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Future Vision

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Complex variables - 1

A number of the form $z = x+iy$ where x, y are real numbers and $i=\sqrt{-1}$ (or) $i^2=-1$ is called a complex no. x is called the real part of z and y is called the imaginary part of z.

Also $\bar{z} = x-iy$ is called the complex conjugate of z.

we have $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

Thus $e^{ix} = \cos x + i\sin x$ ——— ①

$e^{-ix} = \cos x - i\sin x$ ——— ②

Adding and subtracting ① and ② we get

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$\therefore \cos(ix) = \frac{e^{i(ix)}+e^{-i(ix)}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh x$

$\sin(ix) = \frac{e^{i(ix)}-e^{-i(ix)}}{2i} = \frac{e^{-x}-e^{x}}{2i} = \frac{-i(e^{-x}-e^{x})}{2}$

$= \frac{i(e^{x}-e^{-x})}{2} = i\sinh x$

Thus $\boxed{\cos(ix) = \cosh x}$

$\boxed{\sin(ix) = i\sinh x}$

De-moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is a real no.

Properties associated with the modulus and amplitude

1) a) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

b) $amp(z_1 \cdot z_2) = amp\, z_1 + amp\, z_2$

2) a) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

b) $amp\left(\frac{z_1}{z_2}\right) = amp\, z_1 - amp\, z_2$

3) $|z_1 + z_2| \leq |z_1| + |z_2|$

4) $|z_1 - z_2| \geq |z_1| - |z_2|$

Neighbourhood: A nbd of a point z_0 in the complex plane is the set of all points z such that $|z - z_0| < \delta$ where δ is a small positive real no.

Note $z = x + iy$ (or) $z = re^{i\theta}$ we always write

$w = f(z) = u(x, y) + iv(x, y) \longrightarrow$ cartesian form

$w = f(z) = u(r, \theta) + iv(r, \theta) \longrightarrow$ polar form

Limit :- A complex valued function $f(z)$ defined in the nbd of a point z_0 is said to have a limit l as z tends to z_0, if for every $\epsilon > 0$ however small there exists a positive real number

Continuity :- A complex valued function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$

That is to say that $|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta$.

Differentiability :-

A complex valued function $f(z)$ is said to be differentiable at $z = z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is unique. This limit when exists is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$.

Suppose we write $\delta z = z - z_0$, then $z \to z_0$ implies that $\delta z \to 0$

Hence, $f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$

Further $f(z)$ is said to be continuous / differentiable in a domain or a region D if $f(z)$ is continuous / differentiable at every point of D.

These definitions are analogous with the definitions of a real valued function.

***Cauchy - Rieman Eqⁿ (CR - Eqⁿ) in cartesian form.

stmt :- A necessary condⁿ that the function $w = f(z) = u(x, y) + iv(x, y)$ may be analytic at any point $z = x+iy$ is that their exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ thes are known as

C.R - equations.

proof :- Let $f(z)$ be analytic at a point $z = x+iy$

hence by defⁿ of analytic function

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta(z)}$$ exists and is unique.

In cartesian form $f(z) = u(x, y) + iv(x, y)$

Let δz be increments in z corresponding in $\delta x, \delta y$ in x, y

$$\therefore f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta(z)}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - [u(x,y) + iv(x,y)]}{\delta z}$$

$$= \lim_{\delta z \to 0} \left[\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta z} + i \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \right] \quad \text{--- (1)}$$

we have $\delta z = (z+\delta z) - z$

$= (x+\delta x) + i(y+\delta y) - [x+iy]$

$$\boxed{\delta z = \delta x + i\delta y}$$

case i) Let $\delta y = 0$ so that $\delta z = \delta x$

since $\delta z \to 0$, $\delta x \to 0$ $\therefore$ (1) $\Rightarrow$

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{[v(x+\delta x, y) - v(x,y)]}{\delta x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \quad \text{--- (2)}$$

case ii) Let $\delta x = 0$ so that $\delta z = i\delta y$

since $\delta z \to 0$ implies $i\delta y \to 0$ (or) $\delta y \to 0$

(1) $\Rightarrow$

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{[v(x, y+\delta y) - v(x,y)]}{i\delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \quad \text{--- (3)} \qquad \frac{1}{i} = -i$$

Comparing eqn (2) and (3) and equating to RHS

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

equating real and imaginary parts

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

are the CR- eqns in cartesian form.

*** Cauchy Rieman Eq^n (CR-Eq^n) in polar form

Stmt:- if $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z. then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the eq^{n}s. $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ and $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$

These are known as C-R eq^n in polar form.

Proof :- Let f(z) be analytic at a point $z = re^{i\theta}$

hence by def^n $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists

In polar form $f(z) = u(r,\theta) + iv(r,\theta)$

Let δz be increment in z corresponding in $\delta r, \delta\theta$ in r, θ

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i\lim_{\delta z \to 0} \frac{[v(r+\delta r, \theta+\delta\theta) - v(r,\theta)]}{\delta z} \quad \text{—— (1)}$$

Since $z = re^{i\theta}$

$$\delta z = \frac{\partial z}{\partial r}\delta r + \frac{\partial z}{\partial \theta}\delta\theta$$

$$\delta z = \frac{\partial}{\partial r}(re^{i\theta})\delta r + \frac{\partial}{\partial \theta}(re^{i\theta})\delta\theta$$

$$\boxed{\delta z = e^{i\theta}\delta r + ire^{i\theta}\delta\theta}$$

case i) Let $\delta\theta = 0$ so that $\delta z = e^{i\theta}\delta r$

this implies $\delta z \to 0$ so $e^{i\theta}\delta r \to 0$ (or) $\delta r \to 0$

$$(1) \Rightarrow f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}\delta r} + i \lim_{\delta r \to 0} \frac{[v(r+\delta r, \theta) - v(r,\theta)]}{e^{i\theta}\delta r}$$

$$\boxed{f'(z) = \frac{1}{e^{i\theta}}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]} \longrightarrow (2)$$

case ii) Let $\delta r = 0$ so that $\delta z = ire^{i\theta}\delta\theta$

this implies $\delta z \to 0$ so $ire^{i\theta}\delta\theta \to 0$ (or) $\delta\theta \to 0$

$$(1) \Rightarrow f'(z) = \lim_{\delta\theta \to 0} \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{ire^{i\theta}\delta\theta} + i \lim_{\delta\theta \to 0} \frac{[v(r; \theta+\delta\theta) - v(r,\theta)]}{ire^{i\theta}\delta\theta}$$

$$= \frac{1}{ire^{i\theta}}\left[\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right] \qquad \frac{1}{i} = -i$$

$$\boxed{f'(z) = \frac{1}{re^{i\theta}}\left[\frac{\partial v}{\partial\theta} - i\frac{\partial u}{\partial\theta}\right]} \longrightarrow (3)$$

equating RHS of (2) and (3)

$$\frac{1}{e^{i\theta}}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{1}{re^{i\theta}}\left[\frac{\partial v}{\partial\theta} - i\frac{\partial u}{\partial\theta}\right]$$

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta} - \frac{i}{r}\frac{\partial u}{\partial\theta}$$

equating real and imaginary parts we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}} \text{ and } \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}}$$

are the CR - Equations in polar form

Properties of Analytic functions.

Harmonic function - Definition

A function ϕ is said to be harmonic if it satisfies Laplace's eqn $\nabla^2\phi = 0$.

In the cartesian form $\phi(x, y)$ is harmonic

if $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

In the polar form $\phi(r, \theta)$ is harmonic

if $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$

Note :- 1) Harmonic property.

The real and imaginary parts of an analytic function are harmonic.

2) orthogonal property.

if $f(z) = u+iv$ is analytic then the family of curves $u(x, y) = c_1$, $v(x, y) = c_2$, where c_1 and c_2 being constants, intersect each other orthogonally.

3) CR eqns $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are not satisfied.

Hence $u+iv$ is not analytic.

1) S.T $f(z) = e^x(\cos y + i\sin y)$ is Analytic

$u+iv = e^x\cos y + ie^x\sin y$

$u = e^x\cos y$, $v = e^x\sin y$

$u_x = e^x\cos y$ $v_x = e^x\sin y$

$u_y = -e^x\sin y$ $v_y = e^x\cos y$

CR-Eqn $u_x = v_y$ and $v_x = -u_y$ are satisfied

$\therefore$ Thus $f(z)$ is analytic

2) S.T $w = z\bar{z}$ is not analytic $z = x+iy$, $\bar{z} = x-iy$.

$u+iv = x^2+y^2$

$u = x^2+y^2$, $v = 0$

$u_x = 2x$, $u_y = 2y$, $v_x = 0$, $v_y = 0$

$\therefore$ C-R not satisfied

$\therefore$ $w = z\bar{z}$ is not analytic

3) OP S.T $f(z) = \sin z$ is analytic and hence find $f'(z)$

$u+iv = \sin(x+iy)$

$= \sin x\cos iy + \cos x\sin iy$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$= \sin x\cosh y + i\cos x\sinh y$ $\because \cos(ix) = \cosh x$, $\sin(ix) = i\sinh x$

$u = \sin x\cosh y$, $v = \cos x\sinh y$

$u_x = \cos x\cosh y$ | $v_x = -\sin x\sinh y$

$u_y = \sin x\sinh y$ | $v_y = \cos x\cosh y$

$\therefore$ CR-Eqn satisfied

Now $f'(z) = u_x + iv_x$

$f'(z) = \cos x \cosh y + i(-\sin x \sinh y)$

By M.T method

put $x = z$, $y = 0$ in the C-form

$$\boxed{f'(z) = \cos z}$$

4) ST $f(z) = \cosh z$ is analytic and find $f'(z)$

$$u + iv = \cosh z = \left(\frac{e^z + e^{-z}}{2}\right)$$

$$u + iv = \frac{e^{x+iy} + e^{-(x+iy)}}{2}$$

$$= \frac{e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)}{2}$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)\cos y + i\sin y\left(\frac{e^x - e^{-x}}{2}\right)$$

$$u + iv = \cosh x \cos y + i\sin y \sinh x$$

$u = \cosh x \cos y$, $v = \sin y \sinh x$

$u_x = \sinh x \cos y$ | $v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y$ | $v_y = \cos y \sinh x$

C-R are satisfied

Now $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i(\cosh x \sin y)$

By M.T put $x = z$, $y = 0$

$f'(z) = \sinh z$

5) ST $w = z + e^z$ is analytic and hence find $\frac{dw}{dz}$ ②

$$u + iv = (x+iy) + e^{(x+iy)}$$
$$= (x+iy) + e^x(\cos y + i\sin y)$$
$$= (x+iy) + e^x(\cos y + i \sin y)$$
$$= (x + e^x \cos y) + i(y + e^x \sin y)$$

$\underline{u = x + e^x \cos y}$. $\underline{v = y + e^x \sin y}$

$u_x = 1 + e^x \cos y$ | $v_x = e^x \sin y$

$u_y = 0 - e^x \sin y$ | $v_y = 1 + e^x \cos y$

C-R satisfied

$$f'(z) = u_x + i v_x$$
$$= (1 + e^x \cos y) + i(e^x \sin y)$$

By $\underline{\text{M.T}}$ put $x = z$, $y = 0$

$$\boxed{f'(z) = 1 + e^z}$$

6) S.T $w = \log z$; $z \neq 0$ is analytic and hence find $\frac{dw}{dz}$.

$w = \log z$. $z = re^{i\theta}$

$u + iv = \log(re^{i\theta})$

$= \log r + i\theta \log_e e$

$u + iv = \log r + i\theta$

$u = \log r$, $v = \theta$

$u_r = \frac{1}{r}$ | $v_r = 0$

$\Rightarrow r u_r = 1$ | $v_\theta = 1$

$u_\theta = 0$

C-R satisfied

$$f'(z) = e^{-i\theta}\left[u_r + i v_r\right]$$
$$= e^{-i\theta}\left[\frac{1}{r} + i0\right]$$
$$= \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z} \quad z = re^{i\theta}$$
$$\therefore \boxed{f'(z) = \frac{1}{z}}$$

7) S.T $f(z) = z^n$, where n is a +ve integer is analytic and hence find it's derivative

$$u + iv = (re^{i\theta})^n = r^n e^{in\theta}$$
$$u + iv = r^n(\cos n\theta + i\sin n\theta)$$
$$u + iv = r^n\cos n\theta + i r^n \sin n\theta$$
$$u = r^n\cos n\theta, \quad v = r^n \sin n\theta$$

$u_r = nr^{n-1}\cos n\theta$	$v_r = nr^{n-1}\sin n\theta$
$u_\theta = -nr^n \sin n\theta$	$v_\theta = nr^n\cos n\theta$

C-R eq^n $ru_r = v_\theta$ and $rv_r = -u_\theta$ are satisfied

Now $f'(z) = e^{-i\theta}\left[u_r + i v_r\right]$

$$f'(z) = e^{-i\theta}\left[nr^{n-1}\cos n\theta + i\, nr^{n-1}\sin n\theta\right]$$

by MT put $r = z$, $\theta = 0$

$$f'(z) = 1\left[n z^{n-1}(1) + 0\right]$$
$$\boxed{f'(z) = nz^{n-1}}$$

8) S.T $f(z) = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$, (3)

$r \neq 0$ is a regular function of $z = re^{i\theta}$.

also find $f'(z)$.

>> $u = \left(r + \frac{k^2}{r}\right)\cos\theta \qquad v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$u_r = \left(1 - \frac{k^2}{r^2}\right)\cos\theta \qquad v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$

$u_\theta = -\left(r + \frac{k^2}{r}\right)\sin\theta \qquad v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$

C-R satisfied.

$$f'(z) = e^{-i\theta}\left[u_r + i v_r\right]$$

$$f'(z) = e^{-i\theta}\left[\left(1 - \frac{k^2}{r^2}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right]$$

by MT method put $r = z$, $\theta = 0$

$$\boxed{f'(z) = 1 - \frac{k^2}{z^2}}$$

Type-2

9) construct the analytic funⁿ whose real part is $u = \log\sqrt{x^2+y^2}$

$$u = \log(x^2+y^2)^{1/2} = \frac{1}{2}\log(x^2+y^2)$$

$$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$\therefore u_y = \frac{y}{x^2+y^2}$$

consider $f'(z) = u_x + i v_x$ But $v_x = -u_y$ by C-R Eqⁿ

$$\therefore f'(z) = u_x - i u_y$$

$$f'(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

By M.T method put $x = z$, $y = 0$

$$f'(z) = \frac{z}{z^2} = \frac{1}{z}$$

$$f'(z) = \frac{1}{z}$$

$$f(z) = \int \frac{1}{z}\,dz + c = \log z + c$$

$$\therefore \text{Thus } \boxed{f(z) = \log z + c}$$

10) op) Find the analytic funⁿ $f(z) = u + iv$ whose real part is $e^{-x}(x\cos y + y\sin y)$

we give $u = e^{-x}(x\cos y + y\sin y)$

$$u_x = e^{-x}(\cos y + 0) - e^{-x}(x\cos y + y\sin y)$$

$$\therefore u_x = e^{-x}(\cos y - x\cos y - y\sin y)$$

$u_y = e^{-x}(-x\sin y + y\cos y + \sin y)$

$\therefore f'(z) = u_x + i v_x$ But $v_x = -u_y$ by CR-eq^n

$f'(z) = e^{-x}(\cos y - x\cos y - y\sin y) + i\{e^{-x}(-x\sin y + y\cos y + \sin y)\}$

By M.T method

put $x = z,\ y = 0$

$f'(z) = e^{-z}(1 - z - 0) + i(0)$

$f'(z) = e^{-z} - z e^{-z}$

$f(z) = \int e^{-z} dz - \int (z e^{-z}) dz + c$

$= \dfrac{e^{-z}}{-1} - \left[z\dfrac{e^{-z}}{-1} - (1)\dfrac{e^{-z}}{(-1)^2} \right]$

$\int uv\, dx = uv_1 - u'v_{11} + u''v_{111} \mp \cdots$

$= \dfrac{e^{-z}}{-1} + z e^{-z} + e^{-z}$

$= \cancel{-e^{-z}} + z e^{-z} + \cancel{e^{-z}}$

$\boxed{f(z) = z e^{-z} + c}$

ii) Determine the analytic fun $f(z) = u+iv$ given that the real part.

