-2. - SEM-4

SPECIAL FUNCTIONS :-

The solution of Laplace's Equation, $\nabla^2 u = 0$ in cylindrical system & spherical system leads to 2 important ordinary differential Equations namely; Bessel Differential Equation. &
Legendre Differential Equation.

The series solution of the Bessel's differential Equation is a "special function" known as "Bessel's function".

The special polynomial function that occurs in the process of solving in series of Legendre's differential Equation is known as "Legendre's polynomial".

Applications :- The Bessel's function has various applications in solving boundary value problems with axial symmetry and the Legendre polynomial has various application in solving boundary value problems with spherical symmetry.

* SERIES SOLUTION OF DIFFERENTIAL EQUATION :-

(ass.)

* Series solution of Bessel's differential Equation leading to Bessel functions :-

The differential Equation of the form;

$$x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \sim (1).$$

is Known as "Bessel's differential Equation."

where, n is a non-negative real Constant. (parameter).

Now, let us assume that the series solution of (1) of form;

$$y = \sum_{r=0}^{\infty} a_r x^{K+r} \quad \sim (2)$$

$\Rightarrow$ Diff Eqn (2) w.r.t x twice, we get.

$$\frac{dy}{dx} = \sum_{0}^{\infty} a_r (K+r) x^{K+r-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sum_{0}^{\infty} a_r (K+r)(K+r-1) x^{K+r-1-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sum_{0}^{\infty} a_r (K+r)(K+r-1) x^{K+r-2}$$

$\Rightarrow$ using $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in Eqn (1)...

$$(1) \Rightarrow x^2 \cdot \sum_{0}^{\infty} a_r (K+r)(K+r-1)^{K+r-2} + x \cdot \sum_{0}^{\infty} a_r (K+r)^{K+r-1}$$

$$+ x^2 \sum_{0}^{\infty} a_r x^{K+r} - n^2 \sum_{0}^{\infty} a_r x^{K+r} = 0.$$

Case: 1 Assume ; $K = n$ in Eqn (3).

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} \rightarrow // \cancel{n^2} + r^2 + 2nr - \cancel{n^2}$$

$$a_r = \frac{-a_{r-2}}{(2nr + r^2)} \sim (4) \quad , r \geq 2.$$

put ; $r = 2, 3, 4 \ldots$ in Eqn (4)

$r=2 \Rightarrow a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}$

$r=3 \Rightarrow a_3 = \frac{-a_1}{6n+9} = 0$, since, $a_1 = 0$

$r=4 \rightarrow a_4 = \frac{-a_2}{8n+16} = \frac{-1}{8(n+2)}\left[\frac{-a_0}{4(n+1)}\right] = \frac{a_0}{32(n+1)(n+2)} = \frac{a_0}{2^5(n+1)(n+2)}$.

$\Rightarrow$ Substitute the values : $a_0, a_1, a_2, a_3, a_4 \ldots$ in Expanded form of Eqn (2) ;

$$y_1 = \left[x^K [a_0] + x^{K+1}(a_1) + x^{K+2}(a_2) + x^{K+3}(a_3) + x^{K+4}(a_4) + \ldots\right]$$

$$y_1 = x^K \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \}.$$

$$y_1 = x^K \left\{ a_0 + 0 + \left[\frac{-a_0}{4(n+1)}\right] x^2 + \left[\frac{-a_1}{6n+9}\right] x^3 + \left[\frac{-1}{8n+16}\left[\frac{-a_0}{4(n+1)}\right]\right] \ldots \right\}.$$

put, $\boxed{n=K}$

$$y_1 = x^n a_0 \left\{ 1 - \frac{1}{2^2(n+1)} x^2 + \frac{1}{2^5(n+1)(n+2)} x^4 + \ldots \right\}.$$

choose, $a_0 = \frac{1}{2^n \Gamma[n+1]}$

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$$\Rightarrow \sum_0^\infty a_r (K+r)(K+r-1) x^{K+r-2+2} + x.\sum_0^\infty a_r (K+r) x^{K+r-1+1} + \sum_0^\infty a_r x^{K+r+2} - n^2 \sum_0^\infty a_r x^{K+r}$$

Collecting the 1st, 2nd & 4th terms together, we have ;

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\left[(K+r)(K+r-1) + (K+r) - n^2\right] + \sum_0^\infty a_r x^{K+r+2} = 0$$

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\left[(K+r)\left[K+r-\cancel{1}+\cancel{1}\right] - n^2\right] + \sum_0^\infty a_r x^{K+r+2} = 0$$

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\left[(K+r)^2 - n^2\right] + \sum_0^\infty a_r x^{K+r+2} = 0.$$

Now, we shall equate the coefficient of lowest degree term in x, ie, $x^K \to 0$, $r=0$

ie, $a_0(K^2-n^2)=0$.

putting $a_0 \neq 0$ we have ; $K^2 - n^2 = 0$ & hence ; $K = \pm n$

Also, we need to independently equate coefficient of $x^{K+1} \to 0$

ie, $x^{K+1} \to 0$, $r=1$

ie, $a_1\left[(K+1)^2 - n^2\right] = 0$.

$\Rightarrow$ $a_1 = 0$, since & $(K+1)^2 \neq n^2$ $\because$ we accepted already $K = \pm n$

$\Rightarrow$ Coefficient of x^{K+r}.

// $\sum_0^\infty a_r x^{K+r} x^2 = a_{r-2}$

$\Rightarrow x^{K+r}: a_r\left[(k+r)^2 - n^2\right] + a_{r-2} = 0.$

$$\Rightarrow \boxed{a_r = \frac{-a_{r-2}}{\left[(K+r)^2 - n^2\right]}} \sim (3), \quad r \geq 2$$

$\Rightarrow$ Equation (3) is known as Recurrence Relation.

$$\Rightarrow y_1 = \frac{x^n}{2^n \gamma(n+1)} \left\{ 1 - \left(\frac{x^2}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{2(n+1)(n+2)} + \cdots \right\}$$

$$y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\,\gamma(n+1)} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{(n+1)(n+2)\,\gamma(n+1)\,2} + \cdots \right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\gamma(n+1)\,0!} \left(\frac{x}{2}\right)^0 + \left(\frac{x}{2}\right)^2 \frac{(-1)^1}{(n+1)\gamma(n+1)\,1!} + \left(\frac{x}{2}\right)^4 \frac{(-1)^2}{(n+1)(n+2)\,\gamma(n+1)\,2!} \cdots \right]$$

$$y_1 = \cancel{\left(\frac{x}{2}\right)^n} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r} \sim (5)$$

$\Rightarrow$ Equa^n (5) is known as "Bessel's function" of 1st kind of order n, denoted by; $J_n(x)$

$$\therefore \boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r}}$$

$\therefore$ Complete soln is : $\boxed{y = A \cdot J_n(x) + B\, J_{-n}(x)}$

Prove that :-

* $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$

soln:- $J_{1/2}(x)$ // WKT; $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\Rightarrow$ from, the definition of Bessel's fn

$$\therefore J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r} \sim (1)$$

put ; $\boxed{n = \pm 1/2}$

$J_{-1/2}$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(1/2+r+1)\, r!} \left(\frac{x}{2}\right)^{1/2+2r}$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(3/2+r)\, r!} \left(\frac{x}{2}\right)^{2r}.$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\gamma(3/2)} + (-1)\left(\frac{x}{2}\right)^2 \frac{1}{\gamma(3/2+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{\gamma(3/2+2)\cdot 2} + \cdots\right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\gamma(3/2)} - \frac{x^2}{4}\frac{1}{\gamma(5/2)} + \frac{x^4}{16}\frac{1}{\gamma(7/2)\, 2} \cdots\right]$$

// $\gamma(1/2) = \sqrt{\pi}$

$\gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

$\gamma(5/2) = \frac{3}{4}\sqrt{\pi}$

$\gamma(7/2) = \frac{15}{8}\sqrt{\pi}$

$$= \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\frac{8}{15\sqrt{\pi}\,2}\right]$$

$$= \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \left[1 - \frac{2x^2}{12} + \frac{4x^4}{16\times 30}\right]$$

$$= \sqrt{\frac{x}{2}} \cdot \times \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{6} + \frac{4x^4}{120} - \cdots\right]$$

$$= \sqrt{\frac{2x}{\pi}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right]$$

$$\therefore \boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.}$$

2) Proove that ; $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x.$

Soln:- from, the defn of bessel's function ...

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r} \quad \text{— (1)}$$

put, $\boxed{n = -1/2}$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(1/2+r+1)\, r!} \left(\frac{x}{2}\right)^{-1/2+2r}.$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(3/2+r)\, r!} \left(\frac{x}{2}\right)^{2r}.$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16}\left[\frac{4}{3\sqrt{\pi}\,2}\right]\right].$$

$$= \sqrt{\frac{2}{x\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right]$$

$$\therefore \boxed{J_{-1/2}(x) = \sqrt{\frac{2}{x\pi}}\cdot \cos x.}$$

* Orthogonal property of Bessel's function :-

If α & β are 2 distinct roots of $J_n(x)=0$, then PT; $\int_0^1 x J_n(\alpha x).J_n(\beta x)dx = 0$. if $\alpha \neq \beta$.

