FUTURE VISION BIE

One Stop for All Study Materials

& Lab Programs

Future Vision

By K B Hemanth Raj

Scan the QR Code to Visit the Web Page

Or

Visit : https://hemanthrajhemu.github.io

Gain Access to All Study Materials according to VTU,
CSE – Computer Science Engineering,
ISE – Information Science Engineering,
ECE - Electronics and Communication Engineering
& MORE...

MODULE-3 COMPLEX VARIABLES :- (PART-1)

Complex number :- A number of the form : $z = x+iy$, where x, y are real numbers & $i=\sqrt{-1}$ or $i^2=-1$ is called a Complex number.

$\bar{z} = x-iy$ is called the complex conjugate of z.

Note :- 1) If $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

then $\boxed{e^{ix} = \cos x + i\sin x}$, $\boxed{e^{-ix} = \cos x - i\sin x}$

2) $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2}$

3) $\cos(ix) = \cosh x$, $\sin(ix) = i\sinh x$

where, $\cosh x = \frac{e^{x}+e^{-x}}{2}$, $\sinh x = \frac{e^{x}-e^{-x}}{2}$

Complex no in polar form :- $e^{i\theta} = \cos\theta + i\sin\theta$

$z = r \cdot e^{i\theta}$

where; $\boxed{r = \sqrt{x^2+y^2}}$ is called modulus of z.

$\boxed{\theta = \tan^{-1}(y/x)}$

Argument of z :- $\text{Arg}(z) = \theta = \tan^{-1}(y/x)$

* Function of a complex variable :-

w is said to be a function of complex variable, if w is a function of z, defined for a domain D., where : $w = f(z)$

$$w = f(z) = u(x,y) + iv(x,y) \quad // \text{ Cartesian form}$$
$$w = f(z) = u(r,\theta) + iv(r,\theta) \quad // \text{ Polar form}$$

* Limit of Complex variable :- A complex valued function, $f(z)$ defined in the neighbourhood of a point z_0, is said to have a limit l as z tends to z_0, if for every $\varepsilon > 0$ however small, there exists a +ve real no $\delta \ni : |f(z) - l| < \varepsilon$ when ; $|z - z_0| < \delta$.

ie ; $\boxed{\lim_{z \to z_0} f(z) = l}$

Continuity :- A complex valued function $f(z)$ is said to be continuous at $z = z_0$ if $\boxed{f(z_0) \text{ exists}}$ & $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$

Differentiability :- A complex valued fn, $f(z)$ is said to be differentiable at $z = z_0$ if : $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is unique, this lim when exists is called derivative of $f(z)$ at $z = z_0$,

$$\boxed{f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}}$$

* Cauchy-Riemann Equations in Cartesian form :- (C-R Eqn's)

The necessary conditions that the function $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that; there exists 4 continuous 1st order partial derivatives.

ie: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the Equations :-

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ & $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$ These 2 Conditions are known as Cauchy-Riemann (C-R) Equations in Cartesian form.

* Cauchy-Riemann Equations in Polar form :-

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous 1st order partial derivatives : $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfies the equations;

$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}}$ & $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}}$ These equations are known as Cauchy-Riemann Eqns in Polar form,

Harmonic function :- A function ϕ is said to be harmonic if it satisfies Laplace's Equation, $\nabla^2\phi = 0$

In Cartesian form : $\phi(x,y)$ is harmonic if : $\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.}$

In polar form : $\phi(r,\theta)$ is harmonic if : $\boxed{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.}$

Note :- i) The polar family of curves $u(r,\theta) = C$, $v(r,\theta) = C_2$ intersect orthogonally if : $\tan\phi_1 \tan\phi_2 = -1$.

Analytic function :-
A complex valued function ; $w = f(z)$ is said to be analytic at a point $z = z_0$, if $\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique at z_0 and in the neighbourhood of z_0.

(or)

A complex valued function ; $w = f(z)$ is analytic at a point z_0, if it is differentiable at z_0 and in the neighbourhood of z_0.

* Derive Cauchy-Riemann equations in Cartesian form :-

Stmt: The necessary conditions that the function ; $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point, $z = x+iy$ is that, there exist four continuous 1st order partial derivatives ;

: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equations ;

: $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$, these equations are known as Cauchy-Riemann (C-R) Equations.

Proof :- Let $f(z)$ be analytic at a point ; $z = x+iy$ and hence by the definition ; $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is Unique.

In the Cartesian form, $f(z) = u(x,y) + iv(x,y)$.

Let "δz" be the increment in z corresponding to the increments : $\delta x, \delta y$ in x & y.

where ; $\boxed{f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)}$

Considering,

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + i v(x+\delta x, y+\delta y)] - [u(x,y) + i v(x,y)]}{\delta z}$$

where, $\delta x, \delta y \to$ negligibly small

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \sim (1)$$

Now; $\delta z = (z+\delta z) - z$, where, $z = x+iy$.

$\delta z = [(x+\delta x) + i(y+\delta y)] - [x+iy]$

$\delta z = \delta x + i\delta y$. $\sim (*)_1$

Since, δz tends to zero, we have the following 2 possibilities;

Case:1 : let $\delta y = 0$ so that $\overset{(*)_1}{;}$ $\delta z = \delta x$ & $\delta z \to 0$
$\Rightarrow \delta x \to 0$

Now, Eqn (1) becomes;

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}.$$

These limits from the basic definition are the partial derivatives of u and v wrt x...

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \sim (2)$$

Case:2 let $\delta x = 0$ so that; $\delta z = i\delta y$ & $\delta z \to 0$
$\Rightarrow i\delta y \to 0$ (or) $\delta y \to 0$

Now, Eqn (1) becomes;

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y} \sim (*)_2$$

But; $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ $\quad \boxed{\frac{1}{i} = -i}$

$\Rightarrow$ substitute in $(*)_2$

$$f'(z) = \lim_{\delta y \to 0} -i \cdot \frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$f'(z) = -i \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \sim (3)$$

$\therefore$ Equating Eqns (2) & (3) ; we get:

$$\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$\Rightarrow$ Now, Equating the real & imaginary parts ; we get;

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Thus, these Equations are : Cauchy-Riemann Equations (C-R) Eqns in Cartesian form.

Derive Cauchy-Riemann Equations in the polar form :-

Statement :- If $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives : $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the Equations :-

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}.$$

These are known as Cauchy-Riemann Equations in Polar form.

Proof:- Let $f(z)$ be analytic at a point $z = re^{i\theta}$

Hence, by defn ; $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists & is unique

In polar form, $f(z) = u(r,\theta) + iv(r,\theta)$

Let δz be the increment in z corresponding to increments $\delta r, \delta\theta$ in r, θ.

$$\Rightarrow f'(z) = \underset{\delta z \to 0}{\mathrm{lt}} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$f'(z) = \underset{\delta z \to 0}{\mathrm{lt}} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} +$$

where ; $f(z+\delta z) = u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)$

$$+ i \underset{\delta z \to 0}{\mathrm{lt}} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \quad \sim (1)$$

Consider ; $z = re^{i\theta}$, since z is a function of 2 variables r, θ.

we have ; $$\delta z = \frac{\partial z}{\partial r} \cdot \delta r + \frac{\partial z}{\partial \theta} \cdot \delta\theta.$$

$$= \frac{\partial}{\partial r}(re^{i\theta}) \delta r + \frac{\partial}{\partial \theta}(re^{i\theta}) \delta\theta.$$

$$\therefore \delta z = e^{i\theta} \cdot \delta r + ire^{i\theta} \cdot \delta\theta.$$

Since, δz tends to zero, we have the following possibilities....

Case: 1 Let $\delta\theta = 0$ so that ; $\delta z = e^{i\theta} \cdot \delta r$ & $\delta z \to 0$
$\Rightarrow \delta r \to 0$.

Now, Eqn (1) becomes...

$$f'(z) = \lim_{\delta r\to 0} \frac{u(r+\delta r,\theta)-u(r,\theta)}{e^{i\theta}.\delta r} + i \lim_{\delta r \to 0}\frac{v(r+\delta r,\theta)-v(r,\theta)}{e^{i\theta}.\delta r}$$

ie; $f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$ ~ (2).

Case: 2 Let $\delta r = 0$, so that; $\delta z = i r e^{i\theta}.\delta\theta$

$\delta z \to 0 \Rightarrow \delta\theta \to 0$.

Now, Eqn (1) becomes;

$$f'(z) = \underset{\delta\theta\to 0}{lt} \frac{u(r,\theta+\delta\theta)-u(r,\theta)}{ir.e^{i\theta}\delta\theta} + i \underset{\delta\theta\to 0}{lt} \frac{v(r,\theta+\delta\theta)-v(r,\theta)}{ir.e^{i\theta}.\delta\theta}$$

$$= \frac{1}{ir.e^{i\theta}}\left[\underset{\delta\theta\to 0}{lt} \frac{u(r,\theta+\delta\theta)-u(r,\theta)}{\delta\theta} + i \underset{\delta\theta\to 0}{lt} \frac{v(r,\theta+\delta\theta)-v(r,\theta)}{\delta\theta}\right]$$

$$\Rightarrow f'(z) = \frac{1}{ir.e^{i\theta}}\left[\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right] + \frac{1}{r.e^{i\theta}}\left[\frac{1}{i}\cdot\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]$$

But; $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} \Rightarrow \boxed{\frac{1}{i} = -i}$

$$\Rightarrow f'(z) = \frac{1}{r.e^{i\theta}}\left[-i\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right] = e^{-i\theta}\left[\frac{-i}{r}\cdot\frac{\partial u}{\partial\theta} + \frac{1}{r}\cdot\frac{\partial v}{\partial\theta}\right]$$

$$\Rightarrow f'(z) = e^{-i\theta}\left[\frac{1}{r}\cdot\frac{\partial v}{\partial\theta} - \frac{i}{r}\cdot\frac{\partial u}{\partial\theta}\right] \sim (3).$$

Equating RHS of Eqn (2) & (3); we get;

$$\Rightarrow e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = e^{-i\theta}\left[\frac{1}{r}\cdot\frac{\partial v}{\partial\theta} - \frac{i}{r}\cdot\frac{\partial u}{\partial\theta}\right]$$

Equating real & img parts --

$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial\theta}}$ & $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}}$ are C-R Equations in Polar form.