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

$$u_x = e^{2x}(\cos 2y - 0) + 2e^{2x}(x\cos 2y - y\sin 2y)$$

$$u_x = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$$

$$u_y = e^{2x}(-2x\sin 2y - \{2y\cos 2y + \sin 2y\})$$

$$u_y = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

But $v_x = -u_y$ (C.R. Eq^n)

we $f'(z) = u_x + iv_x$

$$f'(z) = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y) + i e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

By $\underline{\text{M.T.}}$ put $x = z$, $y = 0$

$$f'(z) = e^{2z}(\cos 0 + 2z - 0) + i(0)$$

$$= e^{2z}(1 + 2z)$$

$$f'(z) = e^{2z} + 2ze^{2z}$$

$$f(z) = \int e^{2z}dz + 2\int z e^{2z}dz + C$$

$$= \frac{e^{2z}}{2} + 2\left[z\frac{e^{2z}}{2} - \frac{e^{2z}}{4}\right] + C$$

$$= \cancel{\frac{e^{2z}}{2}} + ze^{2z} - \cancel{\frac{e^{2z}}{2}}$$

$$\boxed{f(z) = ze^{2z} + C}$$

op.

12) Find the analytic function $f(z)$ given $u = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

proof $u = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$u_x = e^{-x}\{(2x-0)\cos y + 2y\sin y\} - e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$\Rightarrow u_x = e^{-x}\{2x\cos y + 2y\sin y\} - e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$u_x = e^{-x}[2x\cos y + 2y\sin y - (x^2-y^2)\cos y - 2xy\sin y]$

and $u_y = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$= e^{-x}\{x^2\cos y - y^2\cos y + 2xy\sin y\}$

$= e^{-x}\{-x^2\sin y - [-y^2\sin y + \cos y\, 2y] + 2x[y\cos y + \sin y]\}$

$u_y = e^{-x}\{-x^2\sin y + y^2\sin y - \cos y\, 2y + 2xy\cos y + 2x\sin y\}$

and $f'(z) = u_x + iv_x$ and C.R-eqn $v_x = -u_y$

$f'(z) = u_x - iu_y$

$= e^{-z}[2z + 0 - (z^2-0)(1) - 0] + i\, e^{-z}(-0 + 0 - 0 + 0 + 0)]$

$= e^{-z}[2z - z^2]$

$f'(z) = 2ze^{-z} - z^2e^{-z} = \underline{\underline{(2z-z^2)e^{-z}}}$

$$\therefore f'(z) = (2z - z^2)\,e^{-z}$$

$$f(z) = \int (2z - z^2)\,e^{-z}\,dz + c$$

$$= (2z - z^2)\frac{e^{-z}}{-1} - (2 - 2z)\frac{e^{-z}}{(-1)^2} + (0 - 2)\frac{e^{-z}}{(-1)^3}$$

$$= -(2z - z^2)e^{-z} - (2 - 2z)e^{-z} + 2e^{-z}$$

$$= \cancel{-2z e^{-z}} + z^2 e^{-z} - \cancel{2e^{-z}} + \cancel{2z e^{-z}} + \cancel{2e^{-z}}$$

$$\boxed{f(z) = z^2 e^{-z} + c}$$

OP 13) Find the analytic fun f(z) whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

Let $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\therefore u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(0 - (-2\sin 2x))}{(\cosh 2y - \cos 2x)^2}$$

$$u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y - 0)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{-\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$f'(z) = u_x + i v_x$ But $v_x = -u_y$ C-R eqn ⑥

$f'(z) = u_x - i u_y$

$$f'(z) = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) + 2\sin^2 2x - i\{-\sin 2x(2\sinh 2y)\}}{(\cosh 2y - \cos 2x)^2}$$

By MT put $x = z$, $y = 0$

$$f'(z) = \frac{(1 - \cos 2z)(2\cos 2z) + 2\sin^2 2z - \cancel{i0}}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z + 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(1)}{(1-\cos 2z)^2}$$

$$= \frac{-2\cancel{(1-\cos 2z)}}{(1-\cos 2z)^{\cancel{2}}}$$

$$= \frac{-2}{(1-\cos 2z)}$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{-\cancel{2}}{\cancel{2}\sin^2\theta} = -\frac{1}{\sin^2\theta}$$

$f'(z) = -\operatorname{cosec}^2 z$

$f(z) = -\int \operatorname{cosec}^2 z \, dz = -(-\cot z) + c$

$$\boxed{f(z) = \cot z + c}$$

14) Find the analytic fun whose real part is $\frac{x^4-y^4-2x}{x^2+y^2}$ Hence determine v.

$f(z) = \underline{z^2 - \frac{2}{z} + c}$ after simplification

$$u = \frac{x^4-y^4-2x}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2)(4x^3-0-2) - (x^4-y^4-2x)(2x)}{(x^2+y^2)^2}$$

$$u_x = \frac{(x^2+y^2)(4x^3-2) - 2x(x^4-y^4-2x)}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(0-4y^3-0) - (x^4-y^4-2x)2y}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(-4y^3) - 2y(x^4-y^4-2x)}{(x^2+y^2)^2}$$

$f'(z) = u_x + iv_x$ But $v_x = -u_y$

$f'(z) = u_x - iu_y$ put $x = z, y = 0$

$$f'(z) = \frac{(z^2+0)(4z^3-2) - 2z(z^4-0-2z)}{(z^2+0)^2} + i(0)$$

$$f'(z) = \frac{z^2(4z^3) - 2z(z^4-2z)}{z^4}$$

$$= \frac{4z^5}{z^4} - \frac{2z^5}{z^4} + \frac{4z^2}{z^4}$$

$$= 4z - 2z + \frac{4}{z^2} = \underline{2z + \frac{4}{z^2}}$$

$$f'(z) = 2z + \frac{2}{z^2}$$

$$\therefore f(z) = 2\int z\,dz + 2\int \frac{1}{z^2}\,dz + c$$

$$f(z) = 2\cdot\frac{z^2}{2} + 2\cdot\left(\frac{-1}{z}\right) + c$$

$$\boxed{f(z) = z^2 - \frac{2}{z} + c}$$

To find v, we shall separate the RHS of $f(z)$ into real and imaginary parts.

ie $u+iv = (x+iy)^2 - \frac{2}{x+iy} + c$

$$u+iv = x^2 + (iy)^2 + 2ixy - \frac{2}{x+iy} + c$$

$$= x^2 - y^2 + i\,2xy - \frac{2}{x+iy} + c$$

$$= x^2 - y^2 + i2xy - \frac{2(x-iy)}{x^2+y^2} + c$$

$$= \left(x^2 - y^2 - \frac{2x}{x^2+y^2}\right) + i\left(2xy + \frac{2y}{x^2+y^2}\right) + c$$

$$= \left[\frac{x^4 - y^4 - 2x}{x^2+y^2}\right] + i\left[\frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}\right] + c$$

Equating the real and imaginary parts we observe that the real part u is same as in the given problem and the required Im part is

$$v = \frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}$$

15) S.T the foll^{ng} function u is harmonic. Also determine the corresponding analytic fu^n $f(z)$,

$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$ ——— (1)

$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$ ——— (2)

(1) + (2) $u_{xx} + u_{yy} \Rightarrow$

$\Rightarrow -\sin x \cosh y - 2\cos x \sinh y + 2 + \sin x \cosh y$
$+ 2\cos x \sinh y - 2 = 0$

$\therefore u_{xx} + u_{yy} = 0$. Thus u is harmonic

consider $f'(z) = u_x + i v_x$ But $v_x = -u_y$ by C-R eq^n

$f'(z) = u_x - i u_y$

put $x = z$, $y = 0$. we get

$f'(z) = (\cos z - 2(0) + 2z + 0) - i(0 + 2\cos z + 4z)$

$f'(z) = \cos z + 2z - i(2\cos z + 4z)$

Int

$f(z) = \int \cos z \, dz + 2\int z \, dz - i\left[2\int \cos z \, dz + \int 4z \, dz\right] + c$

$f(z) = \sin z + 2 \cdot \frac{z^2}{2} - i\left[2 \cdot \sin z + 4 \cdot \frac{z^2}{2}\right] + c$

$f'(z) = \sin z + z^2 - 2i\sin z - 2iz^2 + c$

$= (1-2i)\sin z + z^2 - 2iz^2$

$= (1-2i)\sin z + (1-2i)z^2$

$= (1-2i)(z^2 + \sin z) + c$

Thus $f'(z) = (1-2i)(z^2+\sin z)$

16) Find the analytic fu^n $f(z)$ whose imaginary part is $e^x(x\sin y + y\cos y)$ (8)

>> By data $v = e^x(x\sin y + y\cos y)$

$v_x = e^x(\sin y + 0) + e^x(x\sin y + y\cos y)$

$v_x = e^x(\sin y + x\sin y + y\cos y)$ —— (a)

$v_y = e^x(x\cos y + y(-\sin y) + \cos y)$

$v_y = e^x(x\cos y - y\sin y + \cos y)$ —— (b)

Consider $f'(z) = u_x + i v_x$ But $u_x = v_y$ C-R eq^n

$f'(z) = v_y + i v_x$

$f'(z) = e^x(x\cos y - y\sin y + \cos y) + i\, e^x(\sin y + x\sin y + y\cos y)$

by M-T method put $x = z, y = 0$

$f'(z) = e^z(z - 0 + \cos 0) + i\, e^z(0 + 0 + 0)$

$f'(z) = e^z(z+1)$

$f'(z) = ze^z + e^z$

Thus $f(z) = \int ze^z dz + \int e^z dz$

$= z\frac{e^z}{1} - 1\frac{e^z}{(1)^2} + \frac{e^z}{1} + c$

$= ze^z - e^z + e^z$

$\boxed{f(z) = ze^z + c}$

17) Determine the analytic function $\omega = u + iv$ if $v = \log(x^2+y^2) + x - 2y$

>> $v_x = \frac{1}{x^2+y^2} \cdot 2x + 1 = 1 + \frac{2x}{x^2+y^2}$

$v_y = \frac{2y}{x^2+y^2} - 2 = -2 + \frac{2y}{x^2+y^2}$

Consider $f'(z) = u_x + iv_x$ But $u_x = v_y$ by C.R eq^n

$f'(z) = v_y + i v_x$

$f'(z) = \left(-2 + \frac{2y}{x^2+y^2}\right) + i\left(1 + \frac{2x}{x^2+y^2}\right)$

By M.T method put $x = z, y = 0$

$f'(z) = (-2 + 0) + i\left(1 + \frac{2z}{z^2+0}\right)$

$f'(z) = -2 + i\left(1 + \frac{2}{z}\right)$

$f'(z) = -2 + i + \frac{2i}{z}$

$f(z) = -2\int dz + i\int dz + 2i\int \frac{1}{z} dz$

$f(z) = -2z + iz + 2i \log z + c$

$f(z) = -2z + i(z + 2\log z) + c$

$\frac{0}{\text{anything}} = 0$ $\frac{\text{anything}}{0} = \infty$

18) Find an analytic fu^n whose $\underline{Im}$ part is $v = e^x\{(x^2-y^2)\cos y - 2xy\sin y\}$ (9)

$\Rightarrow v_x = e^x\{(2x-0)\cos y - 2y\sin y\} + e^x\{(x^2-y^2)\cos y - 2xy\sin y\}$

$\Rightarrow v_x = e^x\{2x\cos y - 2y\sin y + (x^2-y^2)\cos y - 2xy\sin y\}$

Now $v_y = e^x\{(x^2\cos y - y^2\cos y - 2xy\sin y)\}$

$v_y = e^x\{-x^2\sin y - [y^2\sin y + 2y\cos y] - 2x[y\cos y + \sin y]\}$

$v_y = e^x\{-x^2\sin y + y^2\sin y - 2y\cos y - 2xy\cos y - 2x\sin y\}$

Now $\underline{f'(z) = u_x + iv_x}$

But $u_x = v_y$ by C-R eq^{n}s

$f'(z) = v_y + iv_x$ put $x = z, y = 0$

$f'(z) = e^z\{0 + 0 - 0 - 0 - 0\} + i\{e^z(2z - 0 + (z^2-0)(1) - 0)\}$

$f'(z) = 0 + i\,e^z(2z + z^2)$

$f'(z) = i[2ze^z + z^2e^z]$

$f'(z) = i\{2\int ze^z dz + \int z^2e^z dz\}$ $\{z^2e^z - 2ze^z + 2e^z$

$= i\{2.\{ze^z - e^z\} + \{z^2e^z - 2ze^z + 2e^z\}$

$= i\{\cancel{2ze^z} - \cancel{2e^z} + z^2e^z - \cancel{2ze^z} + \cancel{2e^z}\}$

$f'(z) = iz^2e^z + c$

Q) if $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = (x^2-y^2) + \frac{x}{x^2+y^2}$. find the complex potential as a fun of the complex variable z and hence determine ϕ.

Sol^n assume $\phi + i\psi = u + iv$

$$\therefore \quad v = (x^2-y^2) + \frac{x}{x^2+y^2}$$

$$v_x = (2x - 0) + \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y = (0-2y) + \frac{(x^2+y^2)0 - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Now $f'(z) = u_x + iv_x$ But $u_x = v_y$

$$f'(z) = v_y + i\,v_x$$

$$f'(z) = 0 + i\left(2z + \frac{0 - z^2}{(z^2+0)^2}\right) \quad \text{put } x = z,\ y = 0$$

$$f'(z) = i\left(2z - \frac{z^2}{z^4}\right)$$

$$f'(z) = i\left(2z - \frac{1}{z^2}\right)$$

$$f(z) = i\left[2\int z\,dz - \int \frac{1}{z^2}\,dz\right] + C$$

$$= i\left[2\cdot\frac{z^2}{2} - \left(\frac{-1}{z}\right)\right] + C$$

$$= i\left[z^2 + \frac{1}{z}\right] + C$$

Thus $\boxed{f(z) = i\left(z^2 + \frac{1}{z}\right) + C}$ ——— (1)

To ϕ we shall separate the RHS into real & Im parts.

$\therefore (1) \Rightarrow$

$$u+iv = i\left((x+iy)^2 + \frac{1}{(x+iy)}\right) + c$$

$$= i\left\{x^2-y^2+2ixy + \frac{x-iy}{x^2+y^2}\right\} + c$$

$$= i\left\{\left(x^2-y^2+\frac{x}{x^2+y^2}\right) + i\left(2xy - \frac{y}{x^2+y^2}\right)\right\}$$

$$= \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2-y^2+\frac{x}{x^2+y^2}\right)$$

$$\Rightarrow \boxed{u = -2xy + \frac{y}{x^2+y^2}}$$

20) Construct the analytic fun whose real part is $r^2\cos 2\theta$

put $u = r^2\cos 2\theta$, $\quad u_\theta = -2r^2\sin 2\theta$

$u_r = 2r\cos 2\theta$,

$f'(z) = e^{-i\theta}[u_r + iv_r]$ But $v_r = -\frac{1}{r}u_\theta$

$= e^{-i\theta}\left[u_r - i\frac{1}{r}u_\theta\right]$

$= e^{-i\theta}\left[2r\cos 2\theta - \frac{i}{r}\{-2r^2\sin 2\theta\}\right]$

$= e^{-i\theta}[2r\cos 2\theta + i2r\sin 2\theta]$

put $r = z$, $\theta = 0$ by M.T

$f'(z) = 2z$

$\therefore f(z) = 2\int z\,dz + c = \frac{z^2}{2}\cdot 2 + c = z^2 + c$

Thus $\boxed{f(z) = z^2 + c}$

H.W.

21) Determine the analytic function $f(z)$ whose imaginary part is $\left(r - \frac{k^2}{r}\right)\sin\theta$, $r \neq 0$.

$v = \left(r - \frac{k^2}{r}\right)\sin\theta$ $\qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta.$

$v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$

Now $f'(z) = e^{-i\theta}\left[u_r + i v_r\right]$

But $u_r = \frac{1}{r} v_\theta$ by C-R eq^n

$f'(z) = e^{-i\theta}\left[\frac{1}{r} v_\theta + i v_r\right]$

$f'(z) = e^{-i\theta}\left[\frac{1}{r}\left(r - \frac{k^2}{r}\right)\cos\theta + i\left(1 + \frac{k^2}{r}\right)\sin\theta\right]$

$f'(z) =$ put $r = z$, $\theta = 0$ by M.T

$f'(z) = \left[\frac{1}{z}\left(z - \frac{k^2}{z}\right)\cos\cancel{\theta} + 0\right]$

$f'(z) = \cancel{\frac{z}{z}}\left(1 - \frac{k^2}{z^2}\right)(1)$

$f'(z) = 1 - \frac{k^2}{z^2}$

Now $f(z) = \int dz - k^2 \int \frac{1}{z^2} dz + C$

$f(z) = z - k^2\left(-\frac{1}{z}\right) + c$

$$\boxed{f(z) = \left(z + \frac{k^2}{z}\right) + C}$$

1) Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic funⁿ $f(z)$.

>> $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$; $u_y = -6xy - 6y$

$u_{xx} = 6x + 6$; $u_{yy} = -6x - 6$

$\therefore u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$

Thus u is harmonic.