Proof :- WKT; $y = J_n(\lambda x)$ is soln of Eqn :

$$\therefore \Rightarrow x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0.$$

if $u = J_n(\alpha x)$ & $v = J_n(\beta x)$, then Associated differential

Equations are :-

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \sim (1)$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \sim (2).$$

$\Rightarrow$ x'ly (1) by $\frac{v}{x}$ & (2) by $\frac{u}{x}$, we get ..

$$(1) \Rightarrow xvu'' + vu' + \alpha^2 uvx - n^2\frac{uv}{x} = 0.$$

$$(2) \Rightarrow xuv'' + uv' + \beta^2 uvx - n^2\frac{uv}{x} = 0.$$

$\Rightarrow$ On subtracting the obtained Equations, we get -

$$\Rightarrow \underbrace{x(u''v - uv'') + (vu' - uv')}\ + uvx(\alpha^2 - \beta^2) = 0.$$

$$\Rightarrow \frac{d}{dx}\left[x(vu' - uv')\right] = (\beta^2 - \alpha^2)\, uvx.$$

$\Rightarrow$ Integrating wrt x, b/w $0 \to 1$, we get ..

$$\Rightarrow \left[x(vu' - uv')\right]_{x=0}^{1} = (\beta^2 - \alpha^2)\int_{x=0}^{1} uvx\,dx$$

$$\left[x(vu' - uv')\right]_{x=1} - 0 = (\beta^2 - \alpha^2)\int_{x=0}^{1} uvx\,dx \quad \sim (3)$$

$\rightarrow$ we have;

$\Rightarrow u = J_n(\alpha x)$, $v = J_n(\beta x)$

$\Rightarrow$ diff w.r.t x

$\underline{u' = J_n'(\alpha x).\alpha}$ $\quad$ $\Rightarrow$ diff w.r.t x . .

$\underline{v' = J_n'(\beta x).\beta}$

Sub in (3)

$$\Rightarrow \left[x\{J_n(\beta x)\alpha . J_n'(\alpha x) - J_n(\alpha x).\beta.J_n'(\beta x)\right]_{x=1} = (\beta^2 - \alpha^2)\int_{x=0}^{1} J_n(\alpha x).J_n(\beta x).x\,dx$$

$$\Rightarrow \left[J_n(\beta)\alpha.J_n'(\alpha) - J_n(\alpha).\beta.J_n'(\beta)\right] = (\beta^2 - \alpha^2)\int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x)\,.dx$$

$$\Rightarrow \int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x)\,dx = \frac{1}{(\beta^2 - \alpha^2)}\left[\alpha.J_n(\beta).J_n'(\alpha) - \beta J_n(\alpha)J_n'(\beta)\right]$$

Since; α & β are distinct roots of $J_n(x) = 0$.

$$\Rightarrow \underline{J_n(\alpha) = 0} \quad , \quad \underline{J_n(\beta) = 0}$$

$$\therefore \boxed{\int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x).dx = 0.} \quad \beta^2 \neq \alpha^2$$

$\Rightarrow$. Hence, Orthogonal property is satisfied.

Orthogonal property of bessel's function :-

If α & β are the two distinct root of the Equation ;

$J_n(x)=0$, then ...

$$\int_0^1 x \cdot J_n(\alpha x) \cdot J_n(\beta x) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2} J_{n+1}(\alpha) & \text{if } \alpha = \beta \end{cases}$$

* Series solution of Legendre differential Equation :-

Consider a DE ;

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{— (1)}$$

Eqn (1) is Legendre's diff. Equation.

Let us assume that the soln of Eqn (1) is of form ;

$$y = \sum_{r=0}^{\infty} a_r x^r \quad \text{— (2)}$$

$\Rightarrow$ diff Eqn (2) wrt x ...

$$y' = \sum_{r=0}^{\infty} a_r r \cdot x^{r-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2}$$

$\therefore$ Eqn (1) becomes ...

$$= (1-x^2)\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - 2x\sum_{r=0}^{\infty} a_r r x^{r-1} + n(n+1)\sum_{0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^{r-\cancel{2}+\cancel{2}} - 2\sum_{r=0}^{\infty} a_r r x^{r-\cancel{1}+\cancel{1}}$$

$$+ n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1) x^r - 2\sum_{r=0}^{\infty} a_r r x^r$$
$$+ n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r x^r \left[r(r-1) + 2r - n(n+1) \right] = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r x^r \left[r^2 + r - n(n+1) \right] = 0.$$

$\Rightarrow$ Equate the coefficient of lowest power of x to 0.

$$\left[a_0 (0)(-1) x^{-2} + a_1 (1)(0) x^{-1} + \cdots \right]$$

Coeff of x^{-2} :- $a_0(0)(-1) = 0$, $\Rightarrow$ $\underline{\underline{a_0 \neq 0}}$

x^{-1} :- $a_1(1)(0) = 0$ $\Rightarrow$ $\underline{\underline{a_1 \neq 0}}$

$\Rightarrow$ Now, we equate coeff of $\underline{\underline{x^r \to 0}}$ & Replace $\underline{\underline{r \to r+2}}$ for $\underline{\underline{\text{1st term}}}$

we get,

$\underline{\underline{x^r}}$: $+a_{r+2}(r+2)(r+1) - a_r\left[r(r+1) - n(n+1) \right] = 0.$

$$\Rightarrow a_{r+2} = \frac{r(r+1) - n(n+1)}{(r+1)(r+2)} a_r \sim (3).$$

$\Rightarrow$ Eqn (3) is called as Recurrence relation.

$\Rightarrow$ Substitute : $r = 0, 1, 2 \ldots$ in Eqn (3), we get --

$\underline{\underline{r=0}}$, $a_2 = \left[\frac{-n(n+1)}{(1)(2)} \right] a_0$, $a_2 = \left[\frac{-n(n+1)}{2} \right] a_0$

$-[n^2+n-2]$

$\underline{\underline{r=1}}$, $a_3 = \left[\frac{2 - n(n+1)}{6} \right] a_1$, $a_3 = \left[-\frac{(n-1)(n+2)}{3!} \right] a_1$

$\underline{\underline{r=2}}$, $a_4 = \left[\frac{6 - n(n+1)}{12} \right] a_2$ $\cdot$ $\left\{ \left[\frac{-n(n+1)}{2} \right] a_0 \right\}$

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$$a_4 = -\frac{(n^2+n-6)}{12}\left\{-\frac{n(n+1)a_0}{2}\right\}$$

n^2+r-6

$+3\ -2$

$(n+3)(n-2)$

$$a_4 = \frac{n(n+1)(n+3)(n-2)\,a_0}{4!}$$

(no need)

$r=3$,

$$a_5 = \left[\frac{12-n(n+1)}{20}\right] \times \left[-\frac{(n-1)(n+2)}{3!}\right] a_1$$

$$= \left[-\left[\frac{(n^2+n-12)-(n-1)(n+2)}{120}\right]\right] a_1$$

$$a_5 = \left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}\right] a_1$$

Substitute these values in the expanded form of Eqn (2).

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$= a_0 + xa_1 - \frac{n(n+1)}{2!}a_0x^2 - \left[\frac{(n-1)(n+2)}{3!}\right]a_1x^3$$

$$+ \left[\frac{n(n+1)(n+3)(n-2)a_0}{4!}\right]x^4 + \left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}\right]a_1x^5 + \cdots$$

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)(n-2)}{4!}x^4 + \cdots\right]$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n+4)(n-3)(n-1)(n+2)}{5!}x^5 + \cdots\right] \quad -(4)$$

Let $u(x)$ & $v(x)$ resp represent 2 infinite series in Eqn (4)

$$\Rightarrow \boxed{y = a_0u(x) + a_1v(x)}$$

This is the series soln of "Legendre's differential Equation."

→ [Leading to Legendre's polynomial is Continued if asked].