Diff

7

* Properties of Analytic function :-

Harmonic function :-

A function ϕ is said to be harmonic, if it satisfies Laplace's Equation :- $\nabla^2\phi = 0$

In the cartesian form :- $\phi(x,y)$ is harmonic if : $\boxed{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0}$

In the polar form :- $\phi(r,\theta)$ is harmonic if :

$$\boxed{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.}$$

* Deduce / Proove that :- $\boxed{\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\cdot\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\cdot\dfrac{\partial^2 u}{\partial\theta^2} = 0.}$

(Or) P.T : Real & imaginary parts of analytic function are harmonic.

Proof / Soln :-

Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic.

We shall show that u & v satisfy Laplace's Equation in the polar form;

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

We have ~~from~~ : Cauchy-Riemann Equations in polar form;

$$r\cdot\frac{\partial u}{\partial r} = \frac{\partial v}{\partial\theta} \quad \sim (1)$$

$$r\cdot\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial\theta} \quad \sim (2)$$

→ Diff ① wrt r & ② wrt θ partially, we get -

$$① \Rightarrow r \cdot \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \cdot 1 = \frac{\partial^2 v}{\partial r \cdot \partial \theta}$$

$$② \Rightarrow r \cdot \frac{\partial^2 v}{\partial \theta \partial r} = \frac{-\partial^2 u}{\partial \theta^2} \Rightarrow \frac{-1}{r} \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 v}{\partial \theta \partial r}$$

But ; $\frac{\partial^2 v}{\partial r \cdot \partial \theta} = \frac{\partial^2 v}{\partial \theta \cdot \partial r}$ is always true and hence, we have ;

$$r \cdot \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{-1}{r} \frac{\partial^2 u}{\partial \theta^2}$$

⇒ Dividing by "r", and transposing the term in the RHS, to LHS, we obtain...

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

∴ u satisfies Laplace's Equation in polar form

∴ u is harmonic.

Hence, the result (proof) ⇒ $\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0}$

* If $f(z) = u + iv$ is an analytic function, then proove that u & v both satisfy 2 dimensional Laplace Equation.

ie, $\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \right]$

Proof :- Let $f(z) = u + iv$ is analytic.

$\Rightarrow u$ & v satisfies Cauchy - Reimann Equations...

ie, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ~ (1)

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ~ (2)

$\Rightarrow$ dwt (1) partially wrt x & (2) partially wrt y...

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ~ (3) & $\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$ ~ (4)

Now, Equating Eqn (3) & (4) ; we get..

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{-\partial^2 u}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\therefore u$ satisfies Laplace Equation.

Now, dwt (1) partially wrt y & (2) partially wrt x, ..

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$ ~ (5) & $\frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial x \partial y}$ ~ (6)

$\Rightarrow$ Now, Equating Eqn (5) & (6) ..

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

$\therefore v$ satisfies Laplace Equation.

$\therefore u$ & v satisfies Laplace Equation.., Hence the proof.

— x —

https://hemanthrajhemu.github.io

* If $f(z) = u+iv$ is an analytic function, then prove that the Equations: $u(x,y) = c_1$ & $v(x,y) = c_2$, represent orthogonal family of curves.

Proof :- Let $f(z) = u+iv$ be analytic

$\Rightarrow$ u & v satisfies Cauchy - Riemann Equations.

ie, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Consider; $u(x,y) = c_1$ ~ (1)

$\Rightarrow$ Diff (1) w.r.t x, eqn (1) treating y as a function of x [y is dept on x & u].

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0.$$

$$\frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = -\frac{\partial u}{\partial x}.$$

$$\Rightarrow \frac{dy}{dx} = \left\{ \frac{-\partial u/\partial x}{\partial u/\partial y} \right\} = m_1 \text{ (say)}.$$

III^{ly}, Consider; $v(x,y) = c_2$ ~ (2)

$\Rightarrow$ Diff (2) w.r.t x, keeping y as function of x

$$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = \left\{ \frac{-\frac{\partial v}{\partial x}}{\partial v/\partial y} \right\} = m_2 \text{ (say)}.$$

Now, Consider;

$$m_1 \cdot m_2 = \left\{ \frac{-\partial u/\partial x}{\partial u/\partial y} \right\} \cdot \left\{ \frac{-\partial v/\partial x}{\partial v/\partial y} \right\}$$

$$\Rightarrow \left\{\frac{-\partial v/\partial y}{-\partial v/\partial x}\right\} \cdot \left\{\frac{-\partial v/\partial x}{\partial v/\partial y}\right\}$$

// [By C-R Equations ...

* $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$]

$$\Rightarrow \boxed{m_1 . m_2 = -1}$$

$\therefore$ $u(x,y) = c_1$ & $v(x,y) = c_2$ represents the orthogonal family of curves. Hence, the proof.

Problems & Solutions :- [Construction of analytic function] - TYPE -1

1) Show that : $w = z + e^z$ is analytic and hence find : $\frac{dw}{dz}$ or $f'(z)$

Soln:- By data ; $w = z + e^z$ // $f(z) = w = u + iv$

ie, $u + iv = (x+iy) + e^{x+iy}$

$= (x+iy) + e^x . e^{iy}$ // $e^{ix} = \cos x + i \sin x$

$= (x+iy) + e^x [\cos y + i \sin y]$

$\therefore u + iv = (x + e^x.\cos y) + i(y + e^x \sin y)$ ~ ①

By seperating real & imaginary parts, we get...

$\Rightarrow u = x + e^x.\cos y$ & $v = y + e^x \sin y$.

$\Rightarrow$ diff u partially wrt x & y $\Rightarrow$ diff v partially wrt y & x.

$u_x = 1 + e^x \cos y$ $\qquad v_x = e^x \sin y$

$u_y = -e^x \sin y$. $\qquad v_y = 1 + e^x \cos y$.

We observe that cauchy-Riemann Equations in the cartesian form;

$u_x = v_y$ & $v_x = -u_y$ are satisfied..

Thus, $u_x = v_y = 1 + e^x \cos y$

$v_x = -u_y = e^x \sin y = -- e^x \sin y$

Hence; $w = z + e^z$ is Analytic

Now, to find : $\frac{dw}{dz} = f'(z) = u_x + iv_x$.

ie, $\frac{dw}{dz} = (1 + e^x \cos y) + i(e^x \sin y)$.

$$= 1+e^x\,[\underbrace{\cos y + i\sin y}]$$

$$= 1+e^x \cdot e^{iy}$$

$$\therefore \frac{dw}{dz} = 1+e^{x+iy}.$$

Since; $z = x+iy$

$$\therefore \boxed{\frac{dw}{dz} = 1+e^z}$$

2) S.T ; $f(z) = \sin z$ is analytic.

Soln :- $f(z) = \sin z$.

$u+iv = \sin(x+iy)$ // $\sin(A+B) = \sin A.\cos B + \cos A \sin B.$

$u+iv = \sin x.\cos iy + \cos x\, i \sin iy.$ // $\sin i\theta = i\sinh\theta$, $\cos i\theta = \cosh\theta$.

$u+iv = \sin x.\cosh y + i\cos x.\sinh y.$ ~ (1)

Now, seperating real & imaginary parts..

$u = \sin x.\cosh y$ & $v = \cos x.\sinh y$.

$\Rightarrow$ dwt u partially wrt x & y $\Rightarrow$ dwt v partially wrt x & y

$u_x = \cos x \cosh y$ $v_x = -\sin x.\sinh y.$

$u_y = \sin x\, \sinh y$ $v_y = \cos x.\cosh y$

Since, Cauchy-Riemann Equations . $u_x = v_y$ & $v_x = -u_y$

ie, $u_x = v_y = \cos x \cosh y$ & $v_x = -u_y = \sin x.\sinh y = -\sin x \sinh y$

are satisfied --

$\therefore$ $f(z) = \sin z$ is analytic.

) Show that $f(z) = \cosh z$ is analytic, hence find $f'(z)$.

Soln:- $f(z) = \cosh z$, $z = x+iy$.

$u+iv = \cosh(x+iy)$ // $\cosh\theta = \cos i\theta$.

$u+iv = \cos i(x+iy)$

$= \cos(ix + i^2 y)$ // $i^2 = -1$

$= \cos(ix - y)$. // $\cos(A-B) = \cos A.\cos B + \sin A \sin B$.

$= \cos ix . \cos y + \sin ix . \sin y$. // $\cos ix = \cosh x$.
// $\sin ix = i\sinh x$.

$u+iv = \cosh x . \cos y + i\sinh x . \sin y$.

$\Rightarrow u = \cosh x . \cos y$. $v = \sinh x . \sin y$

$\Rightarrow$ diff u & v partially wrt x & y, we get...

$u_x = \sinh x . \cos y$. $u_x = \cosh x . \sin y$.