Now consider the C-R eqⁿ

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Substitute u_x and u_y

$\therefore \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$; $\frac{\partial v}{\partial x} = -(-6xy - 6y) = 6xy + 6y$

$\partial v = (3x^2 - 3y^2 + 6x)\partial y$; $\partial v = (6xy + 6y)\partial x$

Int

$v = \int (3x^2 - 3y^2 + 6x)\partial y + f(x)$; $v = \int (6xy + 6y)\partial x + g(y)$

$v = 3x^2y - \frac{3y^3}{3} + 6xy + f(x)$; $v = \frac{6x^2}{2}y + 6xy + g(y)$

$v = 3x^2y - y^3 + 6xy + f(x)$; $v = 3x^2y + 6xy + g(y)$

Now $f(x) = 0$

$g(y) = -y^3$

Thus $v = 3x^2y - y^3 + 6xy$

The analytic fun is $f(z) = u+iv$

$\therefore f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy)$

put $x = z,\ y = 0$

$\therefore \boxed{f(z) = z^3 + 3z^2 + 1}$

2) Determine which of the foll^ng fun is harmonic. Find the conjugate harmonic fun and express $u+iv$ as an analytic fun of z.

a) $v = \log\sqrt{x+y}$. b) $v = \cos x \sinh y$.

>> $v = \log(x+y)^{1/2} = \frac{1}{2}\log(x+y)$

$v_x = \frac{1}{2}\cdot\frac{1}{x+y}$. $v_y = \frac{1}{2}\cdot\frac{1}{x+y}$

$v_{xx} = -\frac{1}{2(x+y)^2}$ $v_{yy} = \frac{1}{2}\cdot -\left(\frac{1}{(x+y)^2}\right)$

Now $v_{xx} + v_{yy} = -\frac{1}{2}\cdot\frac{1}{(x+y)^2} - \frac{1}{2}\frac{1}{(x+y)^2} = -\frac{1}{(x+y)^2}$

$\therefore v_{xx} + v_{yy} \neq 0$

Thus v is not harmonic

b) $v = \cos x \sinh y$

$v_x = -\sin x \sinh y$ $v_y = \cos x \cosh y$

$v_{xx} = -\cos x \sinh y$ $v_{yy} = \cos x \sinh y$

$\therefore v_{xx} + v_{yy} = 0$

This v is harmonic

∴ To find the harmonic conjugate, we consider C-R eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Sub $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

∴ $\cancel{\frac{\partial u}{\partial x}} = -$

$$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial u}{\partial y} = -(-\sin x \sinh y)$$

$$\partial u = (\cos x \cosh y)\partial x \qquad \partial u = (\sin x \sinh y)\partial y$$

$$u = \sin x \cosh y + f(y) \qquad u = \sin x \cosh y + g(x)$$

Now $f(y) = g(x) = 0$

∴ $u = \sin x \cosh y$

Thus the analytic fun is $f(z) = u + iv$.

$$f(z) = \sin x \cosh y + i(\cos x \sinh y)$$

put $x = z,\ y = 0$

$$\therefore \boxed{f(z) = \sin z}.$$

H.W 1) ST $u = e^x(x\cos y - y\sin y)$ is harmonic. & find its harmonic conjugate. Also determine the corresponding analytic fun.

2) $u = x^2 + 4x - y^2 + 2y$

3) S.T $u=\left(r+\frac{1}{r}\right)\cos\theta$ is harmonic, Find its harmonic conjugate and also the corresponding analytic fun

>> $u=\left(r+\frac{1}{r}\right)\cos\theta$

we shall S.T $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$ —— ①

$u_r=\left(1-\frac{1}{r^2}\right)\cos\theta$ $\qquad u_\theta=-\left(r+\frac{1}{r}\right)\sin\theta$

$u_{rr}=\frac{2}{r^3}\cos\theta$ $\qquad u_{\theta\theta}=-\left(r+\frac{1}{r}\right)\cos\theta$

① =>

$\frac{2}{r^3}\cos\theta+\frac{1}{r}\left(1-\frac{1}{r^2}\right)\cos\theta+\frac{1}{r^2}\left\{-\left(r+\frac{1}{r}\right)\cos\theta\right\}$

$\cancel{\frac{2}{r^3}\cos\theta}+\cancel{\frac{1}{r}\cos\theta}-\cancel{\frac{1}{r^3}\cos\theta}-\cancel{\frac{1}{r}\cos\theta}-\cancel{\frac{1}{r^3}\cos\theta}=0$

∴ u is harmonic.

C-R eqn $\quad u_r=\frac{1}{r}v_\theta$ and $v_r=-\frac{1}{r}u_\theta$

$\Rightarrow r u_r=v_\theta$ and $r v_r=-u_\theta$

$\therefore v_\theta=r\left(1-\frac{1}{r^2}\right)\cos\theta$

$v_\theta=\left(r-\frac{1}{r}\right)\cos\theta$

$v_r=-\frac{1}{r}\left\{-\left(r+\frac{1}{r}\right)\sin\theta\right\}$

$v_r=\frac{1}{r}\left(r+\frac{1}{r}\right)\sin\theta$

$v_r=\left(1+\frac{1}{r^2}\right)\sin\theta$

$\partial v=\left(r-\frac{1}{r}\right)\cos\theta\,\partial\theta+f(r)$

$v=\left(r-\frac{1}{r}\right)\sin\theta+f(r)$

$v=\left(r-\frac{1}{r}\right)\sin\theta+g(\theta)$

Now $f(r)=0$

$g(\theta)=0$

Thus the required harmonic conjugate

$$V = \left(r - \frac{1}{r}\right)\sin\theta$$

Also we have $f(z) = u + iv$

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

put $r = z,\ \theta = 0$

$\boxed{f(z) = \left(z + \frac{1}{z}\right)}$ is the analytic fun.

Miscellaneous problems

op 4) Find the analytic function $f(z) = u + iv$ given

$$u - v = e^x(\cos y - \sin y)$$

>> $u - v = e^x(\cos y - \sin y)$

Diff w.r.t x and y partially

$$u_x - v_x = e^x(\cos y - \sin y) \quad \text{——— (1)}$$

$$u_y - v_y = e^x(-\sin y - \cos y)$$

using C-R eqns for the LHS of this eqn

in the form $u_x = v_y$ and $v_x = -u_y$

$$-v_x - u_x = e^x(-\sin y - \cos y)$$

$$\Rightarrow \quad u_x + v_x = e^x(\sin y + \cos y) \quad \text{——— (2)}$$

let us solve (1) and (2) simultaneously for u_x and v_x

① + ②

$$u_x - v_x + u_x + v_x = e^x(\cos y) - e^x \sin y + e^x \sin y + e^x \cos y$$

$$2u_x = 2\, e^x \cos y \Rightarrow \boxed{u_x = e^x \cos y}$$

① − ②

$$u_x - v_x - u_x - v_x = e^x \cos y - e^x \sin y - e^x \sin y - e^x \cos y$$

$$-2v_x = -2e^x \sin y$$

$$\boxed{v_x = e^x \sin y}$$

Thus the analytic fun $f'(z) = u_x + iv_x$

$\therefore f'(z) = e^x \cos y + i e^x \sin y$

put $x = z,\ y = 0$

$f'(z) = e^z$

$f(z) = \int e^z dz + c$

Thus $\boxed{f(z) = e^z + c}$

5) if $f(z) = u + iv$ is analytic find $f(z)$ if

$u - v = (x-y)(x^2 + 4xy + y^2)$

$\Rightarrow u - v = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$

$u - v = x^3 + 3x^2y - 3xy^2 - y^3$

$u_x - v_x = 3x^2 + 6xy - 3y^2$ —— ①

$u_y - v_y = 3x^2 - 6xy - 3y^2$

But C-R eqⁿ $u_x = v_y$ and $v_x = -u_y$

$$-v_x - u_x = 3x^2 - 6xy - 3y^2$$

$$\Rightarrow u_x + v_x = -3x^2 + 6xy + 3y^2 \quad \text{—— (2)}$$

(1) + (2)

$$u_x - v_x + u_x + v_x = \cancel{3x^2} + 6xy - \cancel{3y^2} - \cancel{3x^2} + 6xy + \cancel{3y^2}$$

$$2u_x = 12xy$$

$$\boxed{u_x = 6xy}$$

(1) − (2)

$$u_x - v_x - u_x - v_x = 3x^2 + 6xy - 3y^2 - (-3x^2 + 6xy + 3y^2)$$

$$-2v_x = 3x^2 + 6xy - 3y^2 + 3x^2 - 6xy - 3y^2$$

$$-2v_x = 6x^2 - 6y^2$$

$$-v_x = \frac{6x^2 - 6y^2}{2} = 3x^2 - 3y^2$$

$$\boxed{v_x = 3(y^2 - x^2)}$$

Now $f'(z) = u_x + i v_x$

$$f'(z) = 6xy + i\{3(y^2 - x^2)\}$$

put $x = z,\ y = 0$

$$f'(z) = 0 + i\,3\{0 - z^2\} = -3iz^2$$

$$f(z) = -3i \int z^2 dz + c$$

$$f(z) = -iz^3 + c$$

Thus $\boxed{f(z) = -iz^3 + c}$

H.W 1) $u+v = x^3 - y^3 + 3xy(x-y)$

2) ~~u+~~ if $f(z) = u(r,\theta) + iv(r,\theta)$

$$u+v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta), \quad r \neq 0$$

6) Find the analytic fun $f(z) = u+iv$, given that

$$u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y} - 2\cos 2x}$$

The given $u+v$ may be rewritten as

$$u+v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$u_x + v_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(0 + 2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$u_x + v_x = \frac{2\cosh 2y \cos 2x - 2\cos^2 2x - 2\sin^2 2x}{(\quad)^2}$$

$$= \frac{2\cosh 2y \cos 2x - 2}{(\quad)^2}$$

$$u_x + v_x = \frac{2(\cosh 2y \cos 2x - 1)}{(\cosh 2y - \cos 2x)^2} \quad \text{--- (1)}$$

$$u_y + v_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y - 0)}{(\quad)^2}$$

$$u_y + v_y = \frac{-2\sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

By C-R eqns $u_x = v_y$ and $v_x = -u_y$

$$\cancel{+v_x \quad u_x} \quad -v_x + u_x = \frac{-2\sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2} \longrightarrow (2)$$

(1) + (2)

$$u_x + v_x - v_x + u_x = \frac{2(\cos 2x\cosh 2y - 1) - 2\sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\cancel{2}u_x = \frac{(\cos 2x\cosh 2y - 1) - \sin 2x(\sinh 2y)}{(\quad '' \quad)^2}$$

(1) − (2)

$$u_x + v_x + v_x - u_x = \frac{2(\cos 2x\cosh 2y - 1) + 2\sin 2x(\sinh 2y)}{(\quad)^2}$$

$$v_x = \frac{(\cos 2x\cosh 2y - 1) + \sin 2x(\sinh 2y)}{(\quad '' \quad)^2}$$

$f'(z) = u_x + iv_x$ put $x = z,\ y = 0$

$$f'(z) = \frac{(\cos 2z - 1) + i(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= -\frac{(1 - \cos 2z)}{(1 - \cos 2z)^2} - i\frac{(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{-1}{(1 - \cos 2z)} - i\frac{1}{(1 - \cos 2z)}$$

$\because \sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$= \frac{-1}{2\sin^2 z} - i\frac{1}{2\sin^2 z}$$

$$= \frac{-1}{2\sin^2 z}(1 + i)$$

$$= -\frac{1}{2}(1 + i)\cdot\operatorname{cosec}^2 z$$

$$f(z) = \frac{-1}{2}(1+i)\int \operatorname{cosec}^2 z\, dz + c$$

$$f(z) = \frac{-1}{2}(1+i)(-\cot z)$$

$$\boxed{f(z) = \frac{(1+i)}{2}\cot z + c}$$

7) Find the analytic fu^n $f(z) = u+iv$, given

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)} \quad \text{and } f\left(\frac{\pi}{2}\right) = 0$$

>> Sol^n

$$u_x - v_x = \frac{\{2(\cos x - \cosh y)\}(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})\{2(-\sin x - 0)\}}{\{2(\cos x - \cosh y)\}^2}$$

$$u_x - v_x = \frac{\cancel{-2\cos x \sin x} + 2\cos^2 x + 2\cosh y \sin x - 2\cosh y \cos x + \cancel{2\cos x \sin x} + 2\sin^2 x - 2e^{-y}}{(\quad)^2}$$

$$= \frac{2(\cos^2 x + \sin^2 x) + 2\cosh y(\sin x - \cos x) - 2e^{-y}\sin x}{4(\cos x - \cosh y)^2}$$

$$u_x - v_x = \frac{\cosh y(\sin x - \cos x) + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \quad \text{—— (1)}$$

$$u_y - v_y = \frac{\{2(\cos x - \cosh y)\}(0 + 0 + e^{-y}) - (\cos x + \sin x - e^{-y})\{2(0 - \sinh y)\}}{\{2(\cos x - \cosh y)\}^2}$$

$$u_y - v_y = \frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

—— (2)

$\therefore$ By C-R eqn's are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

① & ② $\Rightarrow$ LHS

$$u_x + u_y = \frac{\cosh y(\sin x - \cos x) + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2}$$

$$+u_y - u_x = \frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

Now ① + ②

$u_x + u_y + u_y - u_x = 2u_y$ LHS

$$\Rightarrow u_y = \frac{1}{4(\cos x - \cosh y)^2}\left[\{\cosh y(\sin x - \cos x) + 1 - e^{-y}\sin x\}\right.$$
$$\left. + \frac{1}{4(\cos x - \cosh y)^2}\{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y\}\right]$$

① − ②

$u_x + u_y - u_y + u_x = 2u_x$

$$\therefore u_x = \frac{1}{4(\cos x - \cosh y)^2}\left[\{\cosh y(\sin x - \cos x) + 1 - e^{-y}\sin x\}\right.$$
$$\left. - \frac{1}{4(\cos x - \cosh y)^2}\{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y\}\right]$$

Now $f'(z) = u_x + iv_x$ but $v_x = -u_y$

$f'(z) = u_x - iu_y$ put $x = z, y = 0$

$$f'(z) = \frac{1}{4(\cos z - 1)^2}\left[\{1(0 - \cos z) + 1 - 0\} - \{(\cos z - 1) + 0\}\right]$$

$$- i\frac{1}{4(\cos z - 1)^2}\left[\{1(0 - \cos z) + 1 - 0\} + \{(\cos z - 1) + 0\}\right]$$

$$f'(z) = \frac{1}{4(\cos z-1)^2}\left[-\cos z+1-\cos z+1\right]$$

$$-\ i \cdot \frac{1}{4(\cos z-1)^2}\left[-\cos z+1+\cos z-1\right]^{0}$$

$$f'(z) = \frac{1}{4(\cos z-1)^2}\left[2-2\cos z\right]$$

$$= \frac{-2(\cos z-1)}{4(\cos z-1)^2} = \frac{-1}{2(\cos z-1)}$$

$$= -\frac{1}{2}\cdot\frac{1}{(\cos z-1)} = \frac{1}{2}\cdot\frac{1}{(1-\cos z)}$$

$$= \frac{1}{2}\cdot\frac{1}{2\sin^2(z/2)} \qquad 1-\cos\theta = 2\sin^2\theta/2$$

$$f'(z) = \frac{1}{4}\cdot\operatorname{cosec}^2(z/2)$$

$$f(z) = \frac{1}{4}\int \operatorname{cosec}^2(z/2)\,dz + c$$

$$= \frac{1}{4}\cdot\left(\frac{-\cot z/2}{1/2}\right)+c$$

$$\boxed{f(z) = -\frac{1}{2}\cdot\cot\left(\frac{z}{2}\right)+c}$$

also $f(\pi/2)=0$

$$0 = -\frac{1}{2}\cot\left(\frac{\pi}{4}\right)+c$$

$$0 = -\frac{1}{2}(1)+c$$

$$\boxed{c=\frac{1}{2}}$$

Thus $f(z) = -\frac{1}{2}\cot\left(\frac{z}{2}\right)+\frac{1}{2}$

$$\boxed{f(z) = \frac{1}{2}\left[1-\cot(z/2)\right]}$$

S.T $u = e^x(x\cos y - y\sin y)$ is harmonic & find its harmonic conjugate. Also determine the corresponding analytic fun.