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Legendre Polynomial :-

The solution of Legendre's dE is Given by ;

$$y = a_0 u(x) + a_1 v(x) \sim (*)$$

if 'n' is positive even integer ;

$a_0 u(x)$ reduces to polynomial of degree n.

if 'n' is ~~negative~~ posi tive odd integer ;

$a_1 v(x)$ reduces to polynomial of degree n.

ie, $F(x) = \begin{cases} a_0 , & \text{if n even} \\ a_1 & \text{if n odd} \end{cases}$

Consider Legendre's fn of Second Kind....

$$y = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots F(x) \sim (1)$$

where ; $F(x) = \begin{cases} a_0 & \text{if n is even} \\ a_1 & \text{if n is odd.} \end{cases}$

WKT ; the reccurrance of legendre's is Given by ;

$$a_{r+2} = -\left[\frac{n(n+1) - r(r+1)}{(r+1)(r+2)}\right] . a_r \sim (2)$$

Using Eqn (2) we have to find values of : $(a_{n-2}, a_{n-4} \cdots)$

Replace , $r = n-2$ in Eqn (2) ...

$$a_{n-2+2} = -\left[\frac{n(n+1) - (n-2)(n-2+1)}{(n-2+1)(n-2+2)}\right] a_{n-2}$$

$$a_n = -\left[\frac{n(n+1) - (n-2)(n-1)}{n(n-1)}\right] a_{n-2}$$

$$a_n = -\left[\frac{n^2 + n - n^2 + 3n - 2}{n(n-1)}\right] a_{n-2}.$$

$$a_n = \frac{-2(2n-1)}{n(n-1)}\, a_{n-2}$$

$$\boxed{a_{n-2} = \frac{-n(n-1)}{2(2n-1)} \cdot a_n .}$$

Replace, $r = n-4$ in Eqn 2, we get ...

$$a_{n-4} = -\left[\frac{n(n+1) - (n-4)(n-3)}{(n-3)(n-2)}\right] a_{n-4}$$

$$= -\left[\frac{n^2+n-n^2+3n+4n-12}{(n-3)(n-2)}\right] a_{n-4} = \frac{-4(2n-3)\, a_{n-4}}{(n-3)(n-2)}$$

$$\boxed{a_{n-4} = -\frac{(n-3)(n-2)}{4(2n-3)} \cdot a_{n-2}}$$

$$a_{n-4} = \frac{-(n-3)(n-2)}{4(2n-3)} \cdot \frac{n(n-1)}{2(2n-3)} \cdot a_n .$$

$\Rightarrow$ Now, Eqn 1 becomes ...

$$y = a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-3)(2n-1)} a_n \; x^{n-4}$$

where; $G(x) = \begin{cases} a_0/a_n & \text{if } n \to \text{even} \\ a_1/a_n & \text{if } n \to \text{odd.} \end{cases}$

$$+ a_n \left\{ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-3)(2n-1)} x^{n-4} + \cdots \right\} \quad (3)$$

$\Rightarrow$ Constant, a_n is so choosen such that; $y = f(x)$ becomes 1, when $x=1$, the polynomials so obtained are called Legendre polynomials and denoted by $P_n(x)$

$\Rightarrow$ putting ; $n = 0, 1, 2, 3, 4$ in Eqn (3)

we, get : $p_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$p_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

* Problem :-

Express the following in terms of Legendre polynomial :-

(a) $x)$

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* RODRIGUE'S FORMULA :-

The legendre polynomials $P_n(x)$ of the form ;

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$ is Known as "Rodrigue's formula".

proof :- Let $u = (x^2-1)^n$.

We shall 1st establish that the n^{th} derivative of u, that is u_n is the solution of Legendre's differential Equation.

$$\Rightarrow \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad — (1).$$

$\Rightarrow$ dwt u wrt x, we have...

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} . 2x$$

$$u_1 = 2nx . \frac{(x^2-1)^n}{(x^2-1)}$$

$$(x^2-1)u_1 = 2nx(x^2-1)^n$$

$$\Rightarrow \quad (x^2-1)u_1 = 2nxu$$

$\Rightarrow$ dwt again wrt x, we get ...

$$(x^2-1)u_2 + u_1(2x) = 2n[xu_1 + u(1)]$$

$\Rightarrow$ We shall now dwt, the result n times by applying Leibnitz theorem for n^{th} derivative of product given by ;

$$(uv)_n = uv_n + nu_1v_{n-1} + \frac{n(n-1)}{2!}u_2v_{n-2} + \cdots + u_nv.$$

$$\therefore \quad [(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$$

$$\left\{[(x^2-1)u_2]_n + n[2x]u_{n+1} + \frac{n(n-1)}{2!}(2)(1)u_n\right\} + 2\left[[x\,u_1]_n + n(2(1))u_{1+n-1}\right]$$

1st term

$$= 2n[xu_{n+1} + nu_n] + 2nu_n.$$

ie, $\left[(x^2-1)u_{n+2} + n.2x.u_{n+1} + \frac{n(n-1)}{2}.2.u_n\right] + 2\left[x u_{n+1} + n\,1.u_n\right]$
$= 2n\left[x u_{n+1} + n.1.u_n\right] + 2n u_n.$

$\Rightarrow (x^2-1)u_{n+2} + 2nx\,u_{n+1} + (n^2-n)u_n + 2x u_{n+1} + 2n u_n = 2nx\,u_{n+1} + 2n^2 u_n$
$+ 2n u_n.$

ie, $(x^2-1)u_{n+2} + 2x u_{n+1} - n^2 u_n - n u_n = 0.$

$(x^2-1)u_{n+2} + 2x u_{n+1} - n u_n (n+1) = 0.$

(or) $(1-x^2)u_{n+2} - 2x.u_{n+1} + n(n+1)u_n = 0.$

This can be put in the form;

$\Rightarrow (1-x^2)u_n'' - 2x u_n' + n(n+1)u_n = 0. \sim (2).$

Comparing (2) with (1) we conclude that, u_n is a solution of Legendre's diff. Eqn. *(It may be observed that u is a polynomial of degree $2n$ & hence u_n will be polynomial of deg n)*

Also $P_n(x)$ which satisfies Legendre's dE is also a polynomial of deg. n. Hence u_n is same as $P_n(x)$

$\Rightarrow P_n(x) = K u_n = K\left[(x^2-1)^n\right]_n.$

$P_n(x) = K\left[(x-1)^n (x-1)^n\right]_n.$

Applying Leibneitz thm for RHS;

$P_n(x) = K\left[(x-1)^n \{(x+1)^n\}_n + n.n(x-1)^{n-1}\{(x+1)^n\}_{n-1} + \frac{n(n-1)}{2!}\,n(n-1)(x-1)^{n-2}\{(x+1)^n\}_{n-2}\right.$
$\left. + \cdots\cdots \{(x-1)^n\}_n (x+1)^n\right] \sim (3)$

Since, $z = (x-1)^n$ then;

$Z_1 = n(x-1)^{n-1}$

$Z_2 = n(n-1)(x-1)^{n-2}$ .....

$Z_n = n(n-1)(n-2) \cdots 2.1\,(x-1)^{n-n}$

$Z_n = n!\,(x-1)^0 = n!$

$\therefore \{(x-1)^n\}_n = n!$

putting, $x=1$ in Eqn (3), all terms in RHS become 0, except last term, ie, $n!(1+1)^n = n!\,2^n$.

(3) ⇒ ie ⇒ $P_n(1) = K n!\,2^n$, $P_n(1) = 1$ // By defn of $P_n(x)$.

$1 = K n!\,2^n$

$$\boxed{K = \frac{1}{n!\,2^n}}$$

Since, $P_n(x) = K u_n$, we have; $P_n(x) = \frac{1}{n!\,2^n}\{(x^2-1)^n\}_n$

Therefore; $\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$ is "Rodrigue's Formula".

Note :-

1) $P_0(x) = 1$
2) $P_1(x) = x$
3) $P_2(x) = \frac{1}{2}(3x^2-1)$
4) $P_3(x) = \frac{1}{2}(5x^3-3x)$
5) $P_4(x) = \frac{1}{8}[35x^4-30x^2+3]$

* $1 = P_0(x)$
* $x = P_1(x)$
* $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$
* $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$.
* $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$

(1) Express :- $x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre's polynomial :-

Soln :- $x^4 + 3x^3 - x^2 + 5x - 2$.

$$= \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x) + \frac{6}{5} P_3(x) + \frac{9}{5} P_1(x) - \frac{2}{3} P_2(x)$$

$$- \frac{1}{3} P_0(x) + 5P_1(x) - 2P_0(x).$$

$$= \frac{8}{35} P_4(x) + \left(\frac{12-14}{21}\right) P_2(x) + \frac{6}{5} P_3(x) + \left(\frac{9+25}{5}\right) P_1(x)$$

$$+ \left(\frac{3-5-30}{15}\right) P_0(x).$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \frac{32}{15} P_0(x)$$

* Using Rodrigue's formula, Compute : $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$:-

Soln :- WKT; Rodrigue's formula is ;

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

put ; $n = 0, 1, 2, 3, 4$.

put ; $n=0$, $P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2-1)^0$

$$\boxed{P_0(x) = 1}$$

$n=1$, $P_1(x) = \frac{1}{(2)^1 (1)!} \frac{d}{dx} (x^2-1)^1$

$$= \frac{1}{2} \frac{d}{dx} (x^2-1)$$

$$= \frac{1}{\cancel{2}} (\cancel{2}x)$$

$$\boxed{P_1(x) = x}$$

$\underline{n=2}$ $$P_2(x) = \frac{1}{2^2 . 2!} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{8} \cdot \frac{d}{dx} 2(x^2-1) 2x.$$

$$= \frac{1}{2} \cdot [3x^2-1]$$

$$\therefore \boxed{P_2(x) = \frac{1}{2}(3x^2-1)} \quad \text{(or)} \quad \boxed{x^2 = \frac{2 . P_2(x)+1}{3}}$$

$\underline{n=3}$ $$P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3$$

[Binomial Expression :- $(x-y)^n = x^n - nC_1 x^{n-1} . y . + nC_2 x^{n-2} . y^2 + \dots + x^0 y^n$]

WKT; $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}$, $m \geq n$.