$u_y = -\cosh x . \sin y$. $v_y = \sinh x . \cos y$.

Cauchy-Riemann Equations, $u_x = v_y$ & $v_x = -u_y$ are satisfied.

$\therefore$ $f(z) = \cosh z$ is analytic

$f'(z) = u_x + iv_x$.

$f'(z) = \sinh x \cos y + i \cosh x \sin y$.

$\Rightarrow$ $\times^{ly}$ & $\div$ by i in RHS...

$f'(z) = \frac{1}{i}[i\sinh x . \cos y - \cosh x \sin y]$

$= \frac{1}{i}[\sin ix . \cos y - \cos ix . \sin y]$

$= \frac{1}{i} \sin(ix - y) = \frac{1}{i} \sin i(x+iy)$.

$f'(z) = \frac{1}{i}[i \sinh(x+iy)] = \boxed{\sinh(x+iy) = f'(z)}$

4) S.T ; $w = \log z$, $z \neq 0$ is analytic, hence find $\frac{dw}{dz}$ *

Soln :- $w = \log z$ [It is convenient to do problem in polar form as u & v can be found easily].

$w = \log z$, taking $z = re^{i\theta}$

$u + iv = \log(re^{i\theta}) = \log r + \log_e(e^{i\theta})$

$= \log r + i\theta \cdot \underline{\log_e e}$ // $\underline{\log_e e = 1}$

$u + iv = \log r + i\theta$.

$\Rightarrow u = \log r$ & $v = i\theta$.

$\Rightarrow$ diff u & v wrt r & θ, we get ...

$u_r = \frac{1}{r}$ & $v_r = 0$

$u_\theta = 0$ $v_\theta = i$

C-R Eqns in polar form : $r u_r = v_\theta$ & $r v_r = -u_\theta$ are satisfied.

$\therefore$ $w = \log z$ is Analytic

$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(1/r + i \cdot 0) = \frac{1}{re^{i\theta}} = f'(z) = \frac{1}{z}$

5) S.T ; $w = f(z) = z^n$, n is +ve integer, $f'(z)$. [polar form]

6) S.T ; $w = z + e^{-z}$ is Analytic.

7) S.T ; $w = z + \sinh z$ is Analytic, find :- $f'(z)$

—— * ——

* Construction of Analytic function ;
$f(z)$ given its real or imaginary part :- TYPE - 2

1) Construct analytic function whose real part is :- $u = \log\sqrt{x^2+y^2}$.

Soln :- Given ; Real part of Analytic function is ;

$$u = \log\sqrt{x^2+y^2} = \log(x^2+y^2)^{1/2}$$

$$u = \frac{1}{2}.\log(x^2+y^2)$$

$\Rightarrow$ diff u partially wrt x & y, we get ...

$$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

Consider ; $f'(z) = u_x + i v_x$ ... , But : $v_x = -u_y$ [By :- C-R Equation.]

$$f'(z) = u_x - i u_y$$

$$\therefore f'(z) = \left[\frac{x}{x^2+y^2}\right] - i\left[\frac{y}{x^2+y^2}\right] \sim (1)$$

putting ; $x = z$ & $y = 0$, we get ... // To get $f'(z)$ in terms of z.

$$\therefore (1) \Rightarrow f'(z) = \left[\frac{z}{z^2+0^2}\right] - i\cancel{\left[\frac{0}{z^2+0^2}\right]}^{0}$$

$$\therefore f'(z) = \frac{1}{z} \rightarrow \text{Integrate} \ldots$$

$$\int f'(z) = \int 1/z \, dz$$

$$\therefore \boxed{f(z) = \log z + c.} \text{ is an Analytic fn}$$

2) Determine the analytic function, $f(z) = u+iv$, Given that the real part : $u = e^{2x}.[x.\cos 2y - y \sin 2y]$

Soln :- $u = e^{2x}[x\cos 2y - y\sin 2y]$ ~(1)

$\Rightarrow$ diff (1) partially wrt x & y...

$\Rightarrow u_x = e^{2x}[1.\cos 2y - 0] + [x\cos 2y - y\sin 2y](2)e^{2x}$.

$\therefore u_x = e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y]$

$\Rightarrow u_y = e^{2x}[-2x\sin 2y - 2y\cos 2y - \sin 2y]$

$\therefore u_y = -e^{2x}[2x\sin 2y + 2y\cos 2y + \sin 2y]$

Consider, $f'(z) = u_x + iv_x = u_x - iu_y$ // $\underline{\underline{v_x = -u_y}}$ [By CR Equations]

$f'(z) \Rightarrow$ putting :- $\underline{\underline{x = z}}$, $\underline{\underline{y = 0}}$; we have...

$\Rightarrow f'(z) = e^{2z}(1+2z)$, $\Rightarrow$ Integrate b.s. -

$\Rightarrow f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$

$\Rightarrow \int f'(z) = \int e^{2z}(1+2z).dz$.

$\therefore f(z) = (1+2z)\frac{e^{2z}}{2} - 2.\frac{e^{2z}}{4} = \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2}$.

Thus, $\underline{\underline{f(z) = z.e^{2z} + c}}$

Also; $f(z) = u+iv = (x+iy)\underline{\underline{e^{2(x+iy)}}}$

$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$

$\therefore f(z) = e^{2x}(x\cos 2y - y\sin 2y) + ie^{2x}(x\sin 2y + y\cos 2y)$

Asmt 3) Determine analytic function, $f(z)$ ~~=~~, whose ~~im~~ real part is :

$* \; u = e^{-2xy}.\sin(x^2-y^2)$

4) $u = \log(x^2+y^2)$ ...

5) $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$

6) Determine the analytic function, $f(z)$, whose imaginary part is ; $\left[r - \frac{K^2}{r}\right] \sin\theta$, $r \neq 0$, Hence find the real part of $f(z)$ & PT it is harmonic.

Soln :- Let $v = \left[r - \frac{K^2}{r}\right] \sin\theta$ —(1)

$\Rightarrow$ diff (1) wrt r & θ partially, we get...

$$v_r = \left[1 + \frac{K^2}{r^2}\right] \sin\theta \quad , \quad v_\theta = \left[r - \frac{K^2}{r}\right] \cos\theta .$$

Consider; $f'(z) = e^{-i\theta}(u_r + i v_r)$, But ; $\boxed{\frac{1}{r} . v_\theta = u_r}$ // C-R Equ in polar form

$$\therefore f'(z) = e^{-i\theta}\left[\frac{1}{r} . v_\theta + i v_r\right]$$

$$f'(z) = e^{-i\theta}\left[\left(1 - \frac{K^2}{r^2}\right) \cos\theta + i\left(1 + \frac{K^2}{r^2}\right) \sin\theta\right]$$

$$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{K^2}{r^2}(\cos\theta - i\sin\theta)\right]$$

$$= e^{-i\theta}\left[e^{i\theta} - \frac{K^2}{r^2} e^{-i\theta}\right] = 1 - \frac{K^2}{(re^{i\theta})^2} = 1 - \frac{K^2}{z^2}$$

$f'(z) = 1 - \frac{K^2}{z^2} \Rightarrow$ Integrate...

$$\int f'(z) = \int 1 - \frac{K^2}{z^2} dz \quad \Rightarrow \quad \boxed{f(z) = \left(z + \frac{K^2}{z}\right) + c}$$

Now, to find $u(r, \theta)$, put $z = re^{i\theta}$ in $f(z)$ ...

$$u + iv = (re^{i\theta}) + \frac{K^2}{re^{i\theta}} = r(\cos\theta + i\sin\theta) + \frac{K^2}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{K^2}{r^2}\right) \cos\theta + i\left(r - \frac{K^2}{r^2}\right) \sin\theta .$$

$\therefore \boxed{u = r + \frac{K^2}{r^2} \cos\theta .}$ is Real part.

—— * ——

Type: 3 Finding the Conjugate harmonic function and analytic function :-

1) Show that; $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find its harmonic conjugate, Also find corresponding analytic function, $f(z)$.

Soln :- $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. ~ (1)

$\Rightarrow$ diff (1) partially wrt x & y .. twice ..

$u_x = 3x^2 - 3y^2 + 6x$. $\qquad u_y = -6xy - 6y$

$u_{xx} = 6x + 6$. $\qquad u_{yy} = -6x - 6$.

Consider; $u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$,

$u_{xx} + u_{yy} = 0$, Thus; u is harmonic

Consider; C-R Eqn :- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Substituting for :- $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ we have;

$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$. $\qquad \frac{\partial v}{\partial x} = -(-6xy - 6y)$

$\Rightarrow$ Integrating wrt y.. $\qquad \Rightarrow$ Integrating wrt x..

$\int \frac{\partial v}{\partial y} = \int (3x^2 - 3y^2 + 6x)\,dy + f(x)$ $\qquad ; \int \frac{\partial v}{\partial x} = \int (6xy + 6y)\,dx + g(y)$

$v = \int (3x^2 - 3y^2 + 6x)\,dy + f(x)$ $\qquad v = \int (6xy + 6y)\,dx + g(y)$

$v = 3x^2y - y^3 + 6xy + f(x)$ $\qquad v = 3x^2y + 6xy + g(y)$.