>> $u = e^x(x\cos y - y\sin y)$

$u_x = e^x(\cos y - 0) + e^x(x\cos y - y\sin y)$

$\Rightarrow u_x = e^x(\cos y + x\cos y - y\sin y)$

$u_{xx} = e^x(0 + \cos y - 0) + e^x(\cos y + x\cos y - y\sin y)$

$u_{xx} = e^x(2\cos y + x\cos y - y\sin y)$ ——— (1)

$u_y = e^x(-x\sin y - \{y\cos y + \sin y\})$

$u_y = e^x(-x\sin y - y\cos y - \sin y)$

$u_{yy} = e^x(-x\cos y - \{-y\sin y + \cos y\} - \cos y)$

$= e^x(-x\cos y + y\sin y - \cos y - \cos y)$

$u_{yy} = e^x(-x\cos y + y\sin y - 2\cos y)$ ——— (2)

(1) + (2)

$u_{xx} + u_{yy} = \cancel{2e^x\cos y} + \cancel{xe^x\cos y} - \cancel{ye^x\sin y} - \cancel{xe^x\cos y} + \cancel{ye^x\sin y} - \cancel{2e^x\cos y}$

$\therefore u_{xx} + u_{yy} = 0$

$\Longrightarrow$ u is harmonic

To find harmonic conjugate

CR-eq^ns $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

<u>Sub</u> $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$

$$\frac{\partial v}{\partial y} = e^x(\cos y + x\cos y - y\sin y)$$

$$\partial v = e^x(\cos y + x\cos y - y\sin y)\,\partial y$$

$$V = e^x\left[\int \cos y\,\partial y + x\int \cos y\,\partial y - \int y\sin y\,\partial y\right] + f(x)$$

$$V = e^x\left[\sin y + x\sin y - [y(-\cos y) - (-\sin y)]\right] + f(x)$$

$$\because \int \sin x\,dx = -\cos x$$

$$V = e^x\left[\cancel{\sin y} + x\sin y + y\cos y - \cancel{\sin y}\right] + f(x)$$

$$V = e^x(x\sin y + y\cos y) + f(x)$$

$$\frac{\partial v}{\partial x} = -\{e^x(-x\sin y - y\cos y - \sin y)\}$$

$$\partial v = e^x(x\sin y + y\cos y + \sin y)\,\partial x + g(y)$$

$$V = \sin y\int x e^x\partial x + y\cos y\int e^x\partial x + \sin y\int e^x\,\partial x + g(y)$$

$$V = \sin y\,[xe^x - e^x] + y\cos y\,e^x + \sin y\,e^x + g(y)$$

$$V = \cancel{\sin y}\; xe^x\sin y - e^x\cancel{\sin y} + y\cos y\,e^x + e^x\cancel{\sin y} + g(y)$$

$$V = e^x(x\sin y + y\cos y) + g(y)$$

$$f(x) = 0 = g(y)$$

$f(z) = u + iV$ put $x = z, y = 0$

$$f(z) = e^z(z - 0) + i(e^z(0+0))$$

$$\boxed{f(z) = ze^z}$$ is the required analytic function

Q) if $f(z)$ is analytic. S.T

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$$

Soln) Let $f(z) = u+iv$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2+v^2 = \phi$$

To prove that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\phi = 4|f'(z)|^2$

ie $\phi_{xx}+\phi_{yy} = 4|f'(z)|^2$

Consider $\phi = u^2+v^2$

Diff. wrt x partially

$$\phi_x = 2uu_x + 2vv_x = 2[uu_x + vv_x]$$

$$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2] \quad \text{—— (1)}$$

Similarly

$$\phi_{yy} = 2[uu_{yy} + u_y^2 + vv_{yy} + v_y^2] \quad \text{—— (2)}$$

Add (1) + (2)

$$\phi_{xx}+\phi_{yy} = 2[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2]$$

Since $f(z) = u+iv$ is analytic —— (3)

u and v are harmonic.

$u_{xx}+u_{yy}=0$, $v_{xx}+v_{yy}=0$ and C-R eqns

are $u_x = v_y$ and $v_x = -u_y$

$$= 2\left[u \cdot 0 + v \cdot 0 + u_x^2 + v_x^2 + (-v_x)^2 + u_x^2\right]$$

$$= 2\left[u_x^2 + u_x^2 + v_x^2 + v_x^2\right]$$

$$= 2\left[2\{u_x^2 + v_x^2\}\right]$$

$$= 4\left[u_x^2 + v_x^2\right] \quad \text{--- (4)}$$

since $f'(z) = u_x + i v_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$|f'(z)|^2 = u_x^2 + v_x^2$$

$\therefore$ (4) $\Rightarrow$

$$\boxed{\phi_{xx} + \phi_{yy} = 4\,|f'(z)|^2}$$

is the required proof

Q) If $f(z)$ is a regular fun of z. ST

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

proof. Let $f(z) = u+iv$ is analytic

$$|f(z)| = \sqrt{u^2+v^2} = \phi$$

To prove that

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = |f'(z)|$$

ie $\phi_x^2 + \phi_y^2 = |f'(z)|$

~~(1) Diff. w.r.t x~~

~~2ϕ~~ Consider $\phi^2 = u^2+v^2$

Diff wrt x

$$2\phi\,\phi_x = 2uu_x + 2vv_x$$

$$\phi\,\phi_x = uu_x + vv_x \quad \text{——— (1)}$$

Similarly

$$\phi\,\phi_y = uu_y + vv_y \quad \text{——— (2)}$$

Squaring and adding we get

$$\phi^2(\phi_x^2+\phi_y^2) = (uu_x+vv_x)^2 + (uu_y+vv_y)^2$$

$$\phi^2(\phi_x^2+\phi_y^2) = (u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x)$$

$$+ (u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y)$$

Since by the C-R Eqn $u_x = v_y$ and $v_x = -u_y$

RHS of 2nd form

$$\phi^2(\phi_x^2+\phi_y^2) = \left(u^2u_x^2 + v^2v_x^2 + \cancel{2uv\,u_xv_x}\right)$$
$$+\left(u^2u_x^2 + v^2u_x^2 - \cancel{2uv\,u_xv_x}\right)$$

$$\phi^2(\phi_x^2+\phi_y^2) = u^2(u_x^2+v_x^2) + v^2(u_x^2+v_x^2)$$
$$= (u_x^2+v_x^2)(u^2+v^2)$$

But $\phi^2 = u^2+v^2$

$$\phi^2(\phi_x^2+\phi_y^2) = (u_x^2+v_x^2)\,\phi^2$$
$$(\phi_x^2+\phi_y^2) = u_x^2+v_x^2 \quad \text{—— (8)}$$

But $f'(z) = u_x + iv_x$

$$|f'(z)| = \sqrt{u_x^2+v_x^2}$$
$$|f'(z)|^2 = u_x^2+v_x^2$$

(8) $\Longrightarrow$

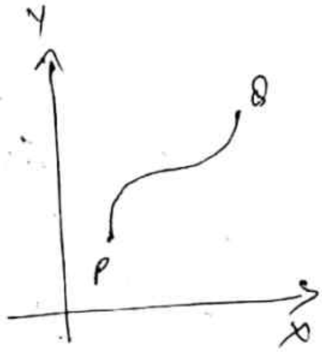
$$\boxed{\phi_x^2+\phi_y^2 = |f'(z)|^2}$$

is called required proof.

Complex Integration

* The complex line integral along the path 'C' equally denoted by $\int_C f(z)dz$.
* If 'C' is a simple closed curve the notation $\oint_C f(z)dz$ is also used.

Properties of complex integral



* If '-C' denotes the curve traversed from Q to P then

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

* If C is split into a no. of parts C_1, C_2, C_3 — then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \text{----}$$

* If λ_1 and λ_2 are constants then

$$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\, dz = \lambda_1 \int_C f_1(z)dz \pm \lambda_2 \int_C f_2(z)dz$$

Line integral of a complex valued function.

Let $f(z) = u(x,y) + iv(x,y)$ be a complex valued funⁿ defined over a region R and C be a curve in the region. Then

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

ie $$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

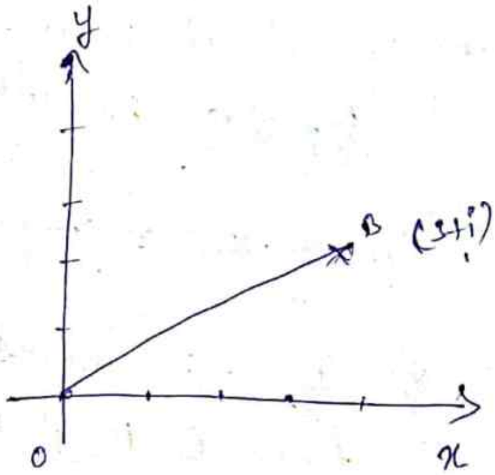
This shows that the evaluation of a line integral of a complex valued funⁿ is nothing but the evaluation of line integrals of real valued functions.

1) Evaluate $\int_C z^2 dz$

a) along the straight line from $z=0$ to $z=3+i$

b) along the curve made up of two line segments, one from $z=0$ to $z=3$ and another from $z=3$ to $z=3+i$.

Sol^n a) $\int_C z^2 dz = \int_{z=0}^{3+i} z^2 dz$



Here z varies from 0 to 3+i means that (x,y) varies from $(0,0)$ to $(3,1)$ The eq^n of the line joining $(0,0)$ and $(3,1)$ is given by

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-0}{x-0} = \frac{1-0}{3-0} \text{ (or) } y = \frac{x}{3} \text{ (or) } \underline{x = 3y}$$

further $z^2 = (x+iy)^2 = x^2 + i^2y^2 + 2ixy = x^2 - y^2 + i(2xy)$

and $dz = dx + i\,dy$

$$\int_C z^2 dz = \int_{(0,0)}^{(3,1)} \{(x^2-y^2) + i(2xy)\}\{dx + i dy\}$$

$$= \int_{(0,0)}^{(3,1)} (x^2-y^2)dx - 2xy\,dy + i\int_{(0,0)}^{(3,1)} \{2xy\,dx + (x^2-y^2)dy\}$$

we have $y = \frac{x}{3}$ (or) $x = 3y$ and we shall convert these integrals into the variable i and integrate w.r.t y from 0 to 1 . we also have $\underline{dx = 3dy}$

$$\therefore \int_C z^2 dz = \int_{y=0}^{1} \{ (9y^2 - y^2)\, 3dy - 2(3y)y\, dy \}$$

$$+ i \int_{y=0}^{1} \{ 2(3y)y\, 3dy + (9y^2 - y^2)\, dy \}$$

$$= \int_{y=0}^{1} (24y^2 - 6y^2)\, dy + i \int_{y=0}^{1} (18y^2 + 8y^2)\, dy$$

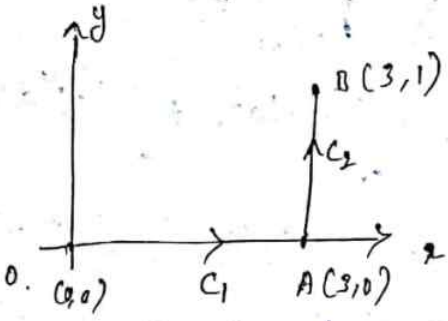
$$= \int_0^1 18y^2 dy + i \int_0^1 26y^2 dy$$

$$= 18 \left[\frac{y^3}{3} \right]_0^1 + 26i \left[\frac{y^3}{3} \right]_0^1$$

$$= 6 + \frac{26}{3} i$$

Thus $\int_C z^2 dz = 6 + \frac{26}{3}i$ along the given path.

b) Segments from $z=0$ to $z=3$ and then from $z=3$ to $3+i$ means that (x, y) varies from $(0,0)$ to $(3,0)$ and then from $(3,0)$ to $(3,1)$ as shown in the fig.



$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz \quad \text{——— (1)}$$

Now along C_1 : $y = 0 \Rightarrow dy = 0$ and

$x \to 0$ to 3 , $z^2 dz \to x^2 dx$

Also along C_2 : $x = 3 \Rightarrow dx = 0$ and $y \to 0$ to 1

$z^2 dz \to (3+iy)^2\, i dy$ $\quad \therefore$

$$(1) \Rightarrow \int_C z^2 dz = \int_{x=0}^{3} x^2 dx + i\int_{y=0}^{1} (3+iy)^2 dy$$

$$= \frac{x^3}{3}\Big|_0^3 + i\int_{y=0}^{1} (9-y^2+6iy)\,dy$$

$$= 9 + i\left[9y - \frac{y^3}{3} + 3iy^2\right]_0^1$$

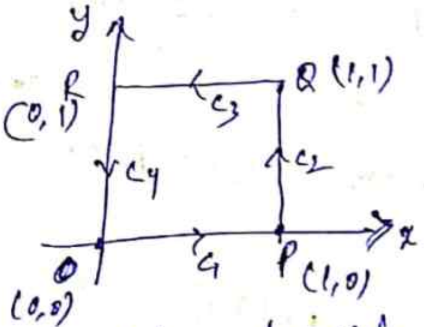
$$= 9 + i\left(9 - \frac{1}{3} + 3i\right)$$

$$= (9-3) + i \cdot \frac{26}{3}$$

Thus $\int_C z^2 dz = 6 + \frac{26}{3}i$ along the given path

2) Evaluate $\int_C |z|^2 dz$ where C is a square with following vertices, $(0,0)\,(1,0)\,,\,(1,1)\,(0,1)$

>> The curve C is as shown in the following fig.



$$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \quad \text{---} (1)$$

we have $|z|^2 dz = (x^2+y^2)\,(dx+idy)$

Along OP C_1, $y=0 \Rightarrow dy=0$, $|z^2|dz = x^2 dx$ where $0 \le x \le 1$

Along PQ C_2, $x=1 \Rightarrow dx=0$, $|z^2|dz = (1+y^2)\,i\,dy$ where $0 \le y \le 1$

Along QR C_3, $y=1 \Rightarrow dy=0$, $|z^2|dz = (x^2+1)\,dx$ where $1 \le x \le 0$

Along RO C_4, $x=0 \Rightarrow dx=0$, $|z^2|\,dz = y^2(i\,dy)$ where $1 \le y \le 0$

using these (1) $\Rightarrow$

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2\,dx + i\int_{y=0}^{1} (1+y^2)\,dy + \int_{x=1}^{0} (x^2+1)\,dx + i\int_{y=1}^{0} y^2\,dy$$

$$= \frac{x^3}{3}\Big|_0^1 + i\left[y+\frac{y^3}{3}\right]_0^1 + \left[\frac{x^3}{3}+x\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= \underline{\underline{-1+i}}$$

Thus $\int_C |z|^2 dz = -1+i$ along the given path.

3) Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along:

a) the line $x=2y$

b) the real axis upto 2 and then vertically to $2+i$.

>> Let $I = \int_0^{2+i} (\bar{z})^2\,dz$

we have $(\bar{z})^2 = (x-iy)^2 = (x^2-y^2) - i(2xy)$ —— (1)

and $dz = dx + i\,dy$ —— (2)

a) Along $x=2y$, $dx = 2dy$

$z=0$ to $2+i$ $\Rightarrow$ (x,y) varies from $(0,0)$ to

$(2,1)$ where $0\le y\le 1$

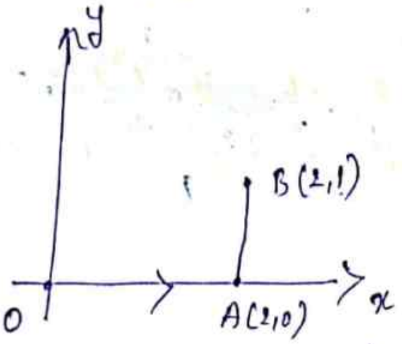
$$\therefore I = \int_{y=0}^{1} \left[(4y^2-y^2) - i\,4y^2\right](2dy+i\,dy)$$

$$= \int_0^1 (3-4i)\,y^2\,(2+i)\,dy$$

$$= \int_0^1 (10-5i)\,y^2\,dy = 5(2-i)\,\frac{y^3}{3}\Big|_0^1 = \frac{5}{3}(2-i)$$

Thus $I = \frac{5}{3}(2-i)$ along the given path.

b) $I = \int_{OA} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 dz$ ---- (3)



Along OA where $O = (0,0)$ and $A = (2,0)$

$y = 0 \Rightarrow dy = 0$ and $0 \le x \le 2$

Along AB where $A = (2,0)$ and $B = (2,1)$

$x = 2 \Rightarrow dx = 0$ and $0 \le y \le 1$

From (1) and (2) we have

along OA, $(\bar{z})^2 dz = x^2 dx$; $0 \le x \le 2$

along AB. $(\bar{z})^2 dz = [(4-y^2) - 4iy]\, i\,dy$; $0 \le y \le 1$

$$\int_{OA} (\bar{z})^2 dz = \int_{x=0}^{2} x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} \quad \text{---} (4)$$

$$\int_{AB} (\bar{z})^2 dz = i \int_{y=0}^{1} [(4-y^2) - 4iy]\, dy$$

$$= i\left[4y - \frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^2}{2}\right]_0^1$$

$$= 2 + \frac{11}{3} i \quad \text{———} (5)$$

using (4) & (5)

eq^n (3) $\Rightarrow$ $I = \frac{8}{3} + \left(2 + \frac{11}{3} i\right)$

Thus $I = \frac{1}{3}(14 + 11\, i)$ along the given path

4) Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along the following path. ④

a) the parabola $x = 2t,\ y = t^2+3$

b) the st line from $(0,3)$ to $(2,4)$

>> a) x varies from 0 to 2 and hence

if $x=0,\ 2t=0 \quad \therefore t=0$

if $x=2,\ 2t=2 \quad \therefore t=1$

$\Rightarrow t \to 0$ to 1

$$I = \int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$$

$$= \int_{t=0}^{1} \{2(t^2+3)+4t^2\}2\,dt + \{3(2t)-(t^2+3)\}2t\,dt$$

$$= \int_0^1 [2(6t^2+6) + (6t-t^2-3)2t]\,dt$$

$$= \int_0^1 (24t^2 - 2t^3 - 6t + 12)\,dt$$

$$= 24\frac{t^3}{3}\Big|_0^1 - 2\frac{t^4}{4}\Big|_0^1 - 6\frac{t^2}{2}\Big|_0^1 + 12t\Big|_0^1$$

$$= 8 - \frac{1}{2} - 3 + 12$$

$$= \frac{33}{2}$$

Thus $I = \frac{33}{2}$ along the given path.

b) Eqn of the st line joining $(0,3)$ and $(2,4)$ is given by $\frac{y-3}{x-0} = \frac{4-3}{2-0}$

ie $\frac{y-3}{x} = \frac{1}{2}$ (or) $x = 2y-6$ hence $dx = 2dy$

Now $I = \int_{y=3}^{4} \{2y + (2y-6)^2\} 2dy + \{3(2y-6) - y\} dy$

$= \int_{3}^{4} \{(4y^2 - 22y + 36)2 + (5y-18)\} dy$

$= \int_{3}^{4} (8y^2 - 39y + 54) dy$

$= \frac{97}{6}$

Thus $I = 97/6$ along the given path.