$3(x^2-1)^2(2x)$

$6(x^2-1)^2 x$

$= 6 . [x . 2(x^2-1) 2x - (x^2-1)^2 (1)]$

$= 6[4x^2(2x) + (x^2-1)(8x) - 2(x^2-1)(2x)]$

$= 24 \times 2 x^3 + 8x^3 - 8x - 4x^3 + 4x$

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To find ; $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

$\underline{n=3}$; $P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3$ // $(a-b)^3 =$ Binomial Expression.

$$= \frac{1}{48} \frac{d^3}{dx^3} \left[(x^2)^3 - 3C_1 (x^2)^2 + 3C_2 (x^2)^1 1^2 - 3C_3 (x^2)^0 \right]$$

$$= \frac{1}{48} \frac{d^3}{dx^3} [x^6 - 3x^4 + 3x^2 - 1]$$

$$= \frac{1}{48} \left[\frac{6!}{3!} x^3 - 3 . \frac{4!}{1!} x^1 + 0 \right]$$

$$= \frac{1}{48} [120x^3 - 72x]$$

$$\boxed{P_3(x) = 5/_2 x^3 - 3/_2 x.} \quad \text{(or)} \quad \boxed{x^3 = \frac{2P_3(x) + 3x}{5}}$$

$\underline{n=4}$ $P_4(x) = \frac{1}{2^4 4!} \frac{d^4}{dx^4} (x^2-1)^4$

$= \frac{1}{384} \frac{d^4}{dx^4} [(x^2-1)^4]$

$= \frac{1}{384} \frac{d^4}{dx^4} [(x^3)^4 - 4C_1 (x^3)^3 + 4C_2 (x^3)^2 - 4C_3 (x^3)^1 + 4C_4 (x^3)^0]$.

$= \frac{1}{384} \frac{d^4}{dx^4} [x^{12} - 4(x^9) + 6x^6 - 4x^3 + 1]$

$= \frac{1}{384} \frac{d^3}{dx^3} [12x^{11} - 9\times4x^8 + 6\times6x^5 - 4\times3x^2 + 0]$

$= \frac{1}{384} \frac{d^2}{dx^2} [12\times11\times x^{10} - 9\times4\times8\times x^7 + 36\times5x^4 - 12\times2x]$

$= \frac{1}{384} \frac{d}{dx} [12\times11\times10x^9 - 36\times8\times7x^6 + 180\times4x^3 - 24]$

$= \frac{1}{384} [120\times11\times9x^8 - 36\times8\times7\times6x^5 + 180\times4\times3x^2 - 0]$

$nC_r = \frac{n!}{r!(n-r)!}$

$4C_2 = \frac{4!}{2!\,2!}$

$\boxed{4C_2 = 6}$

$\Rightarrow \cancel{x^4} /= \cancel{8P}$.

$$\boxed{P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]}$$

(or) $$\boxed{x^4 = \frac{8P_4(x) + 30x^2 - 3}{35}}$$

Formulaes :-

1) $K = K.P_0(x)$
2) $x = P_1(x)$
3) $x^2 = \frac{2P_2(x)+1}{3}$
4) $x^3 = \frac{2P_3(x)+3x}{5x}$
5) $x^4 = \frac{8P_4(x)+30x^2-3}{35}$

Problem 8
1) Express

Problems & Solutions :-

1) Express : x^3+2x^2-x-3 in terms of Legendre's polynomials :-

Soln:- Let $f(x) = x^3+2x^2-x-3$.

We have ;
$$1 = P_0(x) \qquad x^3 = 2/5\, P_3(x) + 3/5\, x.$$
$$x = P_1(x) \qquad x^4 = 8/35\, P_4(x) + 30/35\, x^2 - \frac{3}{35}$$
$$x^2 = 2/3\, P_2(x) + 1/3$$

$$f(x) = \left[\frac{2}{5} P_3(x) + 3/5\, x\right] + 2\left[2/3\, P_2(x) + 1/3\right] - \left[P_1(x)\right] - 3 \cdot P_0(x)$$

$$= 2/5\, P_3(x) + 3/5\, \underline{P_1(x)} + 4/3\, P_2(x) + 2/3\, P_0(x) - \underline{P_1(x)} - 3P_0(x) \Rightarrow \text{Simplify.}$$

$$\therefore f(x) = 2/5\, P_3(x) + 4/3\, P_2(x) - P_1(x) - 2/5\, P_1(x) - 7/3\, P_0(x).$$

✓ 2) Imp. Express : $f(x) = x^4+3x^3-x^2+5x-2$ in terms of Legendre's polynomials.

Let ; $f(x) = x^4+3x^3-x^2+5x-2$ $\qquad x^4 = 8/35\, P_4(x) + 30/35\, x^2 - 3/35.$

We have ; $P_0(x) = 1$, $x^2 = 2/3\, P_2(x) + 1/3$
$x = P_1(x)$ $\qquad x^3 = 2/5\, P_3(x) + 3/5\, x$

$$\therefore f(x) = \left[8/35\, P_4(x) + 30/35\, x^2 - 3/35\right] + 3\left[2/5\, P_3(x) + 3/5\, x\right] - \left[2/3\, P_2(x) + 1/3\right]$$
$$+ 5P_1(x) - 2 \cdot P_0(x).$$

$$= 8/35\, P_4(x) + 30/35\left[2/3\, P_2(x) + 1/3\, P_0(x)\right] - 3/35\, P_0(x) + 6/5\, P_3(x) + 9/5\, P_1(x)$$
$$- 2/3\, P_2(x) - 1/3\, P_0(x) + 5P_1(x) - 2P_0(x).$$

$$f(x) = 8/35\, P_4(x) + \frac{6}{5} P_3(x) - 16/21\, P_2(x) + \frac{34}{105} P_1(x)$$

$$- \frac{32}{105} P_0(x)$$

$\frac{20}{35} - \frac{2}{3} = \frac{20-2}{30}$ $\qquad \frac{60-35\times 2}{105}$ $\qquad \frac{60-70}{105} = -\frac{10}{105}$ $\qquad = \frac{-2}{21} - \frac{2}{3} = \frac{-2-14}{21}$ $\qquad \frac{10}{35} - \frac{3}{35} - \frac{1}{3}$

3) If $x^3+2x^2-x+1 = a p_0(x) + b p_1(x) + c p_2(x) + b p_3(x)$

find values of a, b, c, d

$f(x) = x^3+2x^2-x+1$

we have; $1 = P_0(x)$ $\quad x^2 = 2/3 P_2(x) + 1/3$

$x = P_1(x)$ $\quad x^3 = 2/5 P_3(x) + 3/5 x$.

$f(x) = 2/3 P_3(x) + 3/5 P_1(x) + 2[2/3 P_2(x) + 1/3 P_0(x)] - p_1(x) + p_0(x)$

$= 2/3 P_3(x) + 3/5 p_1(x) + 4/3 p_2(x) + 2/3 p_0(x) - p_1(x) + p_0(x)$

$f(x) = 2/3 P_3(x) + 4/3 P_2(x) - 2/5 p_1(x) + 5/3 p_0(x)$

$\downarrow d \quad \downarrow c \quad \downarrow b \quad \downarrow a$

$\therefore \boxed{a = 5/3} \quad \boxed{b = -2/5} \quad \boxed{c = 4/3} \quad \boxed{d = 2/3}$

4) If $2x^3-x^2-3x+2 = a p_0(x) + b p_1(x) + c p_2(x) + d p_3(x)$ find: a, b, c, d ..

Soln:- $f(x) = 2x^3-x^2-3x+2$, wkt; $1 = p_0(x)$, $x = p_1(x)$, $x^2 = 2/3 P_2(x) + 1/3$

$x^3 = 2/5 P_3(x) + 3/5 x$

$\therefore f(x) = 2[2/5 p_3(x) + 3/5 x] - [2/3 p_2(x) + 1/3] - 3[p_1(x)] + 2p_0(x)$.

$= 4/5 p_3(x) + 6/5 p_1(x) - 2/3 p_2(x) - 1/3 p_0(x) - 3p_1(x) + 2p_0(x)$

$\therefore f(x) = 4/3 p_3(x) - 2/3 p_2(x) - 9/5 p_1(x) + 5/3 p_0(x)$

$\downarrow d \quad \downarrow c \quad \downarrow b \quad \downarrow a$

$\therefore \boxed{a = 5/3} \quad \boxed{b = -9/5} \quad \boxed{c = -2/3} \quad \boxed{d = 4/3}$

MODULE-2 NUMERICAL METHODS-2 - SEM-4

SPECIAL FUNCTIONS :-

The solution of Laplace's Equation, $\nabla^2 u = 0$ in cylindrical system & spherical system leads to 2 important ordinary differential Equations namely; Bessel Differential Equation. &
Legendre Differential Equation.

The series solution of the Bessel's differential Equation is a "special function" known as "Bessel's function".

