13/02/20

1. Complex functions
2. Conformal Transformations
3. probability distributions
4. Statistical methods
5. Joint probability distributions and Sampling theory

Basics of Complex functions

$y = f(x)$

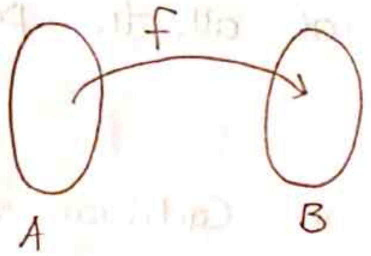
$W = f(z) = u + iv$

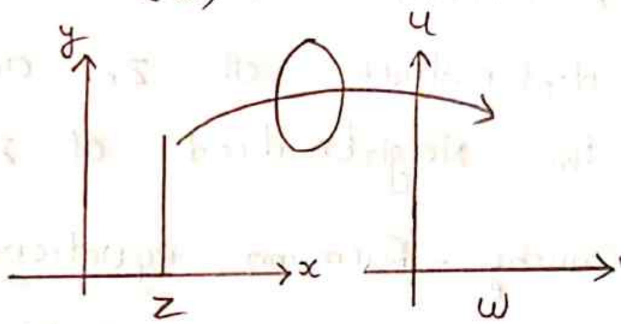
$z = x + iy$

$\bar{z} = x - iy$

$|z| = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$





Continuity

$\lim_{x \to a} f(x) = f(a)$

replace with

$f(z)$ and $z = z_0$

$\lim_{z \to z_0} f(z) = f(z_0)$

Differentiability

h is Same as δx

$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$

$\longrightarrow$ Analytic function differentiable at its Neighbourhood also.

C-R

Cauchy - Riemann equation

Module - 1

Calculus of Complex functions

Continuity :- A Complex function $f(z)$ is Said to be continuous at the point $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Differentiability :- A Complex function $f(z)$ is Said to be differentiable if

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

Analytic functions :- (Regular or polomorphic functions)

A function is Said to be analytic if it is differentiable at z_0 and at all the points in the Neighbourhood of z_0.

Cauchy - Riemann equations in Cartisian form :-

→ The Necessary Condition for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic is that $f(z)$ Satisfies Cauchy - Riemann equations.

i.e: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = \dfrac{-\partial u}{\partial y}$

Let $f(z) = u(x,y) + iv(x,y)$ be Analytic

$f(z)$ is differentiable

$f'(z)$ exists

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - \{u(x,y) + iv(x,y)\}}{\delta z}$$

Combining real and imaginary

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta z} \right\} + i \left\{ \frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta z} \right\} \quad \text{—} (1)$$

But $z = x+iy$

$\delta z = \delta x + i\,\delta y$

Case(i) : $\left.\begin{array}{l}\delta y = 0 \\ \delta z = \delta x\end{array}\right\} \rightarrow$ Substitute in eq (1)

$$f'(z) = \lim_{\delta x \to 0} \left\{ \frac{u(x+\delta x, y) - u(x,y)}{\delta x} \right\} + i \left\{ \frac{v(x+\delta x, y) - v(x,y)}{\delta x} \right\}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \quad \text{—} (2)$$

Case (ii): $\delta x = 0$

$\delta z = i\delta y$ $\qquad$ $i\delta y \to 0$, $\delta y \to 0$

$$f'(z) = \lim_{\delta y \to 0} \left\{ \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} \times \frac{i}{i} + \cancel{i}\,\frac{v(x, y+\delta y) - v(x,y)}{\cancel{i}\,\delta y} \right\}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \longrightarrow (3)$$

As equation 2 and 3 are equal

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

So, Comparing real and imaginary parts

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \longrightarrow \text{C-R equations}$$

These equations are called the Cauchy - Riemann equations in Cartisian form.

Cauchy - Riemann equations in polar form :-

The necessary conditions for a functions $f(z) = u(r,\theta) + iv(r,\theta)$ to be analytic is that $f(z)$ satisfies C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

Proof :-

$f(z) = u(r,\theta) + iv(r,\theta)$ be analytic

$f(z)$ is differentiable

$f'(z)$ exists

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta) - \{u(r,\theta) + iv(r,\theta)\}}{\delta z}$$

So, Combining real and imaginary parts

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \right\} \to (1)$$

we know $z = re^{i\theta}$ By $\frac{d}{dx}(uv) = uv' + vu'$

$$\delta z = r \cdot ie^{i\theta} \cdot \delta\theta + e^{i\theta} \cdot \delta r$$

Case (i): Let $\delta\theta \to 0$

So, $\delta z = e^{i\theta} \cdot \delta r$ $\delta z \to 0$, $e^{i\theta}\delta r \to 0$, $\delta r \to 0$

$$f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta} \cdot \delta r} + i \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta} \cdot \delta r}$$

$$\boxed{f'(z) = e^{-i\theta} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)} \to (2)$$