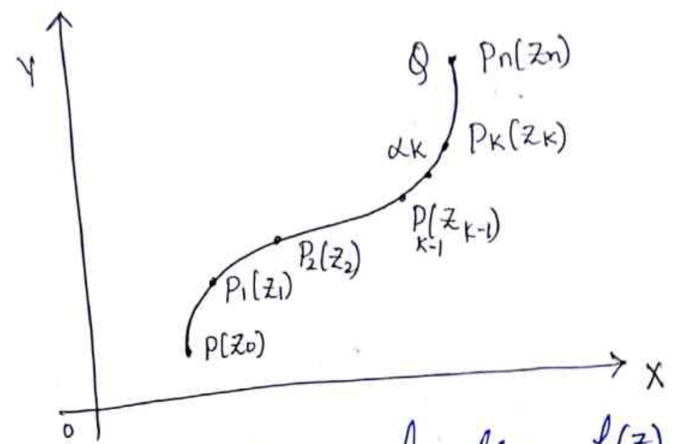
we choose, $f(x) = 0$, .. $g(y) = -y^3$ (so that 1st & 2nd Equa^ns of "v" are same)

$\therefore \boxed{v = 3x^2y - y^3 + 6xy.}$ $\qquad$ put; $\boxed{x = z}$, $\boxed{y = 0}$

The analytic fn : $f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(\ldots)$ [illegible] $\Rightarrow f(z) = z^3 + 3z^2 + 1$

COMPLEX VARIABLES : MODULE - 3 (PART-2)

Complex line integral :-



Consider a continuous function, $f(z)$ of the complex variable $z=x+iy$ defined at all points of curve C extending from P to Q, dividing the curve C into n parts by arbitrarily taking points, $P=p(z_0), p(z_1) \ldots\ldots p_n(z_n)=Q$, then

$\lim_{n\to\infty} \sum_{K=1}^{n} f(\alpha_K)\,\delta z_K$ where $\max|\delta z_K| \to 0$ as $n\to\infty$ is defined as complex line integral along path C denoted by : $\int_c f(z).dz$.

where ; $\delta z_K = z_K - z_{K-1}$. , $\alpha(K)$ is point on arc of Curve.

CAUCHY'S THEOREM :-

Statement :- If $f(z)$ is analytic at all points inside and on a simple closed curve C, then : $\boxed{\int_c f(z).dz = 0.}$

proof :- Let $f(z) = u+iv$. // $dz = dx+idy$.

Then, $\int_c f(z).dz = \int_c (u+iv).(dx+idy)$

$$\int_c f(z)dz = \int_c (udx - vdy) + i\int_c (vdx+udy) \quad -(1)$$

We have ; Green's theorem in a plane stating that, if $M(x,y)$ & $N(x,y)$ are 2 real valued functions having continuous 1st order partial derivatives in a region R bounded by the curve C then ;

Derive the Statement

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \sim (2)$$

Applying this theorem to the two line integrals in RHS of (1) ; we get

$$\int_C f(z).dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right).dx\,dy.$$

Since, $f(z)$ is analytic, we have Cauchy-Riemann Equations :

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$, hence we have ;

$$\int_C f(z)\,dz = \iint_R \underbrace{\left(\cancel{\frac{\partial u}{\partial y}} - \cancel{\frac{\partial u}{\partial y}}\right)}_{0}.dx\,dy + i\iint_R \underbrace{\left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right)}_{0}.dx\,dy.$$

Thus, we get ; $\boxed{\int_C f(z).dz = 0.}$

Hence, the proof of Cauchy's Theorem.

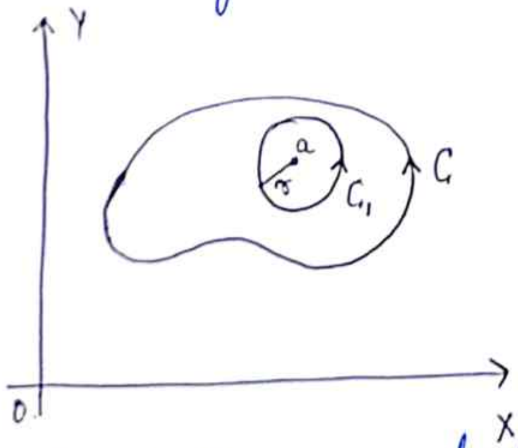
Derive the Cauchy's Integral Formula :- ②

Statement :- If $f(z)$ is analytic inside and on a simple closed curve C and if "a" is any point within C then;

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$

proof :- Since "a" is a point within C, we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C.

The function; $\frac{f(z)}{z-a}$ is analytic and on the boundary of the annular region b/w C & C_1.



Now, By the Consequence of the Cauchy's Theorem;

we have; $$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{— ①}$$

The Equation of C_1 (circle with centre "a" & radius "r"), can be written in the form :- $|z-a| = r$.

Which is equivalent to $\Rightarrow$ $z-a = re^{i\theta}$

$z = a + re^{i\theta}$, $0 \le \theta \le 2\pi$, $dz = ire^{i\theta}\,d\theta$

$\Rightarrow$ using these results in Equation ①, RHS ; we get

$$\int_C \frac{f(z)}{z-a} \cdot dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta} \cdot d\theta.$$

$$\int_C \frac{f(z)}{z-a} \cdot dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\, d\theta.$$

This is true for any ; $r>0$, however small,

Hence ; $r \to 0$ we get ...

$$\int_C \frac{f(z)}{z-a} \cdot dz = i\int_{\theta=0}^{2\pi} f(a) \cdot d\theta = i\, f(a)\, [\theta]_0^{2\pi}.$$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a).$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \cdot dz.}$$

Hence, the proof of Cauchy's Integral theorem

* Singularity / Singular point :-

A point $z=a$, where $f(z)$ fails to be analytic is called singularity or singular point of $f(z)$. $\rightarrow$ Ex :- $f(z) = \frac{z}{z-2}$.

* [Sing. pts are points which make $Dr \rightarrow$ terms $\rightarrow 0$]

$z=2$ is singular point.

Pole :- If the principal part of $f(z)$, consists of only a finite no of terms, say m ; then we say that ; $z=a$ is a "pole of order m".

Simple pole :- A pole of order 1 $(m=1)$ is simple pole.

Residues :- The coefficient of $\frac{1}{z-a}$, that is a_{-1} in the expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z=a$.

Ex :- $f(z) = \frac{\cos z}{z^5}$, then $f(z)$ can be expanded as;

$$f(z) = \frac{1}{z^5}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} \dots\right]$$

$$f(z) = \left[\frac{1}{z^5} - \frac{1}{2!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z} \dots\right]$$

$\therefore$ The residue of $f(z)$ at pole, $z=0$ is coefficent of $\frac{1}{z-0} = \frac{1}{z}$

$\Rightarrow \frac{1}{4!} = \frac{1}{24} \Rightarrow \frac{1}{24}$ is Residue

$\longrightarrow$ Problems after Cauchy's Residue Theorem stmt :-

* [Problems & solns :-

1) For the function ; $f(z) = \frac{2z+1}{z^2-z-2}$, determine the poles & residue at the poles.

soln :- In $f(z) = \frac{2z+1}{z^2-z-2}$

$\rightarrow z^2-z-2$
$z^2-2z+1z-2$
$z(z-2)+1(z-2) = \boxed{(z+1)(z-2)}$

WKT, [The poles are the points, by which by substituting them, we get $Dr \to 0$]

$$f(z) = \frac{2z+1}{(z-2)^1(z+1)^1} \quad m=1 \text{ (Simple pole)}$$

$\therefore$ The poles (simple) are :- $z=2$ & $z=-1$

Now, to find residue : at $z=a=2$]

* Cauchy's Residue Theorem :-

Statement :- If $f(z)$ is analytic inside and on the boun of a simple closed curve C, except for a finite number of poles, a, b, c ...., then integral of $f(z)$ over C is equal to $2\pi i$ times the sum of residues at the poles inside C,

$$\Rightarrow \int_C f(z)dz = 2\pi i (a_{-1} + b_{-1} + C_{-1} + \dots)$$

Important formulaes :-

1) If we obtain simple pole.. then $\boxed{\underset{\text{Residue}}{a_{-1}} = \lim_{z\to a} \{(z-a) f(z)\}}$

2) If $z=a$ is pole of order m, then : $\boxed{R(m,a) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]}$

3) $\int_C f(z)dz = 2\pi i(R_1+R_2)$

→ problem (1) continuation ....

Now, to find Residue at $z=a=2$

$$\Rightarrow \lim_{z\to 2} (z-2).f(z) = \lim_{z\to 2} \cancel{(z-2)} . \frac{2z+1}{\cancel{(z-2)}(z+1)}$$

$$= \lim_{z\to 2} \frac{2z+1}{z+1}.$$

$$= \frac{2(2)+1}{2+1} = \frac{5}{3} \text{ is Residue at } z=2$$

Now, to find residue at $z=a=-1$

$$\Rightarrow \lim_{z\to -1} (z+1).f(z) = \lim_{z\to -1} \cancel{(z+1)} \frac{(2z+1)}{(z-2)\cancel{(z+1)}}$$

$$= \lim_{z\to -1} \frac{(2z+1)}{(z-2)} = \frac{2(-1)+1}{(-1-2)} = \frac{-1}{-3}.$$

$\therefore \frac{1}{3}$ is Residue at $z=-1$

———*———

https://hemanthrajhemu.github.io

Determine residue at pole of function;

$f(z) = \dfrac{\sin z}{(2z-\pi)^2}$

Soln :- $f(z) = \dfrac{\sin z}{(2z-\pi)^2}$, $\underline{\underline{m=2}}$, It is a pole of order; $\underline{\underline{m=2}}$

Now, $2z-\pi = 0$, $2z = \pi$

$\boxed{z = \pi/2}$ is pole of order 2.