The special polynomial function that occurs in the process of solving series of Legendre's differential Equation is known as "Legendre's polynomial."

Applications :- The Bessel's function has various applications in solving boundary value problems with axial symmetry and the Legendre polynomial has various application in solving boundary value problems with spherical symmetry.

* SERIES SOLUTION OF DIFFERENTIAL EQUATION :-

* Series solution of Bessel's differential Equation leading to Bessel functions :-

The differential Equation of the form;

$$x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + (x^2 - n^2) y = 0 \quad \sim (1)$$

is Known as "Bessel's differential Equation."

where, n is a non-negative real Constant. (parameter).

Now, let us assume that the series solution of (1) of form;

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \sim (2)$$

$\Rightarrow$ Diff Eqn (2) w.r.t x twice, we get.

$$\frac{dy}{dx} = \sum_{0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-1-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

$\Rightarrow$ using $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in Eqn (1)...

$$(1) \Rightarrow x^2 \cdot \sum_{0}^{\infty} a_r (k+r)(k+r-1)^{k+r-2} + x \cdot \sum_{0}^{\infty} a_r (k+r)^{k+r-1}$$
$$+ x^2 \sum_{0}^{\infty} a_r x^{k+r} - n^2 \sum_{0}^{\infty} a_r x^{k+r} = 0.$$

Case: 1 Assume ; $\underline{K = n}$ in Eqn (3).

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} \rightarrow // \cancel{n^2} + r^2 + 2nr - \cancel{n^2}$$

$$a_r = \frac{-a_{r-2}}{(2nr + r^2)} \quad \sim (4) \quad , r \geq 2.$$

put ; $r = 2, 3, 4 \ldots$ in Eqn (4)

$\underline{r=2} \Rightarrow a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}$

$\underline{r=3} \Rightarrow a_3 = \frac{-a_1}{6n+9} = 0$, since, $\underline{a_1 = 0}$

$\underline{r=4} \Rightarrow a_4 = \frac{-a_2}{8n+16} = \frac{-1}{8(n+2)}\left[\frac{-a_0}{4(n+1)}\right] = \frac{a_0}{32(n+1)(n+2)} = \frac{a_0}{2^5(n+1)(n+2)}.$

$\Rightarrow$ Substitute the values : $a_0, a_1, a_2, a_3, a_4 \ldots$ in Expanded form of Eqn (2) ;

$$y_1 = \left[x^K[a_0] + x^{K+1}(a_1) + x^{K+2}(a_2) + x^{K+3}(a_3) + x^{K+4}(a_4) + \cdots\right]$$

$$y_1 = x^K\left\{a_0 + \underline{a_1} x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right\}.$$

$$y_1 = x^K\left\{\underline{a_0} + 0 + \left[\frac{-a_0}{4(n+1)}\right]x^2 + \left[\frac{-a_1}{6n+9}\right]x^3 + \left[\frac{-1}{8n+16}\left[\frac{-a_0}{4(n+1)}\right]\right] \cdots\right\}.$$

put, $\boxed{n = K}$

$$y_1 = x^n a_0\left\{1 - \frac{1}{2^2(n+1)}x^2 + \frac{1}{2^5(n+1)(n+2)}x^4 + \cdots\right\}.$$

choose, $a_0 = \frac{1}{2^n \Gamma[n+1]}$

$$\Rightarrow \sum_0^\infty a_r (K+r)(K+r-1) x^{K+r-2+2} + x \cdot \sum_0^\infty a_r (K+r) x^{K+r-1+1} + \sum_0^\infty a_r x^{K+r+2} - n^2 \sum_0^\infty a_r x^{K+r} = 0$$

Collecting the 1st, 2nd & 4th terms together, we have;

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\left[(\underline{K+r})(K+r-1) + (\underline{K+r}) - n^2\right] + \sum_0^\infty a_r x^{K+r+2} = 0$$

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\left[(K+r)[K+r-\cancel{1}+\cancel{1}] - n^2\right] + \sum_0^\infty a_r x^{K+r+2} = 0$$

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\underbrace{\left[(K+r)^2 - n^2\right]}_{\downarrow} + \sum_0^\infty a_r x^{K+r+2} = 0.$$

Now, we shall equate the coefficient of lowest degree term in x, ie, $\underline{x^K \to 0}$, $\underline{\underline{r=0}}$

ie, $a_0(K^2-n^2) = 0$.

Putting $a_0 \neq 0$ ~~we~~ we have; $K^2 - n^2 = 0$ & hence; $\underline{\underline{K = \pm n}}$.

Also, we need to independently equate coefficient of $\underline{\underline{x^{K+1} \to 0}}$

ie, $\underline{\underline{x^{K+1} \to 0}}$, $\underline{\underline{r=1}}$

ie, $a_1\left[(K+1)^2 - n^2\right] = 0$.

$\Rightarrow \underline{\underline{a_1 = 0}}$, ~~since~~ & $(K+1)^2 \neq n^2$ $\because$ we accepted already $\underline{\underline{K = \pm n}}$

$\Rightarrow$ Coefficient of x^{K+r}. // $\sum_0^\infty a_r \underline{\underline{x^{K+r}}} x^2 = a_{r-2}$

$\Rightarrow x^{K+r}: a_r\left[(K+r)^2 - n^2\right] + \underline{\underline{a_{r-2}}} = 0.$

$$\Rightarrow \boxed{a_r = \frac{-a_{r-2}}{\left[(K+r)^2 - n^2\right]}} \sim (3), \quad \underline{\underline{r \geq 2}}$$

$\Rightarrow$ Equation (3) is known as Recurrence Relation.

(3)

$$\Rightarrow y_1 = \frac{x^n}{2^n \gamma(n+1)} \left\{ 1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{2(n+1)(n+2)} \right\}$$

$$y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{n+1\,\gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\,\gamma(n+1)2} \right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\gamma(n+1)\,0!} \left(\frac{x}{2}\right)^0 + \left(\frac{x}{2}\right)^2 \frac{(-1)^1}{(n+1)\gamma(n+1)\,1!} + \left(\frac{x}{2}\right)^4 \frac{(-1)^2}{(n+1)(n+2)\,\gamma(n+1)\,2!} \right]$$

$$y_1 = \cancel{\left(\frac{x}{2}\right)^n} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\,r!} \left(\frac{x}{2}\right)^{n+2r} \quad \text{—(5)}$$

$\Rightarrow$ Equ^n (5) is Known as "Bessel's function" of 1st kind of order n, denoted by; $J_n(x)$

$$\therefore \boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\,r!} \left(\frac{x}{2}\right)^{n+2r}}$$

$\therefore$ Complete soln is : $\boxed{y = A.J_n(x) + B\,J_{-n}(x)}$

Proove that :-

* $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\,\sin x$

soln:- $J_{1/2}(x)$ // wkt; $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \dots$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \dots$

$\Rightarrow$ from, the definition of Bessel's fn

$$\therefore J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\,r!} \left(x/2\right)^{n+2r} \quad \text{—(1)}$$

put ; $\boxed{n = \pm 1/2}$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(1/2+r+1)\, r!} \left(x/2\right)^{1/2+2r}$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(3/2+r)\, r!} \left(x/2\right)^{2r}$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\gamma(3/2)} + (-1)\left(x/2\right)^2 \frac{1}{\gamma(3/2+1)} + \left(x/2\right)^4 \frac{1}{\gamma(3/2+2).2} + \cdots\right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\gamma(3/2)} - \frac{x^2}{4}\frac{1}{\gamma(5/2)} + \frac{x^4}{16}\frac{1}{\gamma(7/2)\, 2} \cdots\right]$$

$\gamma(1/2) = \sqrt{\pi}$

$\gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

$\gamma(5/2) = 3/4\sqrt{\pi}$

$\gamma(7/2) = \frac{15}{8}\sqrt{\pi}$

$$= \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\frac{8}{15\sqrt{\pi}\, 2}\right]$$

$$= \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \left[1 - \frac{2x^2}{12} + \frac{4x^4}{16\times 30}\right]$$

$$= \sqrt{\frac{x}{2}} \times \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{6} + \frac{4x^4}{120} - \cdots\right]$$

$$= \sqrt{\frac{2x}{\pi}} \cdot \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right]$$

$$\therefore \boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sin x.}$$

2) Prove that ; $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x.$

Soln:- From, the defn of bessel's function ....

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\, r!}\left(\frac{x}{2}\right)^{n+2r} \quad \text{—(1)}$$

put, $\boxed{n = -1/2}$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(-1/2+r+1)\, r!}\left(\frac{x}{2}\right)^{-1/2+2r}.$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(3/2+r)\, r!} \left(\frac{x}{2}\right)^{2r}.$$

$$= \left(\frac{x}{2}\right)^{-1/2} \left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16}\left[\frac{4}{3\sqrt{\pi}\,2}\right]\right].$$

$$= \sqrt{\frac{2}{x\pi}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right]$$

$$\therefore \boxed{J_{-1/2}(x) = \sqrt{\frac{2}{x\pi}} \cdot \cos x.}$$

* Orthogonal property of Bessel's function :-

If α & β are 2 distinct roots of $J_n(x) = 0$, then PT; $\int_0^1 x J_n(\alpha x) . J_n(\beta x) dx = 0$. if $\alpha \neq \beta$.