5) Evaluate $\int_c \bar{z}\, dz$ where c represents the foll^ng paths

a) the straight line from $-i$ to i

b) the right half of the unit circle $|z| = 1$ from $-i$ to i

>> a) $z = x+iy$. $\therefore \bar{z} = x - iy$, $dz = dx + idy$

c is the st line joining the points $(0,-1)$ and $(0,1)$

here $x = 0 \Rightarrow dx = 0$, $y \to -1$ to $+1$.

$\int_c \bar{z}\, dz = \int_{y=-1}^{1} (x-iy)(dx+idy)$

$= \int_{-1}^{1} (-iy)\, i dy = \int_{-1}^{1} y\, dy = \left.\frac{y^2}{2}\right|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

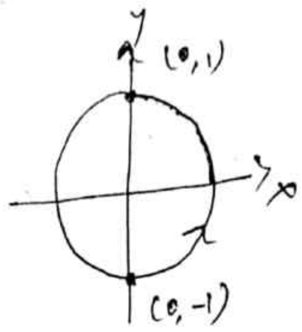
Thus $\int \bar{z} dz = 0$ along the given path

b) The curve C is shown in the foll^{ng} fig. (5)

$C : |z| = 1$. we can take $z = e^{i\theta}$

Also $\bar{z} = e^{-i\theta}$ and $dz = ie^{i\theta}\,d\theta$

from the fig. $y \to -1$ to 1 and $x = 0$



But $x = \cos\theta$, $y = \sin\theta$

$y = -1$ $\sin\theta = -1$ $\therefore \theta = -\pi/2$

$y = +1$. $\sin\theta = 1$ $\therefore \theta = \pi/2$

Now $$\int_C \bar{z}\,dz = \int_{\theta=-\pi/2}^{\pi/2} e^{-i\theta}.ie^{i\theta}d\theta$$

$$= i\int_{-\pi/2}^{\pi/2} 1.d\theta = i[\theta]_{-\pi/2}^{\pi/2} = \pi i$$

Thus $\int_C \bar{z}\,dz = \pi i$ along the given path.

6) if C is a circle with centre 'a' and radius 'r' then S.T

a) $\int_C \frac{dz}{z-a} = 2\pi i$ b) $\int_C (z-a)^n dz = 0$ if $n \neq -1$

(or)

Show that $$\int_C (z-a)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

where C is the circle $|z-a| = r$.

>> on the given circle $|z-a| = r$.

we have $z - a = re^{i\theta}$

hence $dz = ire^{i\theta}\,d\theta$

also $0 \leq \theta \leq 2\pi$

a) $\int_C \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_{\theta=0}^{2\pi} d\theta = i\,\theta\Big|_0^{2\pi} = 2\pi i$

Thus $\int_C \frac{dz}{z-a} = 2\pi i$

b) Also $\int_C (z-a)^n dz = \int_{\theta=0}^{2\pi} (re^{i\theta})^n ire^{i\theta}d\theta$

$$= ir^{n+1}\int_{\theta=0}^{2\pi} e^{i(n+1)\theta}d\theta$$

$$= \cancel{i}r^{n+1}\left[\frac{e^{i(n+1)\theta}}{\cancel{i}(n+1)}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - 1\right]$$

But $e^{i(n+1)2\pi} = \cos(n+1)2\pi + i\sin(n+1)2\pi$

$$= 1 + i.0 = 1$$

$\because \cos 2k\pi = 1$ and $\sin 2k\pi = 0$ for $k = 1, 2, 3 \ldots$

hence $\int_C (z-a)^n dz = \frac{r^{n+1}}{n+1}[1-1] = 0$ when $n \neq -1$

Thus we have proved that,

$$\int_C (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1. \end{cases}$$

Cauchy's theorem

Stmt:- If $f(z)$ is analytic at all points inside and on a simple closed curve C then $\int_C f(z)dz = 0$.

Proof: Let $f(z) = u + iv$

then $\int_C f(z)dz = \int_C (u+iv)(dx+idy)$

ie $\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$ —— ①

we have Green's theorem in a plane stating that if $M(x,y)$ and $N(x,y)$ are two real valued funs having continuous first order p. derivatives in a region R bounded by the curve C then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Applying this theorem to the two line integrals in the RHS of ① we obtain

$$\int_C f(z)dz = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

Since $f(z)$ is analytic, we have C-R eqns.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and hence we have

$$\int_C f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dxdy$$

Thus we get $\int_C f(z)dz = 0$

This proves Cauchy's theorem.

Consequences of Cauchy's theorem

* Stmt 1:- If $f(z)$ is analytic in a region R and if P and Q are any two points in it then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q. That is $\int_P^Q f(z)\,dz$ is same for all curves joining P and Q.

* Stmt 2:- If C_1, C_2 are two simple closed curves such that C_2 lies entirely within C_1 and if $f(z)$ is analytic on C_1, C_2 and in the region bounded by C_1, C_2 then
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

* Stmt 3:- If C is a simple closed curve enclosing non overlapping simple closed curves $C_1, C_2, C_3 \ldots C_n$ and if $f(z)$ is analytic in the annular region b/w C and these curves then
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

Working procedure

* We need to evaluate the integrals of the form $\int_C \frac{f(z)}{z-a}\,dz$; $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$ over a given closed curve C.
* Firstly we have to find out whether the point $z=a$ lies inside (or) outside the given curve C.
* If $z=a$ is inside C then we use Cauchy's integral formula in the forms $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$ and $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$
* If the point $z=a$ is outside C we can conclude that $\int_C f(z)\,dz = 0$ by Cauchy theorem.

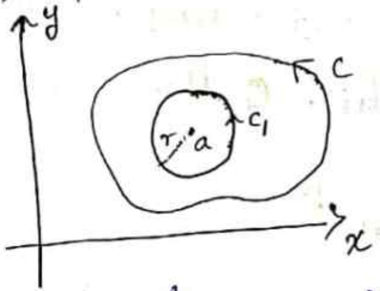
Cauchy's integral formula.

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

Proof: Since 'a' is a point within C, we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C,

The fun $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region b/w C and C_1.



Now, as a consequence of cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}\, dz = \int_{C_1} \frac{f(z)}{z-a}\, dz \quad ---- (1)$$

The eqn of C_1 (circle with centre 'a' and radius r) can be written in the form $|z-a| = r$. That is

$$z-a = re^{i\theta} \quad (\text{or}) \quad z = a+re^{i\theta}$$

$$0 \le \theta \le 2\pi$$

$$dz = ire^{i\theta} d\theta$$

$\therefore (1) \Rightarrow$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}d\theta$$

ie $$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r>0$ however small, hence as $r\to 0$ we get.

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta = i f(a)\,\theta\Big|_0^{2\pi} = 2\pi i f(a)$$

Thus $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$ [Cauchy's integral formula]

Generalized Cauchy's integral formula.

If $f(z)$ is analytic inside and a simple closed curve C and if 'a' is a point within C then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof: we have cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dz \quad \text{---- (1)}$$

Applying Leibnitz rule for diff. under the integral sign we have

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\cdot\frac{\partial}{\partial a}\left[\frac{1}{z-a}\right]dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\cdot\{(-1)\cdot(z-a)^{-2}\cdot(-1)\}\,dz$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2} \quad \text{---- (2)}$$

$\therefore$ Applying L. rule again for (1) we obtain

$$f''(a) = \frac{1!}{2\pi i}\int_C f(z)\cdot\frac{\partial}{\partial a}\left[(z-a)^{-2}\right]dz$$

$$= \frac{1!}{2\pi i}\int_C f(z)\cdot(-2)(z-a)^{-3}(-1)\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

Continuing like this, after diff n times, we get

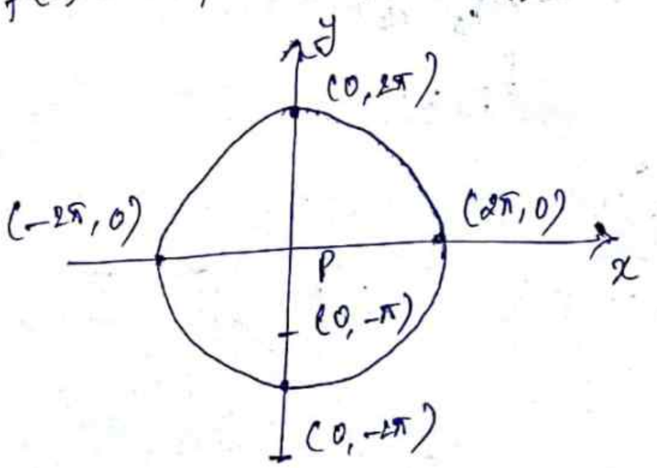
$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Here $f^{(n)}(a)$ denotes the nth derivative of $f(z)$ at $z=a$.

1) Evaluate $\int_C \frac{e^z}{z+i\pi}\,dz$ over each of the foll^{ng} contours C: a) $|z| = 2\pi$ b) $|z| = \pi/2$ c) $|z-1| = 1$

$\Rightarrow$ we have to evaluate the integral which can be written in the form $\int_C \frac{e^z}{z-(-i\pi)}\,dz$ which is of the form $\int_C \frac{f(z)}{z-a}\,dz$

here $f(z) = e^z$, $a = -i\pi$



a) $|z| = 2\pi$ is a circle with centre origin and radius 2π.

The point $z = a = -i\pi$ is the point $(0, -\pi)$ lies within the $\odot^{le}$ $|z| = 2\pi$

we have cauchy's integral formula $\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$

we have $f(z) = e^z$, $a = -i\pi$

$$\therefore \int_c \frac{e^z}{z+i\pi} dz = 2\pi i f(-i\pi) = 2\pi i e^{-i\pi} = 2\pi i(\cos\pi - i\sin\pi)$$
$$= -2\pi i$$

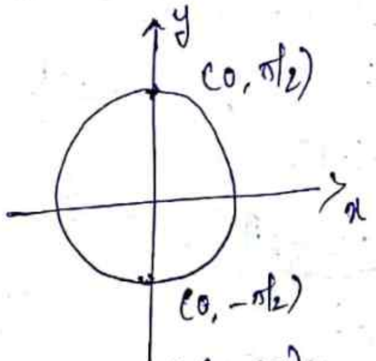
Thus $\int_c \frac{e^z}{z+i\pi} dz = -2\pi i$,

where C is the $\odot^{le}$ $|z| = 2\pi$.

b) $|z| = \pi/2$ is a $\odot^{le}$ with centre origin and radius $\pi/2$,

The point $P(0, -\pi)$ lies outside the $\odot^{le}$ $|z| = \pi/2$

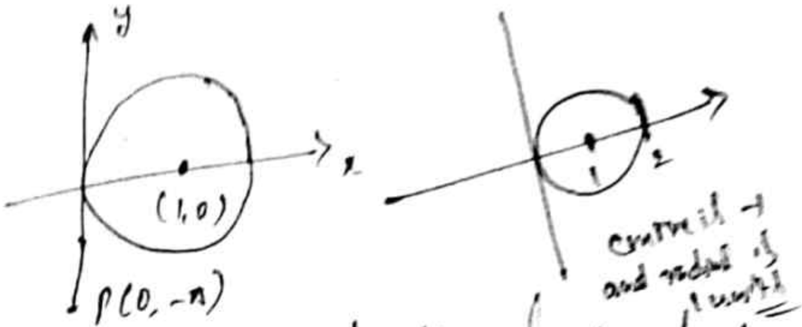
and $\frac{e^z}{z+i\pi}$ is analytic inside and on the $\odot^{le}$ $|z| = \pi/2$.



By Cauchy's theorem

$$\int_c \frac{e^z}{z+i\pi} dz = 0, \quad \text{where } C : |z| = \pi/2$$

c) $|z-1| = 1$ is a cle. with centre at $z = a = 1$ and radius 1. That is a cle. with centre $(1, 0)$ and radius 1.



The point $P(0, -\pi)$ lies outside the cle $|z-1| = 1$ and hence by Cauchy's theorem

$$\int_C \frac{e^z}{z+i\pi}\, dz = 0, \quad \text{where } |z-1| = 1.$$

2) Evaluate $\int_C \frac{dz}{z^2-4}$ over the following curves C.

a) $C : |z| = 1$ b) $C : |z| = 3$ c) $C : |z+2| = 1$

>> Consider $\frac{1}{z^2-4} = \frac{1}{(z^2-2^2)} = \frac{1}{(z+2)(z-2)}$

Resolving into partial fractions,

$$\frac{1}{(z-2)(z+2)} = \frac{A}{(z-2)} + \frac{B}{(z+2)}$$

(or) $1 = A(z+2) + B(z-2)$

putting $z = 2$: $1 = A(4)$ $\therefore A = 1/4$

$z = -2$: $1 = B(-4)$ $\therefore B = -1/4$

Now $\frac{1}{(z-2)(z+2)} = \frac{1}{4} \cdot \frac{1}{z-2} - \frac{1}{4} \frac{1}{z+2}$

$$\therefore \int_C \frac{dz}{(z-2)(z+2)} = \frac{1}{4}\int_C \frac{dz}{z-2} - \frac{1}{4}\int_C \frac{dz}{z-(-2)} \quad \text{—— (1)}$$

a) $c: |z| = 1$;

$\Rightarrow z = a = 2$. and $z = a = -2$ bothe of them lie outside c

Thus by Cauchy's theorem $\int_c \frac{dz}{z^2-4} = 0$ where $c: |z| = 1$

b) $c: |z| = 3$; $z = a = 2$ and $z = a = -2$ lies inside the ○le, Also in each of the integrals as in the RHS of ①,

$f(z) = 1$

Aplying Cauchy's integral formula

$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$ we obtain

$\int_c \frac{dz}{z-2} = 2\pi i f(2) = 2\pi i \cdot (1) = 2\pi i$

$\int_c \frac{dz}{z+2} = 2\pi i f(-2) = 2\pi i \cdot (1) = 2\pi i$

Substituting these in the RHS. of ① we have

$\int_c \frac{dz}{z^2-4} = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) = 0$

Thus $\int_c \frac{dz}{z^2-4} = 0$ where $c: |z| = 3$

c) $c: |z+2| = 1$. This is a ○le with centre $(-2, 0)$ and radius 1.

Let $A = (-2, 0)$ and $P = (2, 0)$ hence $AP = \sqrt{4} = 2 > 1$

$\therefore$ the point $z = a = 2$ lies outside the ○le and clearly the point $z = a = -2$ being $(-2, 0)$ lies inside the ○le.

hence by Cauchy's theorem $\int_c \frac{dz}{z-2} = 0$

$\therefore$ Also by Cauchy's integral formula,

$$\int_C \frac{dz}{z+2} = \int_C \frac{dz}{z-(-2)} = 2\pi i\, f(-2) \text{ where } f(z)=1$$

$$\therefore \int_C \frac{dz}{z+2} = 2\pi i, 1 = 2\pi i$$

Substituting these value in the RHS of (1)

we have,

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}\cdot 0 - \frac{1}{4}\cdot 2\pi i = \frac{-\pi i}{2}$$

Thus $\int_C \frac{dz}{z^2-4} = \frac{-\pi i}{2}$ where $C: |z+2| = 1$

3) Evaluate $\int_C \frac{e^z}{z-i\pi}$ where C is the circle a) $|z| = 2\pi$ b) $|z| = \pi/2$

a) $\int_C \frac{e^z}{z-i\pi} dz = -2\pi i$ for $C: |z| = 2\pi$

b) $\int_C \frac{e^z}{z-i\pi} dz = 0$ for $C: |z| = \pi/2$

Similar to problem (1)

4) Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)} dz$ where C is the circle $|z| = 3$

>> The points $z=a=-1$, $z=a=2$ being $(-1,0)$ $(2,0)$ lies inside $|z| = 3$

Now we shall resolve $\frac{1}{(z+1)(z-2)}$ into p. fractions.