Now, to find the residue of $f(z)$ at $\underline{\underline{z = a = \pi/2}}$

$= \lim_{z\to\pi/2} \dfrac{1}{(2-1)!} \dfrac{d^{2-1}}{dz^{2-1}}\left\{(z-\pi/2)^2 \cdot f(z)\right\}$ // By formula..

$$\boxed{R(m,a) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^m}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}.}$$

$= \lim_{z\to\pi/2} \dfrac{1}{1!} \dfrac{d}{dz}\left\{(z-\pi/2)^2 \cdot \dfrac{\sin z}{(2z-\pi)^2}\right\} = \lim_{z\to\pi/2} \dfrac{d}{dz}\left\{\dfrac{(\cancel{2z-\pi})^2}{2^2} \cdot \dfrac{\sin z}{(\cancel{2z-\pi})^2}\right\}$

$= \dfrac{1}{4} \lim_{z\to\pi/2} \dfrac{d}{dz}\left\{\sin z\right\} = \dfrac{1}{4} \lim_{z\to\pi/2} (\cos z) = \dfrac{1}{4}\left[\cos \pi/2\right]$

$= 1/4 (0)$

$= 0.$

$\therefore$ Thus, the residue of pole is $\underline{\underline{0.}}$

3) Find the residues of the function:-

$f(z) = \dfrac{z}{(z+1)(z-2)^2}$ also find poles.

Soln:- $f(z) = \dfrac{z}{(z+1)^1 (z-2)^2}$

here, $z+1 \Rightarrow \underline{\underline{z=-1}}$ is a pole of "$\underline{\underline{\text{order } 1}}$"

$(z-2)^2 \Rightarrow \underline{\underline{z=2}}$ is a pole of "$\underline{\underline{\text{order } 2}}$"

$\Rightarrow$ Case: 1 To find Residue of $f(z)$ for a simple pole ; ie, $z=-1$ // order "$m=1$"

Demo:- , Evalu

$$\Rightarrow \lim_{z\to -1} (z+1)\cdot f(z) = \lim_{z\to -1} \cancel{(z+1)}\left[\frac{z}{\cancel{(z+1)}(z-2)^2}\right]$$

$$= \lim_{z\to -1}\left\{\frac{z}{(z-2)^2}\right\} = \frac{-1}{(-1-2)^2} = \frac{-1}{9}$$ is Residue at pole $z=-1$

$\Rightarrow$ Case:2 To find Residue of $f(z)$ for pole ; $z=2$ of "order 2."

$$\Rightarrow \lim_{z\to 2} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{\cancel{(z-2)^2}\cdot\left[\frac{z}{(z+1)\cancel{(z-2)^2}}\right]\right\}$$

$$= \lim_{z\to 2}\frac{1}{1!}\frac{d}{dz}\left\{\frac{z}{z+1}\right\}$$ // Quotient rule...

$$= \lim_{z\to 2}\left\{\frac{1}{(z+1)^2}\left[(z+1)(1)-z(1+0)\right]\right\}$$

$$= \lim_{z\to 2}\left[\frac{z+1-z}{(z+1)^2}\right] = \frac{1}{(2+1)^2} = \frac{1}{9}$$ is Residue at pole $z=2$

Amit

4) Determine the poles & residue of functions given :-

① $f(z) = \dfrac{z}{(z-1)^2(z+2)}$

② $f(z) = \dfrac{4z-1}{z^2-z-2}$

③ $f(z) = \dfrac{z}{(z+2)^2}$

https://hemanthrajhemu.github.io

Problems :-

1) Evaluate : $\int_C \frac{e^{2z}}{(z+1)(z-2)} \cdot dz$, where C is the circle $|z|=3$.

Soln :- The poles of the function ; $f(z) = \frac{e^{2z}}{(z+1)(z-2)}$

are :- $\underline{\underline{z=-1}}$ & $\underline{\underline{z=2}}$ which are simple poles and both these poles lie within the circle ; $|z|=3$.

$\therefore$ Residue of $f(z)$ at $\underline{\underline{z=a=-1}}$ is given by ;

$$\lim_{z\to -1} (z+1).f(z) = \lim_{z\to -1} \cancel{(z+1)}\left[\frac{e^{2z}}{\cancel{(z+1)}(z-2)}\right] = \lim_{z\to -1}\left[\frac{e^{2z}}{z-2}\right]$$

$$= \frac{e^{-2}}{-3} = \boxed{\frac{-1}{3e^2} = R_1}, \text{(say)}$$

Also, Residue of $f(z)$ at $\underline{\underline{z=a=2}}$ is given by ;

$$\lim_{z\to 2} \cancel{(z-2)}\left[\frac{e^{2z}}{(z+1)\cancel{(z-2)}}\right] = \lim_{z\to 2}\left[\frac{e^{2z}}{z+1}\right] = \boxed{\frac{e^4}{3} = R_2} \text{(say)}$$

Now, We have by Cauchy's Residue Theorem ;

$$\int_C f(z).dz = 2\pi i \, (R_1+R_2).$$

$$\int_C \frac{e^{2z}}{(z+1)(z-2)} dz = 2\pi i\left[\frac{-1}{3e^2}+\frac{e^4}{3}\right] = 2\pi i\left(e^4-\frac{1}{e^2}\right) //.$$

2) Evaluate : $\int_C \frac{z^2+5}{(z-2)(z-3)}$ using Residue theorem ; $C : |z| = 4$.

3) Evaluate :- $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where $C : |z| = 3$.

Soln :- $f(z) = \frac{e^{2z}}{(z+1)^4}$.

$z = -1$ is a pole of order 4, $m = 4$ which lies inside C

$|z| = 3$.

$\therefore$ The residue of $f(z)$ at $z = a = -1$ is given by ;

$$= \lim_{z \to -1} \frac{1}{(4-1)!} \frac{d^3}{dz^3} \left\{ \cancel{(z+1)^4} \frac{e^{2z}}{\cancel{(z+1)^4}} \right\}$$

$$= \lim_{z \to -1} \frac{1}{3!} \frac{d^3}{dz^3} \left\{ e^{2z} \right\} = \lim_{z \to -1} \left(8 e^{2z} \right)$$

$$= \frac{4}{3} e^{-2}.$$

By Applying Cauchy's Residue thm ;

$$\int_C f(z) \cdot dz = 2\pi i \left(\frac{4}{3} e^{-2} \right)$$

$$\therefore \boxed{\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3e^2}}$$

Asmt : Evaluate :- $\int_C \frac{z^2}{(z-1)^2(z+1)} \cdot dz$, $C : |z| = 2$. (Ans :- $4\pi i$)

Evaluate :- $\int_C \frac{z-3}{z^2+2z+5}$ $C : |z| = 1$ (Ans :- 0)

Evaluate :- $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}.dz$, where

C is the circle $|z|=3$.

Soln :- Let $f(z) = \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}$; $C : |z| = 3$.

$\Rightarrow$ $z=1$ is a pole of order 2 & $z=2$ is a pole of order 1

Both of them lies within the circle; $|z|=3$.

$\therefore$ Residue at $z=1$ be denoted by R_1;

$$\Rightarrow R_1 = \lim_{z\to1} \frac{1}{(2-1)!} \frac{d}{dz}\left\{ \cancel{(z-1)^2} \cdot \frac{(\sin\pi z^2+\cos\pi z^2)}{\cancel{(z-1)^2}(z-2)} \right\}$$

$$= \lim_{z\to1} \frac{d}{dz}\left\{ \frac{\sin\pi z^2+\cos\pi z^2}{z-2} \right\}$$

$$= \lim_{z\to1} (\sin\pi z^2+\cos\pi z^2) \cdot \frac{-1}{(z-2)^2} + \lim_{z\to1} 2\pi z(\cos\pi z^2 - \sin\pi z^2) \cdot \frac{1}{z-2}$$

$\therefore$ $\boxed{R_1 = (1+2\pi)}$ // $\sin\pi = 0$, $\cos\pi = -1$

Now, Residue at $z=2$ be denoted by R_2,

$$R_2 = \lim_{z\to2} \cancel{(z-2)} \left\{ \frac{\sin\pi z^2+\cos\pi z^2}{\cancel{(z-2)}(z-1)^2} \right\} = \lim_{z\to2} \left\{ \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2} \right\} = 1$$

$\boxed{\therefore R_2 = 1}$

Hence, by Cauchy's Residue theorem, $\int_C f(z)dz = 2\pi i\,(R_1+R_2)$

$$= 2\pi i\,(1+2\pi+1) = 4\pi i(1+\pi)$$

$$\boxed{\therefore \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}.dz = 4\pi i(1+\pi).}$$ //..

Ans

5) Evaluate; $f(z) = \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}$, $C, |z| = 3$

* Conformal Transformations :-

If a transformation preserves the angle b/w any 2 curves both in magnitude and sense then it is called a Conformal transformation.

Imp * Discuss the transformation ; $w = e^z$

Consider ; $w = e^z$

ie, $u + iv = e^{x+iy}$ // $w = u + iv$, $z = x + iy$.

$u + iv = e^x \cdot e^{iy}$ // $e^{iy} = \cos y + i \sin y$.

$u + iv = e^x [\cos y + i \sin y]$

$\therefore u = e^x \cos y \quad , \quad v = e^x \sin y \quad \text{—} (1)$

Now, we shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z plane

ie, $x = \text{Constant}$, $y = \text{Constant}$.

Let us eliminate x & y seperately from Eqn (1) ;

$\Rightarrow$ Squaring & Adding ; we get ; $u^2 + v^2 = (e^x \cos y)^2 + (e^x \sin y)^2$

$\Rightarrow u^2 + v^2 = e^{2x} \cos^2 y + e^{2x} \sin^2 y$.