Proof :- WKT ; $y = J_n(\lambda x)$ is soln of Eqn :

$$\Rightarrow x^2 y'' + xy' + (\lambda^2 x^2 - n^2) y = 0.$$

if $u = J_n(\alpha x)$ & $v = J_n(\beta x)$, then Associated differential

Equations are :- $x^2 u'' + xu' + (\alpha^2 x^2 - n^2) u = 0 \sim (1)$

$x^2 v'' + xv' + (\beta^2 x^2 - n^2) v = 0 \sim (2)$.

$\Rightarrow$ x'ly (1) by $\frac{v}{x}$ & (2) by $\frac{u}{x}$, we get ..

(1) $\Rightarrow$ $xvu'' + vu' + \alpha^2 uvx - n^2 \frac{uv}{x} = 0$.

(2) $\Rightarrow$ $xuv'' + uv' + \beta^2 uvx - n^2 \frac{uv}{x} = 0$.

$\Rightarrow$ On subtracting the obtained Equations, we get -

$$\Rightarrow x(u''v - uv'') + (vu' - uv') + uvx(\alpha^2 - \beta^2) = 0.$$

$$\Rightarrow \frac{d}{dx}\left[x(vu' - uv')\right] = (\beta^2 - \alpha^2) uvx.$$

$\Rightarrow$ Integrating wrt x, b/w $0 \rightarrow 1$, we get ..

$$\Rightarrow \left[x(vu'-uv')\right]_{x=0}^{1} = (\beta^2-\alpha^2)\int_{x=0}^{1} uvx.dx.$$

$\rightarrow$ we have;
$$\left[x(vu'-uv')\right]_{x=1} - 0 = (\beta^2-\alpha^2)\int_{x=0}^{1} uvx.dx. \quad \sim (3)$$

$\Rightarrow$ We $u = J_n(\alpha x)$, $v = J_n(\beta x)$

$\Rightarrow$ diff x $u' = J_n'(\alpha x).\alpha$. $\Rightarrow$ diff x.. $v' = J_n'(\beta x).\beta$.

Sub in (3)

$$\Rightarrow \left[x\{J_n(\beta x)\alpha.J_n'(\alpha x) - J_n(\alpha x).\beta.J_n'(\beta x)\right]_{x=1} = (\beta^2-\alpha^2)\int_{x=0}^{1} J_n(\alpha x).J_n(\beta x).x\,dx.$$

$$\Rightarrow \left[J_n(\beta)\alpha.J_n'(\alpha) - J_n(\alpha).\beta.J_n'(\beta)\right] = (\beta^2-\alpha^2)\int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x).dx$$

$$\Rightarrow \int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x)\,dx = \frac{1}{(\beta^2-\alpha^2)}\left[\alpha.J_n(\beta).J_n'(\alpha) - \beta J_n(\alpha)J_n'(\beta)\right]$$

Since; α & β are distinct roots of $J_n(x)=0$.

$$\Rightarrow J_n(\alpha)=0 \quad , \quad J_n(\beta)=0$$

$$\therefore \boxed{\int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x).dx = 0.} \quad \beta^2 \neq \alpha^2$$

$\Rightarrow$. Hence, Orthogonal property is satisfied.

Orthogonal property of bessel's function :-

If α & β are the two distinct root of the Equation ;

$J_n(x)=0$, then ...

$$\int_0^1 x \cdot J_n(\alpha x) \cdot J_n(\beta x) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2} J_{n+1}^2(\alpha) & \text{if } \alpha = \beta \end{cases}$$

* Series solution of Legendre differential Equation :-

Consider a DE ;

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—(1)}$$

Eqn (1) is Legendre's diff. Equation.

Let us assume that the soln of Eqn (1) is of form ;

$$y = \sum_{r=0}^{\infty} a_r x^r \quad \text{—(2)}$$

$\Rightarrow$ diff Eqn (2) wrt x ...

$$y' = \sum_{r=0}^{\infty} a_r r \cdot x^{r-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r r \cdot (r-1) x^{r-2}$$

$\therefore$ Eqn (1) becomes ...

$$= (1-x^2)\sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - 2x \sum_{r=0}^{\infty} a_r r x^{r-1} + n(n+1)\sum_{0}^{\infty} a_r x^r = 0$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2+2} - 2\sum_{r=0}^{\infty} a_r r x^{r-1+1}$$

$$+ n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1) x^r - 2\sum_{r=0}^{\infty} a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r x^r \left[r(r-1) + 2r - n(n+1) \right] = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r x^r \left[r^2 + r - n(n+1) \right] = 0.$$

$\Rightarrow$ Equate the coefficient of lowest power of x to 0.

$$\left[a_0 (0)(-1) x^{-2} + a_1 (1)(0) x^{-1} + \dots \right]$$

Coeff of x^{-2} :- $a_0(0)(-1) = 0$, $\Rightarrow \underline{\underline{a_0 \neq 0}}$

x^{-1} :- $a_1(1)(0) = 0 \quad \Rightarrow \underline{\underline{a_1 \neq 0}}$

$\Rightarrow$ Now, we equate coeff of $\underline{\underline{x^r \to 0}}$ & Replace $\underline{\underline{r \to r+2}}$ for $\underline{\underline{\text{1st term}}}$ we get,

$\underline{\underline{x^r}}$: $+a_{r+2}(r+2)(r+1) - a_r\left[r(r+1) - n(n+1) \right] = 0.$

$$\Rightarrow a_{r+2} = \frac{r(r+1) - n(n+1)}{(r+1)(r+2)} a_r \sim (3).$$

$\Rightarrow$ $\underline{\underline{\text{Eqn}}}$ (3) is called as Recurrence relation.

$\Rightarrow$ Substitute : $r = 0, 1, 2 \dots$ in Eqn (3), we get --

$\underline{\underline{r=0}}$, $\quad a_2 = \left[\frac{-n(n+1)}{(1)(2)} \right] a_0$, $\quad a_2 = \left[\frac{-n(n+1)}{2} \right] a_0$

$-[n^2+n-2]$

$\underline{\underline{r=1}}$, $\quad a_3 = \left[\frac{2 - n(n+1)}{6} \right] a_1$, $\quad a_3 = \left[-\frac{(n-1)(n+2)}{3!} \right] a_1$

$\underline{\underline{r=2}}$, $\quad a_4 = \left[\frac{6 - n(n+1)}{12} \right] a_2 = \cdot \left\{ \left[\frac{-n(n+1)}{2} \right] a_0 \right\}$

$$a_4 = -\frac{(n^2+n-6)}{12}\left\{-\frac{n(n+1)}{2}a_0\right\}$$

$$n^2+n-6 \;/\; n^2-2 \;/\; (n+3)(n-2)$$

$$a_4 = \frac{n(n+1)(n+3)(n-2)}{4!}a_0$$

$r=3$ (no need)

$$a_5 = \left[\frac{12-n(n+1)}{20}\right] \times \left[\frac{-(n-1)(n+2)}{3!}\right] a_1$$

$$= -\left[\frac{(n^2+n-12)-(n-1)(n+2)}{120}\right] a_1$$

$$a_5 = \left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}a_1\right]$$

Substitute these values in the expanded form of eqn (2)

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$= a_0 + xa_1 - \frac{n(n+1)}{2!}a_0x^2 - \left[\frac{(n-1)(n+2)}{3!}\right]a_1x^3$$

$$+ \left[\frac{n(n+1)(n+3)(n-2)a_0}{4!}\right]x^4 + \left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}\right]a_1x^5$$

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)(n-2)}{4!}x^4 + \dots\right]$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n+4)(n-3)(n-1)(n+2)}{5!}x^5 - \dots\right] \quad \text{--- (5)}$$

Let $u(x)$ & $v(x)$ resp represent 2 infinite series in Eqn (5)

$$\Rightarrow \boxed{y = a_0u(x) + a_1v(x)}$$

This is the series soln of "Legendre's differential Equation"

→ [Leading to Legendre polynomials & Continued if asked]

Legendre Polynomial :-

The solution of Legendre's dE is Given by;

$$y = a_0 u(x) + a_1 v(x) \sim (*)$$

if 'n' is positive even integer ;
$a_0 u(x)$ reduces to polynomial of degree n.
if 'n' is ~~negative~~ posi tive odd integer ;
$a_1 v(x)$ reduces to polynomial of degree n.

ie, $F(x) = \begin{cases} a_0, & \text{if } n \text{ even} \\ a_1 & \text{if } n \text{ odd} \end{cases}$

Consider Legendre's fn of second kind....

$$y = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots F(x) \sim (1)$$

where; $F(x) = \begin{cases} a_0 & \text{if } n \text{ is even} \\ a_1 & \text{if } n \text{ is odd.} \end{cases}$