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$

(or) $1 = A(z-2) + B(z+1)$

put $z = 2$, $B = 1/3$

$z = -1$, $A = -1/3$

$$\therefore \int_c \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{1}{3}\left[\int_c \frac{e^{2z}}{z-2}\,dz - \int_c \frac{e^{2z}}{z+1}\,dz\right] \quad \text{--- (1)}$$

we have Cauchy's integral formula.

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

taking $f(z) = e^{2z}$ and $a = 2, -1$ respectively we obtain

$$\int_c \frac{e^{2z}}{z-2}\,dz = 2\pi i f(2) = 2\pi i e^4$$

and $$\int_c \frac{e^{2z}}{z+1}\,dz = 2\pi i f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$$

Substituting these in the RHS of (1) we obtain

$$\int_c \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{1}{3}\left[2\pi i e^4 - \frac{2\pi i}{e^2}\right]$$

Thus $$\int_c \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{2\pi i}{3}\left[e^4 - \frac{1}{e^2}\right]$$

5) Evaluate $\int_c \frac{e^{3z}}{z^2}\,dz$ over $C : |z| = 1$

>> The point $z = 0$ lies within the circle $|z| = 1$ and we have Cauchy's integral formula in the generalized form.

$$\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

taking $f(z) = e^{3z}$, $a = 0$, $n = 1$ in this formula we obtain

$$\int_c \frac{e^{3z}}{z^2}\,dz = \frac{2\pi i}{1!} f'(a) \;; \quad \text{also } f'(z) = 3e^{3z}$$

$$\therefore \int_c \frac{e^{3z}}{z^2}\,dz = 2\pi i\,(3e^0) = 2\pi i(3) = 6\pi i$$

Thus $$\int_c \frac{e^{3z}}{z^2}\,dz = 6\pi i$$

6) Evaluate $\int_C \frac{z^2+z+1}{(z-2)^3}\, dz$ over $C: |z|=3$

>> The point $z=2$ lies inside the circle $|z|=3$

we have generalised Cauchy's integral formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^n(a)$$

taking $f(z) = z^2+z+1$, we obtain $f''(z) = 2$.

$\therefore f''(2) = 2$ also by taking $a=2$, $n=2$ we have

$$\int_C \frac{z^2+z+1}{(z-2)^3}\, dz = \frac{2\pi i}{2!} f''(2) = \frac{2\pi i}{2}.2 = 2\pi i$$

Thus $\int_C \frac{z^2+z+1}{(z-2)^3}\, dz = 2\pi i$

7) Evaluate $\int_C \frac{e^{\pi z}}{(2z-i)^3}\, dz$ where C is the circle $|z|=1$

>> we can write the given integral in the form

$$\int_C \frac{e^{\pi z}}{[2(z-i/2)]^3} = \frac{1}{8}\int_C \frac{e^{\pi z}}{(z-i/2)^3}$$

The point $z=i/2$ being $(0, 1/2)$ lies within the circle $|z|=1$. we have generalised Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^n(a)$$

taking $f(z) = e^{\pi z}$, $a = i/2$, $n=2$ we have

$$\int_C \frac{e^{\pi z}}{(z-i/2)^3}\, dz = \frac{2\pi i}{2!} f''(i/2) = \pi i f''(2)$$

x'ly by $1/8$ we have

$$\frac{1}{8}\int_C \frac{e^{\pi z}}{(z-i/2)^3}\, dz = \frac{1}{8}.\pi i. f''(i/2) \;; \quad \text{But } f''(z) = \pi^2 e^{\pi z}$$

$$\int_C \frac{e^{\pi z}}{(2z-i)^3}\, dz = \frac{\pi i}{8}.\pi^2 e^{\pi i/2}$$

$$= \frac{\pi^3 i}{8}\left(\cos \pi/_2 + i \sin \pi/_2\right)$$

$$= \frac{\pi^3 i}{8}\left(0 + i(1)\right)$$

$$= \frac{\pi^3}{8} i^2 = \frac{-\pi^3}{8} \qquad \because i^2 = -1$$

Thus $\displaystyle\int_C \frac{e^{\pi z}}{(2z-i)^3}\, dz = \frac{-\pi^3}{8}$

8) Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\, dz$ where $C : |z| = 3$

>> we shall first resolve $\dfrac{1}{(z+1)^2(z-2)}$ into p. fractions

Let $\dfrac{1}{(z+1)^2(z-2)} = \dfrac{A}{(z+1)} + \dfrac{B}{(z+1)^2} + \dfrac{C}{(z-2)}$

(or) $1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

put $z = -1$, $\quad B = -1/3$

$z = 2 \quad \therefore C = 1/9$

$z = 0 \quad \cdot\ A = -1/9$

Now

$$\frac{1}{(z+1)^2(z-2)} = -\frac{1}{9}\cdot\frac{1}{z+1} - \frac{1}{3}\cdot\frac{1}{(z+1)^2} + \frac{1}{9}\cdot\frac{1}{z-2}$$

x'ly by e^{2z} and integrating w.r.t z over C we have

$$\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\, dz = -\frac{1}{9}\int_C \frac{e^{2z}}{(z+1)}\, dz - \frac{1}{3}\int_C \frac{e^{2z}}{(z+1)^2}\, dz + \frac{1}{9}\int_C \frac{e^{2z}}{z-2}\, dz \quad (1)$$

The points $z = a = -1$, $z = a = 2$ lies inside the cle $|z| = 3$

we shall consider Cauchy's integral formula in the forms ⑫

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a) \quad \text{and} \quad \int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^n(a)$$

taking $f(z) = e^{2z}$ we obtain $f'(z) = 2e^{2z}$

Now $\displaystyle\int_C \frac{e^{2z}}{z+1}dz = \int_C \frac{e^{2z}}{z-(-1)}dz = 2\pi i f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$

$$\int_C \frac{e^{2z}}{(z+1)^2}dz = \int_C \frac{e^{2z}}{[z-(-1)]^2}dz = \frac{2\pi i}{1!}f'(-1) = 2\pi i(2e^{-2})$$

ie $\displaystyle\int_C \frac{e^{2z}}{(z+1)^2}dz = \frac{4\pi i}{e^2}$

Also $\displaystyle\int_C \frac{e^{2z}}{z-2} = 2\pi i f(2) = 2\pi i \cdot e^4$

Substituting these in the RHS of eqn ①

$$\int_C \frac{e^{2z}}{(z+1)^2(z-2)}dz = -\frac{1}{9}\cdot\frac{2\pi i}{e^2} - \frac{1}{3}\cdot\frac{4\pi i}{e^2} + \frac{1}{9}2\pi i e^4$$

$$= \frac{-7}{9}\frac{2\pi i}{e^2} + \frac{2\pi i}{9}e^4$$

Thus $\displaystyle\int_C \frac{e^{2z}}{(z+1)^2(z-2)}dz = \frac{2\pi i}{9}\left(e^4 - \frac{7}{e^2}\right)$

9) Evaluate $\int_C \frac{dz}{(z^2+4)^2}$ where $C: |z-i| = 2$, by cauchy's integral formula.

>> $C: |z-i| = 2$ is a circle with centre $(0,1)$ and radius 2.

we have $\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

Let $A = (0,1)$ be the centre and $r = 2$ be the radius of C.

if $P_1 = (0,-2)$ and $P_2 = (0,2)$ then $AP_1 = 3 > 2$ and $AP_2 = 1 < 2$

Hence $(0,2)$ (or) $z = 2i$ only lies inside C.

we have cauchy's integral formula in the form

$$f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz \quad \text{——— (1)}$$

Now $\frac{1}{(z^2+4)^2} = \frac{1}{[(z+2i)(z-2i)]^2} = \frac{1/(z+2i)^2}{(z-2i)^2}$.

Taking $f(z) = \frac{1}{(z+2i)^2}$ and $a = 2i$ we have

$$f'(z) = \frac{-2}{(z+2i)^3} \quad ; \quad f'(a) = f'(2i) = \frac{-2}{(4i)^3} = \frac{1}{32i}$$

(1) =>

$$\frac{1}{32i} = \frac{1}{2\pi i} \int_C \frac{1/(z+2i)^2}{(z-2i)^2}\, dz$$

ie $$\frac{\pi}{16} = \int_C \frac{dz}{(z+2i)^2(z-2i)^2}$$

Thus $$\int_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

10) Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\,dz$ where C is the circle (13)

i) $|z| = 3$, ii) $|z| = 1/2$ iii) $|z| = 3/2$

>> we shall first resolve $\frac{1}{(z-1)^2(z-2)}$ by P. fractions

Let $\frac{1}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$ ----- (1)

(or) $1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$

put $z=1$ $\therefore B = -1$

$z=2$ $C = 1$

Eq. z^2 on. b. s $0 = A + C$ or $A = -C$ $A = -1$

Let $f(z) = \sin\pi z^2 + \cos\pi z^2$

x'y (1) by $f(z)$ and int w.r.t z over C by using the value of the constants obtained we have.

$$I = \int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = -\int_C \frac{f(z)}{z-1}\,dz - \int_C \frac{f(z)}{(z-1)^2}\,dz + \int_C \frac{f(z)}{z-2}\,dz \quad \text{--- (2)}$$

$\Rightarrow I = I_1 + I_2 + I_3$ (say)

Case i) $C : |z| = 3$

The points $z=1$ and $z=2$ both lie within C.

Hence by Cauchy's integral formula.

$I_1 = -[2\pi i f(1)] = -2\pi i[\sin\pi + \cos\pi] = -2\pi i(0-1) = 2\pi i$

$I_2 = -[2\pi i f'(1)]$ But $f'(z) = 2\pi z(\cos\pi z^2 - \sin\pi z^2)$

Hence $I_2 = -[2\pi i \cdot 2\pi(\cos\pi - \sin\pi)] = 4\pi^2 i$

$I_3 = 2\pi i f(2) = 2\pi i [\sin 4\pi + \cos 4\pi] = 2\pi i (0+1) = 2\pi i$

Hence from (2),

$$I = 2\pi i + 4\pi^2 i + 2\pi i = 4\pi i + 4\pi^2 i$$

Thus $\underline{I = 4\pi i (1+\pi)}$ where $C: |z| = 3$

Case ii) $C: |z| = 1/2$

The points $z=1$ and $z=2$ both lie outside C and hence $I_1 = 0 = I_2 = I_3$

Thus $I = 0$, where $C: |z| = 1/2$

Case iii) $C: |z| = 3/2$

The points $z=1$ lies inside C and $z=2$ lies outside C.

Hence $I_1 = 2\pi i f(1) = 2\pi i$

$I_2 = 2\pi i f'(1) = 4\pi^2 i$

and $I_3 = 0$

Now $I = 2\pi i + 4\pi^2 i + 0$

$= 2\pi i (1 + 2\pi)$

Thus $\underline{I = 2\pi i (1+2\pi)}$

where $C: |z| = 3/2$

11) Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ where C is the circle $|z|=1$ (14)

we have. $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$ ——— (1)

The point $z = a = \pi/6 \simeq 0.5$ lies within the circle $|z|=1$

Now by putting $n=2$ in (1) we have

$$f^{(2)}(a) = f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

taking $f(z) = \sin^6 z$ we have with $a=\pi/6$

$$f''(\pi/6) = \frac{1}{\pi i}\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz \quad \text{——— (2)}$$

Consider $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z\cos z$;

$f''(z) = -6\sin^6 z + 30\sin^4 z\cos^2 z$

Now $f''(\pi/6) = -6\sin^6(\pi/6) + 30\sin^4(\pi/6)\cos^2(\pi/6)$

ie $f''(\pi/6) = -6\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2$

$$= -\frac{6}{64} + \frac{90}{64}$$

$$= \frac{84}{64}$$

$$= \frac{21}{16}$$

Thus by sub. this value in (2) we have

$$\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{21\pi i}{16}$$

Singularity and Residue

* A point $z=a$ where $f(z)$ fails to be analytic is called a singularity or a singular point of $f(z)$.

* A point $z=a$ is called an isolated singularity of $f(z)$ if there exists a neighbourhood of a point 'a' which enclose no other singularities of $f(z)$.

Examples.

1) If $f(z) = \frac{z}{z-2}$ then $f(z)$ is not analytic at $z=2$ which is called the singular point of $f(z)$.

2) If $f(z) = \frac{z^2}{(z-1)(z+1)(z-2)}$ then the points $z=1$, $z=-1$, $z=2$ are ~~all~~ called singular points of $f(z)$.

It may be noted that the singular points of $f(z)$ are identified from the factors present in the denominator of $f(z)$ and the singular points are the points which make these factors zero.

Suppose $f(z)$ is expanded as a Laurent series about the point $z=a$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n} \quad \text{--- (1)}$$

then the first term is called the analytic part of $f(z)$ and the 2nd term is called the principal part of $f(z)$. if the principal part of $f(z)$ consists of only a finite no of terms, say m, then we say that $z=a$ is a pole of order m, In particular a pole of order 1 $(m=1)$ is called a simple pole

If the principal part of $f(z)$ at $z=a$ contain an infinite no of terms then $z=a$ is called an essential singularity of $f(z)$. Also if the principal part of $f(z)$ is completely absent (ie $a_{-n}=0$) then $z=a$ is called a removable singularity of $f(z)$.

Examples.

1) If $f(z) = \dfrac{z^2}{(z-1)(z+1)^2(z-2)}$ then $z=1,2$ are poles of order 1 (simple poles) and $z=-1$ is a pole of order 2.

2) If $f(z) = \dfrac{e^z}{z^3(z^2+1)}$ then $z=0$ is a pole of order 3 and solving $z^2+1=0$ we get $z=\pm i$ which are simple poles.

3) If $f(z) = \dfrac{z+1}{(z^2+1)^2(4z^2-1)}$ then $z=\pm i$ are poles of order 2 and $z=\pm 1/2$ are simple poles.

Residues

The coefficient of $\frac{1}{z-a}$ that is a_{-1} in the expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z=a$.

Formula for the residue at the pole.

If $z=a$ is a pole of order m of $f(z)$ then the residue of $f(z)$ at $z=a$ is denoted by $R[m,a]$ and is given by

$$R[m,a] = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

Cauchy's Residue Theorem.

Stmt :- If $f(z)$ is analytic inside and on the boundary of a simple closed curve C except for a finite numbers of poles $a, b, c, \ldots$ then the integral of $f(z)$ over C is equal to $2\pi i$ times the sum of the residues at the poles inside C. That is

$$\int_C f(z)\,dz = 2\pi i \left(a_{-1} + b_{-1} + c_{-1} + \cdots \right)$$

- working procedure for problems to find $\int_c f(z)dz$ by using Cauchy's residue theorem

→ we locate all the poles of $f(z)$ along with their orders by looking at the denominator of the given $f(z)$.

→ we identify the poles lying inside c.

→ we compute the residue for these poles using appropriate formula.

→ Finally we apply Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i \sum R$$

where $\sum R$ denote the sum of the residue at the poles lying in C.

1). Find the residues of the fu^n

$$f(z) = \frac{z}{(z+1)(z-2)^2} \text{ at i) } z=-1 \text{ ii) } z=2$$

>> $z=-1$ is a pole of order 1 (simple pole) and $z=2$ is a pole of order 2.

The residue of $f(z)$ for a pole of order m at $z=a$ is given by

$$R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

case i) Residue at $z=a=-1$ is given by $m=1$, $0!=1$

$$\lim_{z\to -1} \cancel{(z+1)} \cdot \frac{z}{\cancel{(z+1)}(z-2)^2}$$

$$= \lim_{z\to -1} \frac{z}{(z-2)^2} = \frac{-1}{(-1-2)^2} = \frac{-1}{(-3)^2} = \frac{-1}{9}$$

case ii) Residue at $z=a=2$ where $m=2$ is given by

$$\lim_{z\to 2} \frac{1}{1!} \frac{d}{dz}\left\{(z-2)^2 \frac{z}{(z+1)(z-2)^2}\right\}$$

$$= \lim_{z\to 2} \frac{d}{dz}\left(\frac{z}{z+1}\right)$$

$$= \lim_{z\to 2} \frac{(z+1)-z}{(z+1)^2} = \lim_{z\to 2} \frac{1}{(z+1)^2}$$

$$= \lim_{z\to 2} \frac{1}{(z+1)^2} = \frac{1}{9}$$

Thus the required residues are $-\frac{1}{9}$ and $\frac{1}{9}$.