$= e^{2x} [\sin^2 y + \cos^2 y] \Rightarrow u^2 + v^2 = e^{2x} \sim (2)$

Also, by dividing ; we get ...

$\Rightarrow \dfrac{u}{v} = \dfrac{e^x \cos y}{e^x \sin y} \Rightarrow \dfrac{u}{v} = \tan y \sim (3)$

Case : 1 Let $x=c_1$ where c_1 is Constant.

Equation (2) becomes ; $u^2+v^2 = e^{2x} = e^{2c_1} = \text{Constant} = r^2$, (say)

This represents a circle with centre origin & radius r in the w-plane.

Case : 2 Let $y=c_2$ where c_2 is Constant.

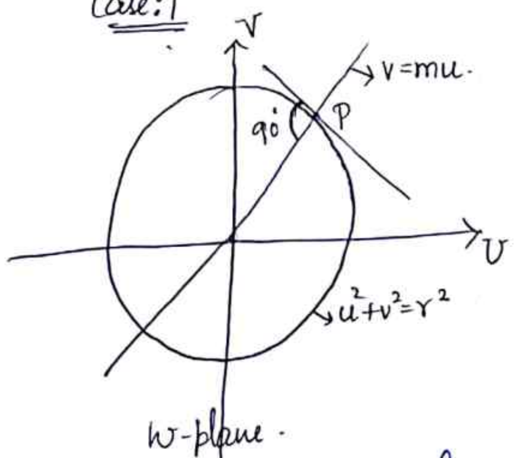
Equation (3) becomes ; $\frac{u}{v} = \tan y = \tan c_2 = m$, (say)

$\therefore v = mu$

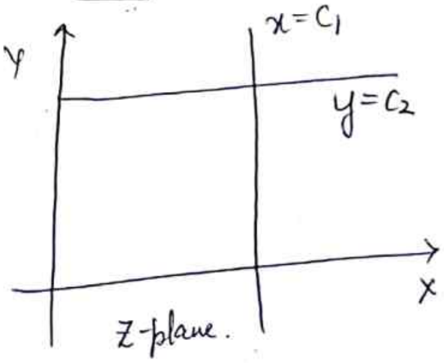
This represents a straight line passing through the origin in the w-plane.

Conclusion :- The straight line parallel to the x-axis $(y=c_2)$ in

Case : 1



Case : 2



the plane maps onto straight line passing through origin in w-plane the straight line parallel to y-axis $(x=c_1)$ in z-plane maps onto circle with centre origin & radius r, where $r=e^{c_1}$ in w-plane.

Suppose, we draw tangent ~~to~~ at pt of intersection of these 2 curves in w-plane, the angle subtended is equal to 90°.

Hence 2 Curves can be regarded as orthogonal trajectories of each other.

* Discuss the transformation :- $w = z + a^2/z$, $z \neq 0$. since,

Consider, $w = z + (a^2/z)$.

putting; $z = re^{i\theta}$, we have, ... $w = u + iv$.

$\Rightarrow u + iv = re^{i\theta} + (a^2/r) \cdot e^{-i\theta}$ // $r = \cos\theta + i\sin\theta$.

ie, $u + iv = r[\cos\theta + i\sin\theta] + (a^2/r)(\cos\theta - i\sin\theta)$

$u + iv = [r\cos\theta + a^2/r \cos\theta] + i[r\sin\theta - \frac{a^2}{r}\sin\theta]$

$\Rightarrow u + iv = [r + a^2/r]\cos\theta + i[r - a^2/r]\sin\theta$

$\therefore u = [r + a^2/r]\cos\theta \qquad v = [r - a^2/r]\sin\theta \sim (1)$.

Now, we shall elliminate r & θ seperately from (1) ...

To eliminate θ, let us put (1) in the form ..

$$\frac{u}{[r + a^2/r]} = \cos\theta \; ; \; \frac{v}{[r - (a^2/r)]} = \sin\theta.$$

$\Rightarrow$ squaring & adding, we obtain...

$$\frac{u^2}{[r + a^2/r]^2} + \frac{v^2}{[r - a^2/r]^2} = 1 \quad , r \neq a.$$

To eliminate r, let us put (1) in form...

$$\frac{u}{\cos\theta} = [r + a^2/r] \quad , \quad \frac{v}{\sin\theta} = [r - (a^2/r)]$$

$\Rightarrow$ squaring & subtracting, we obtain...

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = [r + (a^2/r)]^2 - [r - (a^2/r)]^2 = 4a^2$$

$$\frac{u^2}{(2a\cos\theta)^2} - \frac{v^2}{(2a\sin\theta)^2} = 1 \sim (3)$$

$z \neq 0$.

Since, $z = re^{i\theta}$, $|z| = r$ & $amp(z) = \theta$

$|z| = r \Rightarrow \sqrt{x^2+y^2} = r$ (or) $x^2+y^2 = r^2$

This represents a ~~straight~~ "circle" with centre origin & radius r, in the z-plane, when r is constant.

$\Rightarrow amp(z) = \theta$, $\tan^{-1}(y/x) = \theta$ or $y/x = \tan\theta$.

This represents "straight line" in z plane when θ is Constant.

We shall discuss the image in the w-plane, corresponding to r = Const, (circle) & θ = Const. (straight line) in z-plane.

Case:1 Let r = Constant

Equation (2) is of the form;

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1, \text{ where } A = [r + (a^2/r)], \ B = [r - (a^2/r)]$$

This represents an ellipse in the w-plane with $f = (\pm\sqrt{A^2-B^2}, 0)$ $= (\pm 2a, 0)$.

Since; $\sqrt{A^2-B^2} = \sqrt{(r+(a^2/r))^2 - [r-(a^2/r)]^2} = \sqrt{4a^2} = \pm 2a$

Hence, we conclude that $\bigcirc^{le}$, $|z| = r$ = Const in z-plane maps onto "ellipse" in w-plane

Case:2 Let θ - Constant.

Equation (3) is of the form...

$$\Rightarrow \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1, \text{ where } A = 2a\cos\theta, \ B = 2a\sin\theta.$$

This, represents hyperbola in w-plane with

$f = (\pm\sqrt{A^2+B^2}, 0) = (\pm 2a, 0)$.

Hence, we conclude, that "straight line" passing through in z-plane maps onto a "hyperbola in w-plane".

3) Discuss the transformation; $w = z + 1/z$.

9

Conformal transformation :-

1) Discuss the transformation of : $w = z^2$

Soln :- $w = z^2$

$\Rightarrow u + iv = (x+iy)^2$ where ; $w = u+iv$

$z = x+iy$.

$\Rightarrow u+iv = x^2 - y^2 + i2xy$

$\Rightarrow$ Seperating Real & imaginary parts...

$\therefore u = x^2 - y^2 \sim (1)$

$\therefore v = 2xy \sim (2)$

Case : 1 $x = c_1$ (Constant), Replace in Equation (1) & (2) ;

$\therefore u = c_1^2 - y^2$

$\therefore v = 2c_1 y$.

$\Rightarrow y = \frac{v}{2c_1}$

$\Rightarrow u = c_1^2 - \frac{v^2}{4c_1^2} \Rightarrow 4uc_1^2 = 4c_1^2 \cdot c_1^2 - v^2$

$v^2 = -4c_1^2 \cdot c_1^2 - \cancel{v^2}\ 4uc_1^2$

$\Rightarrow v^2 = -4c_1^2 [u - c_1^2] \sim (3)$ // $y^2 = 4ax$ is parabola..

$\Rightarrow$ Equation (3) represents the Equation of the parabola, which is symmentrical about real-axis & focus at the origin.

Case : 2 $y = c_1$, (Constant) Replace in Eqn (1) & (2),...

$\therefore u = x^2 - c_1^2$ & $\therefore v = 2c_1 x$

$\Rightarrow x = \frac{v}{2c_1}$

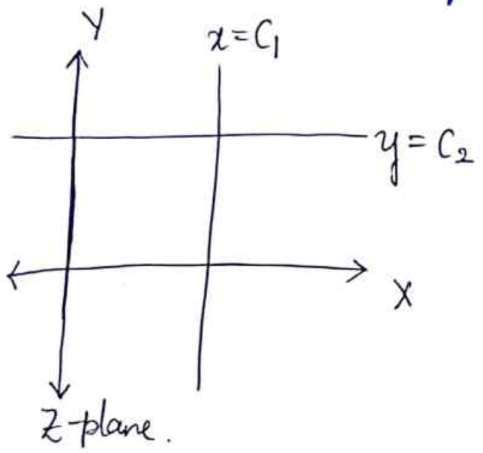
$\Rightarrow u = \frac{v^2}{4c_1^2} - c_1^2$

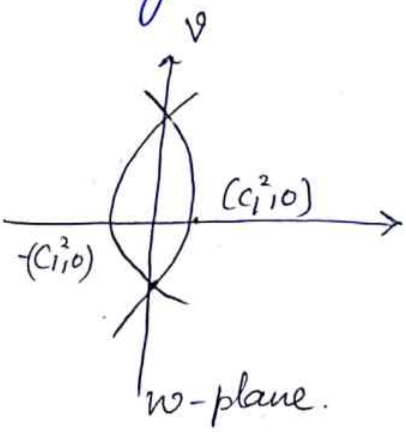
$u = \frac{v^2 - 4c_1c_1^2}{4c_1^2} \Rightarrow 4c_1^2 u = v^2 - 4c_1c_1^2$

$v^2 = 4c_1^2 u + 4c_1c_1^2$

$$\therefore v^2 = 4c_1^2[u + c_1^2] \sim (4)$$

$\Rightarrow$ Eqn (4) represents equation of the parabola, which is symmetric about the real axis & focus at the origin.