WKT; the reccurrance of legendre's is Given by;

$$a_{r+2} = \frac{-[n(n+1) - r(r+1)]}{(r+1)(r+2)} \cdot a_r \sim (2)$$

Using Eqn (2) we have to find values of : $(a_{n-2}, a_{n-4} \cdots)$

Replace, $r = n-2$ in Eqn (2) ...

$$a_{n-2+2} = -\left[\frac{n(n+1) - (n-2)(n-2+1)}{(n-2+1)(n-2+2)}\right] a_{n-2}$$

$$a_n = -\left[\frac{n(n+1) - (n-2)(n-1)}{n(n-1)}\right] a_{n-2}$$

$$a_n = -\left[\frac{n^2 + n - n^2 + 3n - 2}{n(n-1)}\right] a_{n-2}.$$

$$a_n = \frac{-2(2n-1)}{n(n-1)} a_{n-2}$$

$$\boxed{a_{n-2} = \frac{-n(n-1)}{2(2n-1)} \cdot a_n.}$$

Replace, $r = n-4$ in Eqn (2), we get...

$$a_{n-2} = -\left[\frac{n(n+1) - (n-4)(n-3)}{(n-3)(n-2)}\right] a_{n-4}$$

$$= -\left[\frac{n^2+n-n^2+3n+4n-12}{(n-3)(n-2)}\right] a_{n-4} = \frac{-4(2n-3)}{(n-3)(n-2)} a_{n-4}$$

$$\boxed{a_{n-4} = -\frac{(n-3)(n-2)}{4(2n-3)} \cdot a_{n-2}}$$

$$a_{n-4} = \frac{-(n-3)(n-2)}{4(2n-3)} \cdot \frac{n(n-1)}{2(2n-3)} \cdot a_n.$$

⇒ Now, Eqn (1) becomes...

$$y = a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-3)(2n-1)} a_n x^{n-4}$$

where; $G(x) = \begin{cases} a_0/a_n & \text{if } n \to \text{even} \\ a_1/a_n & \text{if } n \to \text{odd.} \end{cases}$

$$= a_n \left\{ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-3)(2n-1)} x^{n-4} + \cdots \right\} \quad \text{—(3)}$$

⇒ Constant, a_n is so choosen such that; $y = f(x)$ becomes 1, when $x=1$, the polynomials so obtained are called Legendre polynomials and denoted by $P_n(x)$.

⇒ putting; $n = 0, 1, 2, 3, 4$ in Eqn 3

we get: $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}[35x^4-30x^2+3]$

(Problems :-

Express the following in terms of Legendre polynomial :-

(i) x),

* RODRIGUE'S FORMULA :-

The legendre polynomials $P_n(x)$ of the form :

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$ is known as "Rodrigue's formula".

proof :- Let $u = (x^2-1)^n$.

We shall 1st establish that the n^{th} derivative of u, that is u_n is the solution of Legendre's differential Equation.

$$\Rightarrow \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad — (1)$$

$\Rightarrow$ dwt u wrt x, we have...

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} \cdot 2x$$

$$u_1 = 2nx \cdot \frac{(x^2-1)^n}{(x^2-1)}$$

$$(x^2-1)u_1 = 2nx(x^2-1)^n$$

$$\Rightarrow \quad (x^2-1)u_1 = 2nxu$$

$\Rightarrow$ dwt again wrt x, we get...

$$(x^2-1)u_2 + u_1(2x) = 2n[xu_1 + u(1)]$$

$\Rightarrow$ We shall now dwt, the result n times by applying Leibnitz theorem for n^{th} derivative of product given by;

$$(uv)_n = uv_n + nu_1 v_{n-1} + \frac{n(n-1)}{2!} u_2 v_{n-2} + \cdots + u_n v.$$

$$\therefore [(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$$

$$\left\{[(x^2-1)u_2]_n + n[2x]u_{n+2-1} + \frac{n(n-1)}{2!}(2(1))u_{n+2-2}\right\} + 2\left[x u_1]_n + n(2(1))u_{1+n-1}\right] \ldots 2nu_n.$$

ie, $\left[(x^2-1)u_{n+2} + n.2x\, u_{n+1} + \frac{n(n-1)}{2}.2u_n\right] + 2\left[x u_{n+1} + n.1.u_n\right]$

$= 2n\left[x u_{n+1} + n.1 u_n\right] + 2n u_n$

$\Rightarrow (x^2-1)u_{n+2} + 2nx\, u_{n+1} + (n^2-n)u_n + 2x u_{n+1} + 2n u_n = 2nx\, u_{n+1} + 2n^2 u_n + 2n u_n$

ie, $(x^2-1)u_{n+2} + 2x u_{n+1} - n^2 u_n - n u_n = 0$

$(x^2-1)u_{n+2} + 2x u_{n+1} - n u_n(n+1) = 0$

(or) $(1-x^2)u_{n+2} - 2x.u_{n+1} + n(n+1)u_n = 0$

This can be put in the form;

$\Rightarrow (1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \sim (2)$

Comparing (2) with (1) we conclude that, u_n is a solution of Legendre's diff. Eqn. *(It may be observed that u is a polynomial of degree $2n$ & hence u_n will be polynomial of deg n)*

Also $P_n(x)$ which satisfies Legendre's dE is also a polynomial of deg. n. Hence u_n is same as $P_n(x)$

$\Rightarrow P_n(x) = K u_n = K\left[(x^2-1)^n\right]_n$

$P_n(x) = K\left[(x-1)^n(x+1)^n\right]_n$

Applying Leibneitz thm for RHS;

$$P_n(x) = K\left[(x-1)^n\left\{(x+1)^n\right\}_n + n.n(x-1)^{n-1}\left\{(x+1)^n\right\}_{n-1} + \frac{n(n-1)}{2!}n(n-1)(x-1)^{n-2}\left\{(x+1)^n\right\}_{n-2} + \cdots\cdots \left\{(x-1)^n\right\}_n(x+1)^n\right] \sim (3)$$

Since, $z = (x-1)^n$ then;

$$Z_1 = n(x-1)^{n-1}$$

$$Z_2 = n(n-1)(x-1)^{n-2} \quad \cdots$$

$$Z_n = n(n-1)(n-2) \cdots 2.1\,(x-1)^{n-n}$$

$$Z_n = n!\,(x-1)^0 = n!$$

$$\therefore \{(x-1)^n\}_n = n!$$

putting, $x=1$ in eqn (3), all terms in RHS become 0, except last term, ie, $n!(1+1)^n = n!\,2^n$.

(3) ie $\Rightarrow$ $P_n(1) = K n!\,2^n$, $P_n(1) = 1$ // By defn of $P_n(x)$.

$$1 = K n!\,2^n$$

$$\boxed{K = \frac{1}{n!\,2^n}}$$

Since, $P_n(x) = K u_n$, we have; $P_n(x) = \frac{1}{n!\,2^n}\{(x^2-1)^n\}_n$.

Therefore; $\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$ is "Rodrigue's Formula".

Note :-

1) $P_0(x) = 1$
2) $P_1(x) = x$
3) $P_2(x) = \frac{1}{2}(3x^2-1)$
4) $P_3(x) = \frac{1}{2}(5x^3-3x)$
5) $P_4(x) = \frac{1}{8}[35x^4-30x^2+3]$

* $1 = P_0(x)$
* $x = P_1(x)$
* $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$
* $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$.
* $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$

(1) Express :- $x^4+3x^3-x^2+5x-2$ in terms of Legendre's polynom...

× Soln :- $x^4+3x^3-x^2+5x-2$.

$$= \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x) + \frac{6}{5}P_3(x) + \frac{9}{5}P_1(x) - \frac{2}{3}P_2(x)$$
$$- \frac{1}{3}P_0(x) + 5P_1(x) - 2P_0(x).$$

$$= \frac{8}{35}P_4(x) + \left(\frac{12-14}{21}\right)P_2(x) + \frac{6}{5}P_3(x) + \left(\frac{9+25}{5}\right)P_1(x)$$
$$+ \left(\frac{3-5-30}{15}\right)P_0(x).$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{32}{15}P_0(x)$$

* Using Rodrigue's formula, Compute : $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$:-

Soln :- WKT; Rodrigue's formula is ;

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

put ; $n=0,1,2,3,4$.

put ; $n=0$, $P_0(x) = \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0$.

$$\boxed{P_0(x)=1}$$

$n=1$, $P_1(x) = \frac{1}{(2)^1(1)!}\frac{d}{dx}(x^2-1)^1$

$$= \frac{1}{2}\frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2}(2x)$$

$$\boxed{P_1(x)=x}$$

$\underline{n=2}$ $\quad P_2(x) = \dfrac{1}{2^2 \cdot 2!} \dfrac{d^2}{dx^2} (x^2-1)^2$

$= \dfrac{1}{8} \cdot \dfrac{d}{dx}\, 2(x^2-1)\, 2x.$

$= \dfrac{1}{2} \cdot [3x^2 - 1]$

$\therefore \boxed{P_2(x) = \frac{1}{2}(3x^2-1)}$ (or) $\boxed{x^2 = \frac{2 P_2(x) + 1}{3}}$

$\underline{n=3}$ $\quad P_3(x) = \dfrac{1}{2^3 3!} \dfrac{d^3}{dx^3} (x^2-1)^3$

[Binomial Expression :- $(x-y)^n = x^n - nC_1 x^{n-1} \cdot y + nC_2 x^{n-2} \cdot y^2 + \dots + x^0 y^n$]

WKT; $\dfrac{d^n}{dx^n}(x^m) = \dfrac{m!}{(m-n)!} x^{m-n}$, $m \geq n$.