2) For the fu^n $f(z) = \frac{2z+1}{z^2-z-2}$ determine the poles and the residue at the poles.

>> In $f(z) = \frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z-2)(z+1)}$

$z=2$, $z=-1$ are simple poles.

i) Residue at $z=a=2$ is given by

$$\lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \cancel{(z-2)} \cdot \frac{2z+1}{\cancel{(z-2)}(z+1)}$$

$$= \lim_{z\to 2} \frac{2z+1}{z+1} = \frac{5}{3}$$

ii) Residue at $z=a=-1$ is given by

$$\lim_{z\to -1} (z+1) f(z) = \lim_{z\to -1} \cancel{(z+1)} \frac{2z+1}{(z-2)\cancel{(z+1)}}$$

$$= \lim_{z\to -1} \frac{2z+1}{(z-2)}$$

$$= \frac{2(-1)+1}{-1-2}$$

$$= \frac{-2+1}{-3}$$

$$= \frac{-1}{-3}$$

$$= \frac{1}{3}$$

Thus the residue at the poles are 5/3 and 1/3

3) Determine the residue at the pole of the fu^n $\frac{\sin z}{(2z-\pi)^2}$

>> Let $f(z) = \frac{\sin z}{(2z-\pi)^2}$

Now $2z-\pi = 0$

$\Rightarrow 2z = \pi \Rightarrow z = \pi/2$

$\therefore z = a = \pi/2$ is a pole of order 2.

The residue of $f(z)$ at $z = a = \pi/2$ $(m=2)$ is given by

$$\lim_{z\to\pi/2} \frac{1}{1!} \frac{d}{dz}\left\{ (z-\pi/2)^2 \cdot \frac{\sin z}{(2z-\pi)^2} \right\}$$

$$= \lim_{z\to\pi/2} \frac{d}{dz}\left\{ \frac{\cancel{(2z-\pi)^2}}{2^2} \cdot \frac{\sin z}{\cancel{(2z-\pi)^2}} \right\}$$

$$= \lim_{z\to\pi/2} \frac{1}{4} \frac{d}{dz}(\sin z)$$

$$= \frac{1}{4} \lim_{z\to\frac{\pi}{2}} \cos z$$

$$= \frac{1}{4} \cos\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{4}(0)$$

$$= 0$$

Thus the residue at the pole is 0.

4) Determine the residue at the poles for the fu^n $f(z) = \frac{z}{(z+1)^2(z^2+4)}$

>> $z = -1$ is a pole of order 2.

Also, $(z^2+4) = 0 \Rightarrow (z+2i)(z-2i) = 0$

$\therefore z = 2i, -2i$ are simple poles.

Let $R[m, a]$ denote the residue of $f(z)$ at $z = a$ for a pole of order m and we have

$$* \; R[2,-1] = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} \left\{ (z+1)^2 \frac{z}{(z+1)^2 (z^2+4)} \right\}$$

$$= \lim_{z \to -1} \frac{d}{dz} \left\{ \frac{z}{z^2+4} \right\}$$

$$= \lim_{z \to -1} \frac{(z^2+4)1 - z(2z)}{(z^2+4)^2}$$

$$R[2,-1] = \lim_{z \to -1} \frac{4-z^2}{(z^2+4)^2} = \frac{4-1}{(1+4)^2} = \underline{\frac{3}{25}}$$

$$* \; R[1, 2i] = \lim_{z \to 2i} (z-2i) \frac{z}{(z+1)^2 (z^2+4)}$$

$$= \lim_{z \to 2i} \cancel{(z-2i)} \frac{z}{(z+1)^2 (z+2i) \cancel{(z-2i)}}$$

$$= \lim_{z \to 2i} \frac{z}{(z+1)^2 (z+2i)}$$

$$= \frac{2i}{(2i+1)^2 4i}$$

$$= \frac{1}{2} \frac{1}{4i^2 + 1 + 4i} = \frac{1}{2} \cdot \frac{1}{4i-3} \times \frac{4i+3}{4i+3}$$

$$= \frac{1}{2} \cdot \frac{4i+3}{(4i-3)(4i+3)} = \frac{1}{2} \cdot \frac{4i+3}{16i^2 - 9} \qquad i^2 = -1$$

$$R[1, 2i] = \underline{\frac{-1}{50}(4i+3)}.$$

Also $R[1, -2i] = \lim_{z \to -2i} \cancel{(z+2i)} \cdot \frac{z}{(z+1)^2 \cancel{(z+2i)} (z-2i)}$

$$= \lim_{z \to -2i} \frac{z}{(z+1)^2 (z-2i)}$$

$$= \frac{-2i}{(1-2i)^2(-4i)}$$

$$= \frac{1}{2} \cdot \frac{1}{1+4i^2-4i} = \frac{1}{2} \cdot \frac{1}{-3-4i}$$

$$= -\frac{1}{2} \cdot \frac{(3-4i)}{(3-4i)(3+4i)}$$

$$= -\frac{1}{2} \cdot \frac{3-4i}{25} = \frac{4i-3}{50}$$

$$R[1, -2i] = \frac{4i-3}{50}$$

5) Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)} dz$ where C is the circle $|z| = 3$

>> The poles of the fun $f(z) = \frac{e^{2z}}{(z+1)(z-2)}$

are $z = -1$, $z = 2$ which are simple poles and both these lie within the circle $|z| = 3$.

$\therefore$ residue of $f(z)$ at $z = a = -1$ is given by

$$\lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} (z+1) \frac{e^{2z}}{(z+1)(z-2)}$$

$$= \lim_{z \to -1} \frac{e^{2z}}{(z-2)} = \frac{e^{-2}}{-3} = \frac{-1}{3e^2} = R_1$$

∴ Also residue of $f(z)$ at $z = a = 2$ is given by

$$\lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} (z-2) \frac{e^{2z}}{(z+1)(z-2)}$$

$$= \lim_{z\to 2} \frac{e^{2z}}{z+1}$$

$$= \frac{e^4}{3} = R_2$$

we have Cauchy's Residue theorem

$$\int_c f(z)\,dz = 2\pi i [R_1 + R_2]$$

Thus $$\int_c \frac{e^{2z}}{(z+1)(z-2)} dz = 2\pi i \left(\frac{-1}{3e^2} + \frac{e^4}{3}\right)$$

$$= \frac{2\pi i}{3}\left(e^4 - \frac{1}{e^2}\right)$$

6) Evaluate $\int_c \frac{(z^2+5)}{(z-2)(z-3)} dz$ using residue theorem.

$C : |z| = 4$

>> The poles of the fun $f(z) = \frac{z^2+5}{(z-2)(z-3)}$

are $z=2$, $z=3$ and both the poles lie

within the ⊙le $|z| = 4$

∴ residue at $z = 2 = a$ which is a simple

pole is given by

$$\lim_{z \to 2} (z-2) f(z) = \lim_{z \to 2} \cancel{(z-2)} \frac{z^2+5}{\cancel{(z-2)}(z-3)}$$

$$= \lim_{z \to 2} \frac{z^2+5}{z-3}$$

$$= \frac{2^2+5}{z-3} = -9 = \underline{\underline{R_1}}$$

Similarly residue at $z = a = 3$ is given by

$$\lim_{z \to 3} (z-3) f(z) = \lim_{z \to 3} (z-3) \frac{z^2+5}{(z-2)(z-3)}$$

$$= \lim_{z \to 3} \frac{z^2+5}{z-2}$$

$$= \frac{3^2+5}{3-2} = 14 = R_2 \text{ (say)}$$

we have by Cauchy's residue theorem

$$\int_c f(z)\, dz = 2\pi i \left[R_1 + R_2 \right]$$

$$= 2\pi i \left[-9 + 14 \right]$$

$$= 2\pi i \left[5 \right]$$

$$= \underline{\underline{10\pi i}}$$

Thus $\displaystyle \int_c \frac{z^2+5}{(z-2)(z-3)} dz = \underline{10\pi i}$

7) Evaluate $\int_C \frac{dz}{z^3(z-1)}$ where C is the circle $|z|=2$.

>> Let $f(z) = \frac{1}{z^3(z-1)}$ and the poles of $f(z)$ are $z=0$, $z=1$. Both the poles lie within $|z|=2$

$\therefore$ residue at $z=a=0$, being a pole of order 3 $(m=3)$ is given by

$$R_1 = \lim_{z\to 0} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left\{(z-0)^3 \frac{1}{z^3(z-1)}\right\}$$

$$= \lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2}\left\{\frac{1}{z-1}\right\}$$

$$= \frac{1}{2}\lim_{z\to 0} \frac{2}{(z-1)^3} = \frac{-2}{2} = -1$$

Also residue at $z=a=1$, being a simple pole is given by

$$R_2 = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\cancel{(z-1)}\cdot\frac{1}{z^3\cancel{(z-1)}}$$

$$= \lim_{z\to 1}\frac{1}{z^3}$$

$$= \frac{1}{1} = 1$$

$$\int_C f(z)\,dz = 2\pi i\left[R_1+R_2\right]$$

$$= 2\pi i\left[-1+1\right]$$

$$= 0$$

Thus $\int_C \frac{dz}{z^3(z-1)} = 0$

8). Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where $C: |z|=3$

>> Let $f(z) = \frac{e^{2z}}{(z+1)^4}$

$z=-1$ is a pole of order 4 $(m=4)$ which lies inside $C: |z|=3$

$\therefore$ The residue of $f(z)$ at $z=a=-1$ is given by

$$\lim_{z\to -1} \frac{1}{(4-1)!} \frac{d^3}{dz^3}\left\{ \cancel{(z+1)^4} \frac{e^{2z}}{\cancel{(z+1)^4}} \right\}$$

$$= \lim_{z\to -1} \frac{1}{3!} \frac{d^3}{dz^3}\left\{ e^{2z} \right\}$$

$$= \lim_{z\to -1} \frac{1}{3!} (8e^{2z})$$

$$= \frac{1}{6} 8e^{-2}$$

$$= \frac{4}{3}e^{-2}$$

Applying cauchy's residue theorem we have

$$\int_C f(z)\,dz = 2\pi i\left[\frac{4}{3}e^{-2}\right] = \frac{8\pi i}{3e^2}$$

9) Using Cauchy's residue theorem evaluate

$\int_C \frac{z\cos z}{(z-\pi/2)^3} dz$ where $C: |z-1|=1$

>> Let $f(z) = \frac{z\cos z}{(z-\pi/2)^3}$ $\quad C: |z-1|=1$

here $z=\pi/2$ is a pole of order 3.

C is the circle with centre at the point $P(1,0)$ and radius 1.

Let $z = \pi/2$ be the point $Q(\pi/2, 0)$

Distance $PQ = (\pi/2, -1) < 1$ and hence

$z = \pi/2$ lies within the given circle C

$\therefore$ The residue (R) at $z = \pi/2$ is given by

$$\lim_{z \to \pi/2} \frac{1}{(3-1)!} \frac{d^2}{dz^2} \left\{ (z-\pi/2)^3 \frac{z \cos z}{(z-\pi/2)^3} \right\}$$

$$R = \lim_{z \to \pi/2} \frac{1}{2} \frac{d^2}{dz^2} \{z \cos z\}$$

$$R = \lim_{z \to \pi/2} \frac{1}{2} \{-z \cos z - 2 \sin z) = -1$$

hence Cauchy's residue Theorem

$$\int_C f(z)\, dz = 2\pi i (R)$$

Thus $\displaystyle\int_C \frac{z \cos z}{(z-\pi/2)^3} dz = -2\pi i$

Bilinear Transformation (BLT)

The transformation $w = \frac{az+b}{cz+d}$, where, a, b, c, d are complex constants such that $ad - bc \neq 0$ is called a bilinear transformation.

Note:- Bilinear transformations preserve the cross-ratio of four points. z_1, z_2, z_3 and w_1, w_2, w_3

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Ex:- 1) Find the BLT that maps (transforms) the points $z_1 = 0$, $z_2 = -i$, $z_3 = -1$ on to the points $w_1 = i$, $w_2 = 1$, $w_3 = 0$

The required BLT is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Substitute z_1, z_2, z_3 and w_1, w_2, w_3

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{(w-i)}{w(1-i)} = \frac{z(1-i)}{(z+1)(-i)}$$

$$\frac{w-i}{w} = \frac{z}{z+1} \cdot \frac{(1-i)^2}{-i} = \frac{z}{z+1} \frac{[1+i^2-2i]}{-i}$$

$$\frac{w-i}{w} = \frac{z}{z+1} \cdot \frac{[1-1-2i]}{-i}$$

$$\frac{w-i}{w} = \frac{z}{z+1} \cdot \frac{-2i}{-i} = \frac{z}{z+1} \cdot 2 = \frac{2z}{z+1}$$

$$\frac{w-i}{w} = \frac{2z}{z+1}$$

$$(z+1)(w-i) = 2wz$$

$$wz + w - iz - i = 2wz$$

$$wz + w - 2wz = iz + i$$

$$w - wz = i(1+z)$$

$$w[1-z] = i[1+z]$$

$$\boxed{w = \frac{i[z+1]}{-z+1}}$$ is the required transformation.

2) Find the bilinear transformation [BLT] that maps the points $z = -1, i, 1$ on the points $w = 1, i, -1$ respectively. Find the fixed points of the transformation. (or) invariant points.

>> $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\therefore \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

$$\frac{(w-1)}{(w+1)} = \frac{(z+1)}{(z-1)} \cdot \frac{(i-1)^2}{(i+1)^2} = \frac{z+1}{z-1} \cdot \frac{i^2+1-2i}{i^2+1+2i}$$

$$= \frac{(z+1)}{(z-1)} \cdot \frac{-1+1-2i}{-1+1+2i} = \frac{z+1}{z-1} \cdot \frac{-\cancel{2i}}{\cancel{2i}}$$

$$\frac{(w-1)}{(w+1)} = -\frac{(z+1)}{(z-1)}$$

$$(w-1)(z-1) = -(z+1)(w+1)$$

$$wz - w - z + 1 = -[wz + z + w + 1]$$

$$wz - w - z + 1 = -wz - z - w - 1$$

$$wz - \cancel{w} - \cancel{z} + 1 + wz + \cancel{z} + \cancel{w} + 1 = 0$$

$$2wz + 2 = 0$$

$$2wz = -2$$

$$wz = -2/2 = -1$$

$$wz = -1$$

$$\boxed{w = -1/z} \quad -①$$ is the required transforms.

For a fixed point, we have $w = z$

① ⟹ $z = -1/z$

$z^2 = -1$

$z = \pm i$

Thus $i, -i$ are its fixed point.

3) Find the BLT that transforms the points $z_1 = 1,\ z_2 = i,\ z_3 = -1$ onto the points $w_1 = 2,\ w_2 = i,\ w_3 = -2$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-2)}{(w+2)} = \frac{(z-1)}{(z+1)} \cdot \frac{(i+1)(i-2)}{(i-1)(i+2)} = \frac{z-1}{z+1} \cdot \frac{i^2-2i+i-2}{i^2+2i-i-2}$$

$$= \frac{(z-1)}{(z+1)} \cdot \frac{-1-i-2}{-1+i-2} = \frac{(z-1)}{(z+1)} \cdot \frac{-3-i}{-3+i}$$

$$= \frac{(z-1)}{(z+1)} \cdot \frac{-(3+i)}{-(3-i)}$$

$$\frac{w-2}{w+2} = \frac{(z-1)}{(z+1)} \cdot \frac{(3+i)}{(3-i)} = P$$

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$$\frac{w-2}{w+2} = p$$

$$\Rightarrow w-2 = p(w+2)$$

$$w-2-pw-2p=0$$

$$w-wp-2-2p=0$$

$$w(1-p) = 2(1+p)$$

$$w\left[1-\frac{(z-1)}{(z+1)}\cdot\frac{(3+i)}{3-i}\right] = 2\left[1+\frac{(z-1)(3+i)}{(z+1)(3-i)}\right]$$

$$w\left[\frac{(z+1)(3-i)-(z-1)(3+i)}{''}\right] = 2\left[\frac{(z+1)(3-i)+(z-1)(3+i)}{''}\right]$$

$$w\left[\frac{3z-iz+3-i-(3z+iz-3-i)}{''}\right] = 2\left[\frac{3z-i\cancel{z}+\cancel{3}-i+3z+i\cancel{z}-\cancel{3}-i}{''}\right]$$

$$w\left[\frac{\cancel{3z}-iz+3-\cancel{i}-\cancel{3z}-iz+3+\cancel{i}}{''}\right] = 2\left[\frac{6z-2i}{(z+1)(3-i)}\right]$$

$$w\left[\frac{(-2iz+6)}{(z+1)(3-i)}\right] = 2\left[\frac{(6z-2i)}{(z+1)(3-i)}\right]$$

$$w = \frac{(6z-2i)}{(6-2iz)}\cdot\frac{(z+1)\cancel{(3-i)}}{(z+1)(3-i)}$$

$$\boxed{w = \frac{6z-2i}{6-2iz}} \Rightarrow \frac{2(3z-i)}{2(3-iz)}$$

$$\boxed{w = \frac{(3z-i)}{3-iz}}$$ is the required BLT

HW $1, i, -1$ onto the points $i, 0, -i$ respectively also find fixed points.