Conclusion:- Hence, we conclude that the line which is parallel to co-ordinate axis in z-plane which maps onto equation of parabola in w-plane.

2) Discuss the transformation; $w = e^z$

Soln:- $u+iv = e^z$ // $w = u+iv$, $z = x+iy$.

$u+iv = e^{x+iy}$.

$u+iv = e^x \cdot e^{iy}$ // $e^{iy} = \cos y + i\sin y$.

$u+iv = e^x \cdot [\cos y + i\sin y]$

$\Rightarrow$ Separating real & imaginary parts:-

$\Rightarrow u = e^x \cos y$ & $v = e^x \sin y$.

Consider; $u^2+v^2 = e^{2x}\cos^2 y + e^{2x}\sin^2 y$.

$= e^{2x}(\cos^2 y + \sin^2 y)$

$$\therefore \boxed{u^2+v^2 = e^{2x}} \sim (1)$$

Consider;

$$\frac{u}{v} = \frac{e^x \cos y}{e^x \sin y}.$$

$$\therefore \boxed{\frac{u}{v} = \cancel{\tan y}} \sim (2)$$

$\cot y$

$$\frac{v}{u} = \tan y \sim (2).$$

(10)

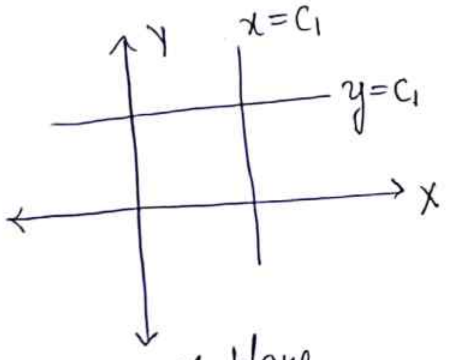
Case : 1 $x = c_1$, Constant, sub in Eqn ①, we get...

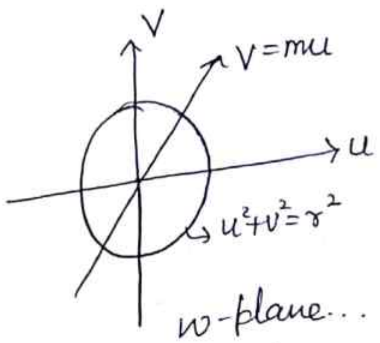
$$\Rightarrow u^2+v^2 = e^{2c_1} = r^2 \text{ (say)}. \quad \text{—} ③$$

Case : 2 $y = c_1$, constant, sub in Eqn ②, we get ...

$$\Rightarrow \frac{v}{u} = \tan c_1$$

$$\Rightarrow v = mu, \text{ (say)} \quad \text{—} ④$$





Conclusion :- The line which is ||el to y-axis [$x = c_1$] in z-plane, maps onto Equation of ⊙le [$u^2+v^2=r^2$] in w-plane. ||ly, the line parallel to x-axis [$y = c_1$] in z-plan maps on to Equation of straight line [$v = mu$] in w-plane...

✓3) Discuss the transformation of : $w = z + 1/z$.

* [If we solve this in Cartesian form, it will be complicated, Hence we solve it by Polar form].

Soln :- $w = z + 1/z$. // $w = u+iv$ // $z = re^{i\theta}$

$$\Rightarrow u+iv = re^{i\theta} + 1/re^{i\theta}$$

$$\Rightarrow u+iv = re^{i\theta} + \frac{1}{r}.e^{-i\theta} \quad // \; e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow u+iv = r[\cos\theta + i\sin\theta] + \frac{1}{r}[\cos\theta - i\sin\theta]$$

$$u+iv. = (r+1/r)\cos\theta + i[r - 1/r]\sin\theta.$$

$\Rightarrow$ Now, by separating real & imaginary parts..

$$\Rightarrow u = (r + 1/r)\cos\theta \quad \& \quad v = (r - 1/r)\sin\theta.$$

$$\Rightarrow \frac{u}{\cos\theta} = r + 1/r. \sim (*)_1 \quad \& \quad \frac{v}{\sin\theta} = r - 1/r. \sim (*)_2$$

$\Rightarrow$ SBS $\qquad\qquad$ $\Rightarrow$ SBS

$$\Rightarrow \frac{u^2}{\cos^2\theta} = (r + 1/r)^2 \quad \& \quad \frac{v^2}{\sin^2\theta} = (r - 1/r)^2$$

$$\frac{u^2}{\cos^2\theta} = r^2 + 1/r^2 + 2.\cancel{r}.1/\cancel{r} \quad \& \quad \frac{v^2}{\sin^2\theta} = r^2 + 1/r^2 - 2.\cancel{r}.1/\cancel{r}$$

Consider; $\dfrac{u^2}{\cos^2\theta} - \dfrac{v^2}{\sin^2\theta} = \cancel{r^2} + \cancel{1/r^2} + 2 - \cancel{r^2} - \cancel{1/r^2} + 2$

$$\therefore \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$$

$$\Rightarrow \frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1. \sim (1).$$

Since, by $(*)_1$ & $(*)_2$ $\Rightarrow$ $\dfrac{u}{r+1/r} = \cos\theta \sim (2)$ & $\dfrac{v}{r-1/r} = \sin\theta \sim (3)$.

~~$\Rightarrow$ using in (1)~~ $\Rightarrow$
S.B.S of (3) & (2) $\quad \dfrac{u^2}{(r+1/r)^2} + \dfrac{v^2}{(r-1/r)^2} = \cos^2\theta + \sin^2\theta.$

$$\Rightarrow \frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = 1 \sim (4).$$

$\Rightarrow$ $z = re^{i\theta}$, $\quad r = \sqrt{x^2+y^2}$

$$r^2 = (x^2+y^2)$$

$\theta = \tan^{-1}(y/x)$ $\Rightarrow$ $\dfrac{y}{x} = \tan\theta$

$$y = x.\tan\theta.$$

Case :1 Suppose : $r = C_1$ (or) $x^2+y^2 = C_1^2$ [Eqn of circle]

$\Rightarrow$ Substituting in Eqn ① ...

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \sim (5) \quad \text{where; } A^2 = \left[C_1 + \frac{1}{C_1}\right]^2$$
$$B^2 = \left[C_1 - \frac{1}{C_1}\right]^2$$

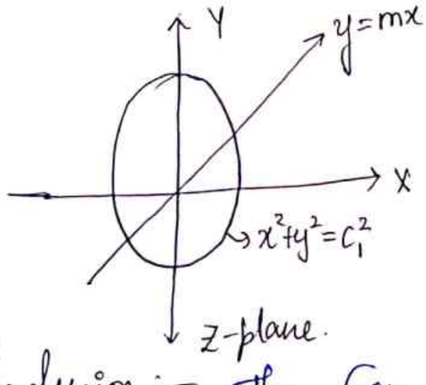
$\Rightarrow$ Eqn ⑤ represent Eqn of Ellipse with the focii $\left[\pm\sqrt{A^2-B^2}, 0\right] = \left[\pm 2a, 0\right]$

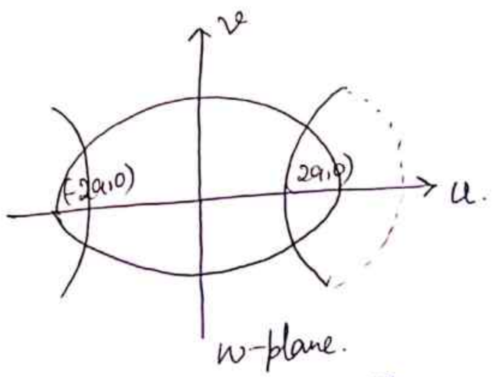
Case :2 Suppose : $\theta = C_1$ (or) $y = x \tan C_1 = mx$, (say).

Eqn ① becomes ;

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \sim (6) \quad \text{where; } (2\cos C_1)^2 = A^2$$
$$(2\sin C_1) = B^2$$

Eqn ⑥ represents ~~Eqn~~ of the ellipse with focci $\left[\pm\sqrt{A^2+B^2}, 0\right] = \left[\pm 2a, 0\right]$.





Conclusion :- The Eqn of circle in z-plane $[x^2+y^2=C_1^2]$, maps onto the Equation of ellipse in $\left[\frac{u^2}{A^2}+\frac{v^2}{B^2}=1\right]$ in W-plane.

|||ly, The Eqn of Straight line in Z-plane $[y=mx]$, maps onto the Equation of the Ellipse $\left[\left[\frac{u^2}{A^2}-\frac{v^2}{B^2}\right]=1\right]$ in w-plane.

——— * ———

BILINEAR TRANSFORMATION :- [B.T]

The transformation, $w = \dfrac{az+b}{cz+d}$, where a, b, c, d are real/Complex constant such that ; $ad-bc \neq 0$. is called bilinear transformation.

Invariant points :-
If the point z, maps itself ie ; $w=z$ under the bilinear transformation, then the point is called as invariant point (or) fixed point.

To find the bilinear transformations ;

We have ;

$$w = \frac{az+b}{cz+d}.$$

(or)

$$\boxed{w = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}}$$

Problems : on Bilinear transformations :-

1) Find the bilinear transformation which maps the points, $z = 1, i, -1$ into $w = i, 0, -i$.

Soln:- Let $w = \frac{az+b}{cz+d}$ be the required bilinear transformation.