$3(x^2-1)^2(2x)$

$6(x^2-1)^2 x$

$= 6 \cdot [x \cdot 2(x^2-1)2x - (x^2-1)^2(1)]$

$= 6[4x^2(2x) + (x^2-1)(8x) - 2(x^2-1)(2x)]$

$= \underset{48}{24 \times 2}\, x^3 + 8x^3 - 8x - 4x^3 + 4x$

To find ; $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n} (x^2-1)^n$

$\underline{n=3}$; $P_3(x) = \dfrac{1}{2^3 3!} \dfrac{d^3}{dx^3} (x^2-1)^3$ // $(a-b)^3 =$ Binomial Expression.

$= \dfrac{1}{48} \dfrac{d^3}{dx^3} \left[(x^2)^3 - 3C_1 (x^2)^2 + 3C_2 (x^2)^1 1^2 - 3C_3 (x^2)^0 \right]$

$= \dfrac{1}{48} \dfrac{d^3}{dx^3} \left[x^6 - 3x^4 + 3x^2 - 1 \right]$

$= \dfrac{1}{48} \left[\dfrac{6!}{3!} x^3 - 3 \cdot \dfrac{4!}{1!} x^1 + 0 \right]$

$= \dfrac{1}{48} [120x^3 - 72x]$

$\boxed{P_3(x) = 5/2\, x^3 - 3/2\, x.}$ (or) $\boxed{x^3 = \frac{2P_3(x) + 3x}{5}}$

https://hemanthrajhemu.github.io

Problem

$\underline{n=4}$

$$P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{384}\frac{d^4}{dx^4}\left[(x^2-1)^4\right]$$

$$= \frac{1}{384}\frac{d^4}{dx^4}\left[(x^3)^4 - 4C_1(x^3)^3 + 4C_2(x^3)^2 - 4C_3(x^3)^1 + 4C_4(x^3)^0\right].$$

$$= \frac{1}{384}\frac{d^4}{dx^4}\left[x^{12} - 4(x^9) + 6x^6 - 4x^3 + 1\right]$$

$$= \frac{1}{384}\frac{d^3}{dx^3}\left[12x^{11} - 9\times4x^8 + 6\times6x^5 - 4\times3x^2 + 0\right]$$

$$= \frac{1}{384}\frac{d^2}{dx^2}\left[12\times11\times x^{10} - 9\times4\times8\times x^7 + 36\times5x^4 - 12\times2x\right]$$

$$= \frac{1}{384}\frac{d}{dx}\left[12\times11\times10x^9 - 36\times8\times7x^6 + 180\times4x^3 - 24\right]$$

$$= \frac{1}{384}\left[120\times11\times9x^8 - 36\times8\times7\times6x^5 + 180\times4\times3x^2 - 0\right]$$

$$nC_r = \frac{n!}{r!(n-r)!}$$

$$4C_2 = \frac{4!}{2!\,2!}$$

$$\boxed{4C_2 = 6}$$

$\Rightarrow x^4 /= 8P.$

$$\boxed{P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]}$$

(or)

$$\boxed{x^4 = \frac{8P_4(x) + 30x^2 - 3}{35}}$$

Formulaes :-

1) $K = K.P_0(x)$

2) $x = P_1(x)$

3) $x^2 = \dfrac{2P_2(x)+1}{3}$

4) $x^3 = \dfrac{2P_3(x)+3x}{5}$

5) $x^4 = \dfrac{8P_4(x)+30x^2-3}{35}$

Problems & Solutions :-

1) Express : x^3+2x^2-x-3 interms of legendre's polynomials :-

Soln :- Let $f(x) = x^3+2x^2-x-3$.

We have ;
$1 = P_0(x)$
$x = P_1(x)$
$x^2 = \frac{2}{3}P_2(x)+\frac{1}{3}$
$x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}x$
$x^4 = \frac{8}{35}P_4(x)+\frac{30}{35}x^2-\frac{3}{35}$

$$f(x) = \left[\frac{2}{5}P_3(x)+\frac{3}{5}x\right]+2\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]-\left[P_1(x)\right]-3\cdot P_0(x)$$

$$= \frac{2}{5}P_3(x)+\frac{3}{5}\underline{P_1(x)}+\frac{4}{3}P_2(x)+\frac{2}{3}P_0(x)-\underline{P_1(x)}-3P_0(x) \Rightarrow \text{Simplify}$$

$$\therefore f(x) = \frac{2}{5}P_3(x)+\frac{4}{3}P_2(x)-P_1(x)-\frac{2}{5}P_1(x)-\frac{7}{3}P_0(x).$$

2) Express : $f(x) = x^4+3x^3-x^2+5x-2$ in terms of legendre's polynomials.

Let ; $f(x) = x^4+3x^3-x^2+5x-2$

We have ; $P_0(x)=1$, $x^2 = \frac{2}{3}P_2(x)+\frac{1}{3}$
$x = P_1(x)$ $\quad x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}x$
$x^4 = \frac{8}{35}P_4(x)+\frac{30}{35}x^2-\frac{3}{35}$.

$$\therefore f(x) = \left[\frac{8}{35}P_4(x)+\frac{30}{35}x^2-\frac{3}{35}\right]+3\left[\frac{2}{5}P_3(x)+\frac{3}{5}x\right]-\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]$$
$$+5P_1(x)-2\cdot P_0(x)$$

$$= \frac{8}{35}P_4(x)+\frac{30}{35}\left[\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)\right]-\frac{3}{35}P_0(x)+\frac{6}{5}P_3(x)+\frac{9}{5}P_1(x)$$
$$-\frac{2}{3}P_2(x)-\frac{1}{3}P_0(x)+5P_1(x)-2P_0(x)$$

$$f(x) = \frac{8}{35}P_4(x)+\frac{6}{5}P_3(x)-\frac{16}{21}P_2(x)+\frac{34}{105}P_1(x)$$
$$-\frac{32}{[illegible]}P_0(x)$$

$\frac{20}{35}-\frac{2}{3}$; $= -\frac{2}{21}-\frac{2}{3}$; $\frac{[illegible]}{35}-\frac{3}{35}-\frac{1}{3}$

3) If $x^3+2x^2-x+1 = a\,p_0(x) + b\,p_1(x) + c\,p_2(x) + b\,p_3(x)$

find values of a, b, c, d

$f(x) = x^3+2x^2-x+1$

we have ; $\quad 1 = P_0(x) \qquad x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$

$\qquad x = P_1(x) \qquad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x.$

$$f(x) = \frac{2}{3}P_3(x) + \frac{3}{5}P_1(x) + 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - p_1(x) + p_0(x)$$

$$= \frac{2}{3}P_3(x) + \frac{3}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - p_1(x) + p_0(x)$$

$$f(x) = \underset{\downarrow\, d}{\frac{2}{3}P_3(x)} + \underset{\downarrow\, c}{\frac{4}{3}P_2(x)} - \underset{\downarrow\, b}{\frac{2}{5}p_1(x)} + \underset{\downarrow\, a}{\frac{5}{3}p_0(x)}.$$

$\therefore \boxed{a = 5/3} \quad \boxed{b = -2/5} \quad \boxed{c = 4/3} \quad \boxed{d = 2/3}$

4) If $2x^3-x^2-3x+2 = a\,p_0(x)+b\,p_1(x)+c\,p_2(x)+d\,p_3(x)$ find : a, b, c, d . .

Soln:- $f(x) = 2x^3-x^2-3x+2$, wkt; $1 = p_0(x)$, $x = p_1(x)$, $x^2 = \frac{2}{3}p_2(x) + \frac{1}{3}$

$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$

$$\therefore f(x) = 2\left[\frac{2}{5}p_3(x) + \frac{3}{5}x\right] - \left[\frac{2}{3}p_2(x) + \frac{1}{3}\right] - 3\left[p_1(x)\right] + 2p_0(x).$$

$$= \frac{4}{5}p_3(x) + \frac{6}{5}p_1(x) - \frac{2}{3}p_2(x) - \frac{1}{3}p_0(x) - 3p_1(x) + 2p_0(x)$$

$$\therefore f(x) = \underset{\downarrow\, d}{\frac{4}{3}P_3(x)} - \underset{\downarrow\, c}{\frac{2}{3}p_2(x)} - \underset{\downarrow\, b}{\frac{9}{5}p_1(x)} + \underset{\downarrow\, a}{\frac{5}{3}p_0(x)}$$

$\therefore \boxed{a = 5/3} \quad \boxed{b = -9/5} \quad \boxed{c = -2/3} \quad \boxed{d = 4/3}$