4) Find the BLT which maps the points 0, 1, ∞ onto the points −5, −1, 3 respectively. (10)(24)

here $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$, so that $\frac{1}{z_3} = 0$

and $w_1 = -5$, $w_2 = -1$, $w_3 = 3$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w+5)(-1-3)}{(w-3)(-1+5)} = \frac{(z-0)\left(\frac{1}{\infty}-1\right)}{\left(\frac{z}{\infty}-1\right)(1-0)}$$

$$\frac{(w+5)(-4)}{(w-3)(4)} = \frac{z(0-1)}{(0-1)(1-0)} = \frac{-z}{-1} = z$$

$$\frac{(w+5)}{(w-3)} = \frac{4z}{-4} = -z$$

Note $\frac{0}{\text{anything}} = 0$ & $\frac{\text{anything}}{0} = \infty$

$(w+5) = -[z(w-3)]$

$(w+5) = -[wz-3z]$

$w+5 = -wz+3z$

$w+5+wz-3z = 0$

$w+wz+5-3z = 0$

$w+wz-3z+5 = 0$

~~$w[1+z-3] = -5$~~

$w[1+z] = 3z-5$

$$w = \frac{3z-5}{1+z} = \frac{3z-5}{z+1}$$ is the required BLT

5) Find the BLT which maps the points $z = 0, i, \infty$ onto the points $w = 1, -i, -1$ respectively. Find the invariant points.

Here $z_1 = 0,\ z_2 = i,\ z_3 = \infty$ so that $\frac{1}{z_3} = \frac{1}{\infty} = 0$

$w_1 = 1,\ w_2 = -i,\ w_3 = -1$

Hence $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\cdot z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-1)(-i+1)}{(w+1)(-i-1)} = \frac{(z-0)\left(\frac{i}{\infty}-1\right)}{\left(\frac{z}{\infty}-1\right)(i-0)} = \frac{z(0-1)}{(0-1)(i-0)} = \frac{-z}{-i}$$

$$\frac{(w-1)}{(w+1)} = \frac{(-i-1)}{(-i+1)}\cdot\frac{z}{i} = \frac{-iz-z}{-i^2+i} = \frac{-iz-z}{1+i}$$

$$= \frac{-z(i+1)}{(i+1)} = -z$$

$$\frac{(w-1)}{(w+1)} = -z$$

$(w-1) = -z[w+1]$

$w-1 = -wz-z$

$w-1+wz+z=0$

$w-1+wz+z=0$

$w+wz+z-1=0$

$w[1+z] = 1-z$

$\boxed{w = \frac{1-z}{1+z}}$ is the required BLT

To find the fixed points

$w = z$

$z(z+1) = 1-z$

$z^2+z-1+z=0$

$z^2+2z-1=0$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-2\pm\sqrt{4+4}}{2}$

$= \frac{-2\pm\sqrt{8}}{2}$

$= \frac{-2\pm\sqrt{4\times 2}}{2}$

$= \frac{-2\pm2\sqrt{2}}{2}$

$= -1\pm\sqrt{2}$

$-1+\sqrt{2}$ & $-1-\sqrt{2}$ are the fixed points

④ Find the BLT that transforms the points $z_1 = i$, $z_2 = 1$, $z_3 = -1$ onto the points $w_1 = 1$, $w_2 = 0$, $w_3 = \infty$ respectively.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,w_3(w_2/w_3-1)}{w_3(w/w_3-1)\,(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2/w_3-1)}{(w/w_3-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)(0/\infty-1)}{(w/\infty-1)(0-1)} = \frac{(z-i)(1+1)}{(z+1)(1-i)}$$

$$\frac{(w-1)(0-\cancel{1})}{(0-1)(0-1)} = \frac{(z-i)\,2}{(z+1)(1-i)}$$

$$\frac{(w-1)(-1)}{(-1)} = \frac{(z-i)\,2}{(z+1)(1-i)}$$

$$(w-1) = \frac{-2(z-i)}{(z+1)(1-i)}$$

$$w = 1 - \frac{2(z-i)}{(z+1)(1-i)} = \frac{(z+1)(1-i)-2(z-i)}{(z+1)(1-i)}$$

$$= \frac{\cancel{z}-iz+1-\cancel{i}-2\cancel{z}+\cancel{2}i}{(z+1)(1-i)}$$

$$= \cancel{\frac{-z-3i-iz+1}{(z+1)(1-i)}} = \frac{-z+i+1-iz}{''}$$

$$= \cancel{-z(1+i)} \qquad = \frac{-z-iz+(1+i)}{''}$$

$$= \frac{-z(1+i)+(1+i)}{''}$$

$$= \frac{(1+i)(1-z)}{''}$$

$\times^{4}$ and $\div$ by $(1+i)$

$$w = \frac{(1+i)^2(1-z)}{(1+i)(1-i)(1+z)}$$

$$w = \frac{[1-1+2i](1-z)}{[1-i^2](1+z)}$$

$$= \frac{2i(1-z)}{2(1+z)}$$

$$= i\frac{(1-z)}{(1+z)}$$ is the required tranlformation.

H.W

(*) Find the bilinear BLT which has 1 and i as fixed points and which maps to 0 to -1.

>> Since 1 and i are fixed points of the required tranlformation. the required tranlformation maps the points $z_1 = 1$ and $z_2 = i$ to the points $w_1 = 1$, $w_2 = i$ respectively. Also, it is given that the tranlformation maps the points $z_3 = 0$ to the point $w_3 = -1$. Thus the required tranlformation maps the points $z_1 = 1$, $z_2 = i$ & $z_3 = 0$ to the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$.

$\times^{4}$ & $\div$ by $(1+i)$

Ans. $$\frac{(1+2i)z - i}{2+i}$$

Discussion of Conformal Transformations.

Given the transformation $w = f(z)$, we put $z = x+iy$ (or) $z = re^{i\theta}$ to obtain u and v as functions of x, y (or) r, θ we find the image in w-plane corresponding to the given curve in the z-plane. Some times we need to make some judicious elimination from u and v for obtaining the image in the w-plane.

1) Discussion of $w = e^z$

(OR)

Show that the transformation $w = e^z$ map straight lines parallel to the co-ordinate axes in the z-plane into orthogonal trajectories in the w-plane and sketch the region.

Proof :- Consider $w = e^z$

ie $u + iv = e^{x+iy}$

$= e^x e^{iy}$

$= e^x(\cos y + i \sin y)$ $\because e^{i\theta}$

$= e^x \cos y + i e^x \sin y$

$\therefore u = e^x \cos y$ and $v = e^x \sin y$ ——— (1)

Separating the Re and Im parts

we shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z-plane.
ie $x = \underline{constant}$ and $y = \underline{constant}$.

Let us eliminate x and y separately from (1)

Squaring and adding we get

$$u^2+v^2 = e^{2x} \quad \text{——— (2)}$$

Also by dividing we get

$$\frac{v}{u} = \frac{e^x \sin y}{e^x \cos y} = \tan y \quad \text{——— (3)}$$

Case i) Let $x = c_1$ where c_1 is a constant.

Eq^n (2) $\Rightarrow$

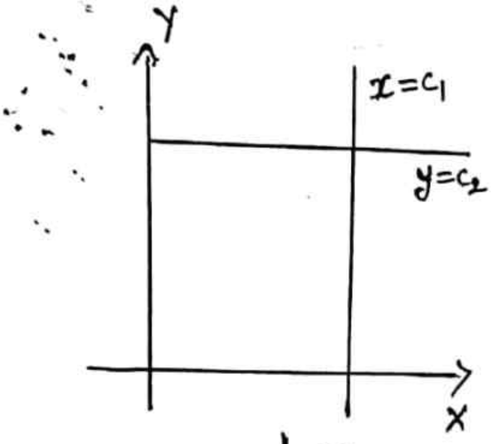
$$u^2+v^2 = e^{2c_1} = \text{constant} = r^2$$

ie $\underline{u^2+v^2=r^2}$ represents a circle with centre $\underline{\text{origin}}$ and radius $\underline{r}$ in the w-plane.

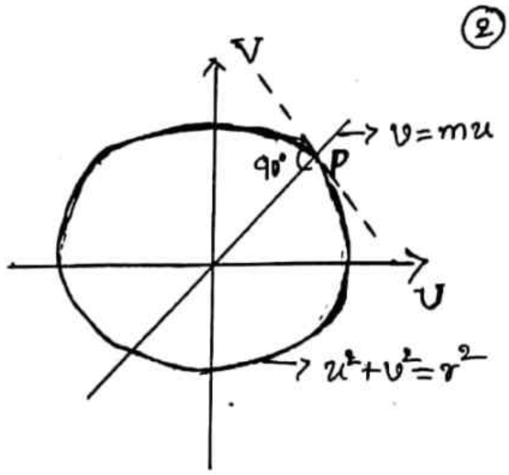
Case ii) Let $y = c_2$ where c_2 is a constant.

Eq^n (3) $\Rightarrow$ $\frac{v}{u} = \tan c_2 = m$

$\therefore$ $\underline{v = mu}$ represents a straight line passing through the origin in the w-plane.



z-plane



w-plane

Conclusion: The straight line parallel to the x-axis ($y=c_2$) in the z-plane maps onto a St line passing through the origin in the w-plane. The St line parallel to y-axis ($x=c_1$) in the z-plane maps onto a circle with centre origin and radius r where $r=e^{c_1}$ in the w-plane

Suppose we draw a tangent at the point of intersection of these two curves in the w-plane (ie, at P as in the above fig) the angle subtended is equal to 90°. Hence these two curves can be regarded as orthogonal trajectories of each other.

2) Discussion of $w = z^2$

(OR)

Find the images in the w-plane corresponding to the straight lines $x = c_1$, $x = c_2$, $y = k_1$, $y = k_2$, under the transformation $w = z^2$. Indicate the region with sketches.

Proof:- Consider $w = z^2$

ie $u + iv = (x+iy)^2$

$= x^2 + (iy)^2 + 2(xiy)$ but $i^2 = -1$

$= (x^2 - y^2) + i(2xy)$

ie $u = (x^2 - y^2)$ and $v = 2xy$ —— (1)

Case i) Let us consider $x = c_1$, c_1 is a constant.

The set of equations (1) ⇒

$u = c_1^2 - y^2$; $v = 2c_1 y$

Now $y = v/2c_1$ and substituting this in u

$u = c_1^2 - (v^2/4c_1^2)$

(or) $v^2/4c_1^2 = c_1^2 - u$

(or) $v^2 = -4c_1^2(u - c_1^2)$

This is a parabola in the w-plane symmetrical about the real axis with its vertex at $(c_1^2, 0)$

and focus at the origin. It may be observed that the line $x=-c_1$ is also transformed into the same parabola.

<u>Case ii)</u> Let us consider $y=c_2$, c_2 is a constant.

The set of Eqⁿs (1) ⟹

$$u=x^2-c_2^2, \quad v=2xc_2$$

Now $x=v/2c_2$ and substituting this in u

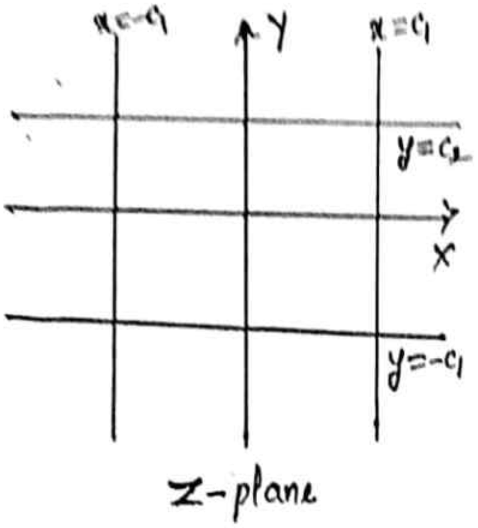
$$u=(v^2/4c_2^2)-c_2^2$$

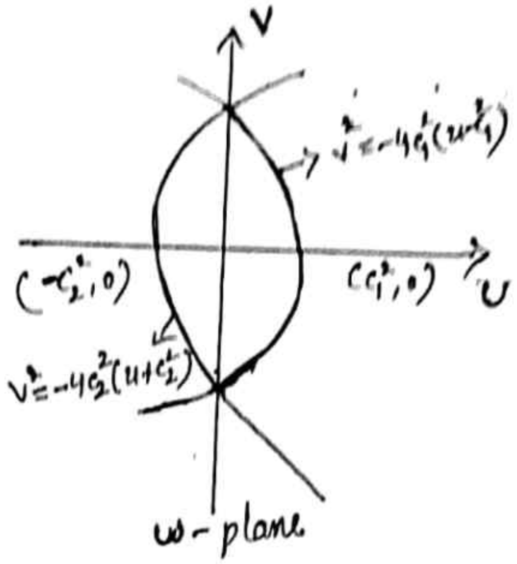
(or) $$v^2/4c_2^2 = u+c_2^2$$

(or) $$v^2=4c_2^2(u+c_2^2)$$

This is also a parabola in the w-plane symmetrical about the real axis whose vertex is at $(-c_2^2, 0)$ and focus at the origin. Also the line $y=-c_2$ is transformed into the same parabola.

Hence from these two cases we conclude that the <u>st</u>-lines parallel to the co-ordinate axes in the z-plane map onto parabolas in the w-plane.





3) Discussion of $w = z+\left(\frac{1}{z}\right)$, $z \neq 0$

Consider the transformation

$$w = z + \frac{1}{z} \quad \text{———} \quad (1)$$

Here $f'(z) = 1-\frac{1}{z^2}$, From this, we note that $f'(z)$ exists and not zero when $z \neq 0$ and $z^2 \neq 1$.

$\therefore$ the transformation (1) is conformal at all points except at '0' and '± 1'. This transformation is known as the Joukowski's tranformation.

Taking $z = re^{i\theta}$ in (1) we obtain

$$u+iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$u+iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\therefore \quad u = \left(r+\frac{1}{r}\right)\cos\theta, \quad v = \left(r-\frac{1}{r}\right)\sin\theta \quad \text{——} \quad (2)$$

∴ From ellipse we get

$$\frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = \cos^2\theta + \sin^2\theta = 1 \quad \text{——} (3)$$

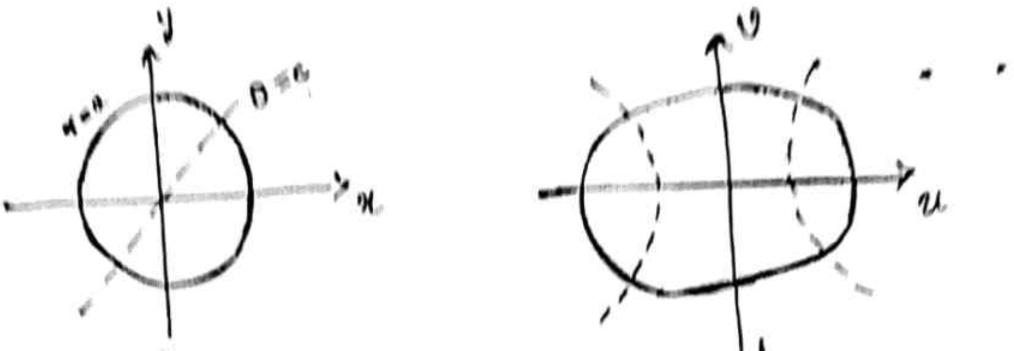
Consider the polar eqⁿ $r = a\ (\neq 1)$, a constant, which represents a ⊙le centred at the origin in the z-plane. Then eqⁿ (3) represents an ellipse having centre at the origin of the w-plane and u and v-axes as it's axes.

Thus, under the transformation ① the ⊙le $r=a$ centred at the origin in the z-plane is transformed into the ellipse ③ in the w-plane.

From relations ②, we also obtain

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4 \quad \text{——} (4)$$

For $\theta = c$, a constant, eqⁿ ④ represents a hyperbola having centre at the origin of the w-plane and u-axis and v-axis as axes. Thus under the transformation ① the radial line $\theta = c$ in the z-plane is transformed to the hyperbola ④ in the w-plane.



z-plane w-plane

a takes different constant values, the eq^n $r=a$ represents a family of concentric circles in the z-plane and eq^n (1) represents a family of ellipses in the w-plane all of which have the origin as their centre and u- and v- axes as their axes. Thus under the transformation (1), a family of concentric circles having their centres at the origin in the z-plane transform to the family of concentric and coaxial ellipses having their centres at the origin in the w-plane.