Now, we shall substitute the given values of z & w to obtain 3 equivalent equations as follows :-

$w_1 = i$, $w_2 = 0$, $w_3 = -i$

$z_1 = 1$, $z_2 = i$, $z_3 = -1$

$z=1, w=i$; $w_1 = \frac{az_1+b}{cz_1+d}$

$$\Rightarrow i = \frac{a(1)+b}{c(1)+d} \Rightarrow a+b-ic-id = 0 \sim (1)$$

$z_2 = i$, $w_2 = 0$; $w_2 = \frac{az_2+b}{cz_2+d}$

$$\Rightarrow 0 = \frac{a(i)+b}{c(i)+d} \Rightarrow ai+b = 0 \sim (2)$$

$z_3 = -1$, $w_3 = -i$; $w_3 = \frac{az_3+b}{cz_3+d}$

$$\Rightarrow -i = \frac{a(-1)+b}{c(-1)+d} \Rightarrow -a+b-ic+id = 0 \sim (3)$$

Consider ; Eqn (1) + (3) ... $\cancel{a}+b-ic-\cancel{id}-\cancel{a}+b-ic+\cancel{id} = 0$

$$\Rightarrow 2b-2ic = 0$$

$$b-ic = 0 \sim (4)$$

Now, we shall solve; 2 & 4;

$$ia + b + 0c = 0$$
$$0a + 1b - ic = 0.$$

Applying the rule of Cross multiplication, we have...

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}} = \frac{c}{\begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\Rightarrow \frac{a}{-i} = \frac{-b}{i^2} = \frac{c}{i} \quad \text{(or)} \quad \frac{a}{-i} = \frac{b}{-1} = \frac{c}{i} = K, \text{ say.}$$

$$\therefore \underline{\underline{a = -iK}}, \quad \underline{\underline{b = -K}}, \quad \underline{\underline{c = iK}}$$

$\Rightarrow$ sub in Eqn 1, we get..

$$1 \Rightarrow iK - K + K - di = 0.$$
$$\Rightarrow -(di + iK) = 0.$$
$$\Rightarrow \underline{\underline{d = -K}}$$

$\Rightarrow$ Sub, values of a, b, c, d in assumed bilinear transformations, we get..

$$\therefore w = \frac{-iKz - K}{+iKz - K} = \frac{-K(1+iz)}{-K(1-iz)}.$$

Thus, $\boxed{w = \frac{1+iz}{1-iz}.}$ is Required bilinear transformation

2) Find ... (6)

2) Find the Bilinear transformation, which maps the points ; $z = 1, i, -1$, into $w = 2, i, -2$. , also find invariant points (or) fixed pt of transformation.

Soln :- Let $w = \dfrac{az+b}{cz+d}$.

$z = 1, i, -i$, $w = 2, i, -2$.

$\Rightarrow$ $z=1$, $w=2$ $\Rightarrow$ $w_1 = \dfrac{az_1+b}{cz_2+d}$.

$2 = \dfrac{a(1)+b}{c(1)+d}$ $\Rightarrow$ $2c+2d = a+b$.

$a+b-2c-2d=0 \sim (1)$.

$\Rightarrow$ $z=i$, $w=i$ $\Rightarrow$ $w_2 = \dfrac{az_2+b}{cz_2+d}$

$i = \dfrac{a(i)+b}{c(i)+d}$ $\Rightarrow$ $ci^2+di = ai+b$

$ai+b+c-di=0 \sim (2)$.

$\Rightarrow$ $z=-i$, $w=-2$ $\Rightarrow$ $-2 = \dfrac{-a+b}{-c+d}$.

$\Rightarrow a-b+2c-2d=0 \sim (3)$.

$\Rightarrow$ Sub Eqn in (1) & (3).

$$\begin{array}{l} a+\cancel{b}-\cancel{2c}-2d=0 \\ a-\cancel{b}+\cancel{2c}-2d=0 \\ \hline 2a-4d=0 \sim (4) \end{array}$$

$\Rightarrow$ Sub Eqn in (2) & (4) ...

$ai+b+c-id=0$.

$2a+0b+0c-4d=0$.

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}.$$

$$\Rightarrow \frac{(w-2)(i+2)}{(2-i)(-2-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}. \quad \to (-\text{Common})$$

$$\Rightarrow \frac{(w-2)(i+2)}{-(2-i)(2+w)} = \frac{(z-1)(i+1)}{-(1-i)(1+z)}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{z-1}{z+1}\left\{\frac{(i+1)(2-i)}{(1-i)(i+2)}\right\}$$

$$\frac{w-2}{w+2} = \frac{z-1}{z+1}\left\{\frac{2i-i^2+2-i}{i+2-i^2-2i}\right\}$$

$$\frac{w-2}{w+2} = \frac{(z-1)}{(z+1)}\frac{(i+3)}{(-i+3)}.$$

$$= \frac{(w-2)(-i+3)}{(w+2)(i+3)} = \frac{z-1}{z+1}.$$

$$\Rightarrow (w-2)(-i+3)(z+1) = (z-1)(w+2)(i+3).$$

$$\Rightarrow (w-2)[-iz-i+3z+3] = (w+2)[zi+3z-i-3]$$

$$\Rightarrow -iwz - \cancel{iw} + 3\cancel{w}z + 3w + 2\cancel{i}z + 6z - 2i - \cancel{6}.$$

$$-[iwz + 3\cancel{w}z - \cancel{iw} - 3w + 2\cancel{i}z + 6z - 2i - \cancel{6}]$$

$$\Rightarrow -iwz + 3w + 2i - 6z = iwz - 3w - 2i + 6z. = 0.$$

$$\Rightarrow -2iwz + 6w + 4i - 12z = 0.$$

$$iwz - 3w - 2i + 6z = 0.$$

$$w(iz-3) = 2i - 6z.$$

$$\therefore \boxed{w = \frac{2i-6z}{iz-3}.}$$

For invariant pts, Consider; $z = \dfrac{2i-6z}{iz-3}.$

$$z = \frac{2i - 6z}{iz - 3}$$

$$z(iz-3) = 2i - 6z$$

$$iz^2 - 3z = 2i - 6z$$

$$z^2 + 3z - 2i = 0 \qquad b = 3,\ a = 1,\ c = -2$$

$$z = \frac{-3 \pm \sqrt{9 - 4(i)(-2i)}}{2(i)} = \frac{-3 \pm \sqrt{9-8}}{2i}$$

$$z = \frac{-3 \pm 1}{2i} \quad \therefore z = \frac{-3+1}{2i} \quad (\text{or}) \quad z = \frac{-3-1}{2i}$$

$$\therefore \boxed{z = \frac{-1}{i}} \quad (\text{or}) \quad \boxed{z = \frac{-2}{i}}$$

Ans

3) Find the Bilinear transformations; which maps; $z = 0, -i, -1$. & $w = i, 1, 0$. $\left[w = \frac{i(1+z)}{1-z}\right.$, 1

4) Find the Bilinear transformation; which maps: $z = 0, i, \infty$, $w = 1, -i, -1$., also find invariant point.

Soln:- Consider; $\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$

$$= \left\{ \frac{z_3\left[\frac{z_2}{z_3} - 1\right]}{z_3\left[\frac{-z}{z_3} + 1\right]} \right\} = \frac{\frac{1}{\infty} - 1}{1 - 1/\infty} = \frac{0-1}{1-0} = -1$$

WKT; $z = 0, i, \infty$; $w = i, 1, 0$

$$\Rightarrow \frac{(w-1)(-i+1)}{(1+i)(-1-w)} = -1\left[\frac{z-0}{0-i}\right]$$

$$\Rightarrow \frac{(w-1)(1-i)}{-(i+1)(w+1)} = +\frac{z}{i} \qquad // \ \frac{1}{i} = -i$$

$$\Rightarrow \frac{(w-1)(1-i)}{-(i+1)(w+1)} = -zi$$

$$\Rightarrow (w-1)(1-i) = zi(1+i)(w+1)$$

$$\Rightarrow w - iw - 1 + i = (zi - z)(w+1)$$

$$\Rightarrow w - iw - 1 + i = zwi + zi - zw - z.$$

$\Rightarrow -w+1-iw+i = z+iz+wz+iwz.$

$w[-1-i-z-iz] = z+iz-1-i \Rightarrow -w(1+i+z+iz) = (z-1)(1+i)$

$\Rightarrow -w(1+i+z+iz) = (z+1)(i+1).$

$\Rightarrow -w[(1+i)(z+i)] = [(z-1)(i+1)]$

$\therefore \boxed{w = \frac{(1-z)}{1+z}}$

$\Rightarrow z = \frac{1-z}{1+z}. \quad \Rightarrow z+z^2-1+z=0.$

$z^2+2z-1=0.$

$z = \frac{-2\pm\sqrt{4+4}}{2} \Rightarrow z = -1\pm\sqrt{2}$

$\therefore \boxed{z=-1+\sqrt{2}}$ (or) $\boxed{z=-1-\sqrt{2}}$

—— * ——

Assignments

1) Find B.T. for :- $z = -1, i, 1$, $w = 1, i, -1$., also find inv pt.
2) Find B.T for : $z = 0, 1, \infty$, $w = -5, -1, 3$. " —
3) Find B.T for :- $z = i, +1, -1$, $w = 1, 0, \infty$. " —
4) Find B.T for :- $z = -1, i, 1$, $w = 1, i, -i$.
5) Find B.T for :- $z = \infty, i, 0$, $w = -1, -i, 1$.

—— * ——