Case(ii): Let $\delta r \to 0$

$$\text{So, } \delta z = r \cdot i e^{i\theta} \delta\theta \qquad \text{as } \delta z \to 0, \quad r \cdot i e^{i\theta} \cdot \delta\theta \to 0, \quad \delta\theta \to 0$$

$$f'(z) = \lim_{\delta\theta \to 0} \left\{ \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{r \cdot i e^{i\theta} \cdot \delta\theta} \times \frac{i}{i} + \cancel{i} \frac{v(r, \theta+\delta\theta) - v(r,\theta)}{r \cdot \cancel{i} \cdot e^{i\theta} \cdot \delta\theta} \right\}$$

$$\boxed{f'(z) = \frac{1}{r} \cdot e^{-i\theta} \left\{ -i \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} \right\}} \to (3)$$

As equation (2) and (3) are equal

$$\cancel{e^{-i\theta}} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) = \frac{\cancel{e^{-i\theta}}}{r} \left(-i \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} \right)$$

Comparing real and imaginary parts

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r} \frac{\partial u}{\partial \theta}}$$

These are required equations C-R equations in polar form.

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Cosequences:

(i) if $f(z) = u + iv$ is an analytic function, then u and v satisfies Laplace's equation i.e:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

proof is:

Then CR- equations is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—} (1)$$

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \quad \text{—} (2)$$

differentiate (1) partially w.r.t x and (2) w.r.t y

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{-\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-\partial^2 u}{\partial y^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Thus, u satisfies the Laplace equation.

Then for v term equation

differentiate ① w.r.t y and ② w.r.t x

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y^2} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial x\,\partial y}$$

So,

$$\frac{\partial^2 v}{\partial y^2} = \frac{-\partial^2 v}{\partial x^2}$$

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0} \longrightarrow$$ So, Thus v also Satisfies Laplace equation

(ii) If $f(z) = u + iv$ is an analytic function then $u(x,y) = c_1$ and $v(x,y) = c_2$ represent the Orthogonal family of curves.

$u(x,y) = c_1$

differentiate w.r.t x

$$\frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = \frac{-\partial u}{\partial x}$$

$$\frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1$$

$V(x,y) = c_2$

differentiate w.r.t x

$$\frac{\partial v}{\partial x}\frac{dx}{dx} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2$$

$$m_1 \times m_2 = \frac{-\cancel{\partial u}/\partial x}{\partial u/\cancel{\partial y}} \times \frac{-\partial v/\cancel{\partial x}}{\cancel{\partial v}/\partial y}$$

$$= -1 \qquad \text{as } \frac{\partial u}{\partial x} = \frac{+\partial v}{\partial y}$$

Therefore $u(x,y)=c_1$ and $v(x,y)=c_2$ represents the Orthogonal family of curves

(iii) Finding the derivative of an analytic function

Step(i) – given $w=f(z)$ substitute $z=x+iy$ or $z=re^{i\theta}$

step(ii) – verify C-R equations to conclude $f(z)$ is analytic

step(iii) – To find the derivative use the result

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$f'(z)=e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)$$

Step(iv) – put $x=z$ and $y=0$

($r=z$ and $\theta=0$)

to get $f'(z)$ in terms of z.

Q Show that $w=z+e^z$ is analytic and hence find $\frac{dw}{dz}$

Sol: Given $w=z+e^z$ $\quad z=x+iy$

$$u+iv=x+iy+e^{x+iy}$$

$$u+iv=x+iy+e^x\cdot e^{iy} \qquad e^{i\theta}=\cos\theta+i\sin\theta$$

$$u+iv=x+iy+e^x(\cos y+i\sin y)$$

$u = x + e^x \cos y$ $\qquad$ $v = y + e^x \sin y$

$\frac{\partial u}{\partial x} = 1 + e^x \cos y$, $\qquad$ $\frac{\partial v}{\partial y} = 1 + e^x \cos y$

$\frac{\partial v}{\partial x} = e^x \sin y$ $\qquad$ $-\frac{\partial u}{\partial y} = -e^x(-\sin y)$

$= e^x \sin y$

as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

u and v Satisfies C-R equations

Therefore $w = f(z)$ is analytic

So, $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$= (1 + e^x \cos y) + i\,(e^x \sin y)$

So, put $x = 2$ and $y = 0$

$f'(z) = 1 + e^2(1) + i\,(e^2 \cdot (0))$

$\frac{dw}{dz} = 1 + e^2$

Q Show that $f(z) = \sin z$ is analytic and hence find $f'(z)$

Sol: Given $f(z) = \sin z$

$u + iv = \sin(x + iy)$

$u + iv = \sin x \cos iy + \cos x \sin iy$

$\cos(i\theta) = \cosh\theta$
$\sin(i\theta) = i\sinh\theta$ } we can verify by substituting in $\cos x = \frac{e^{ix}+e^{-ix}}{2}$
$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$u+iv = \sin x \cosh y + \cos x \, i \sinh y$

$u = \sin x \cosh y$ $\quad$ $v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y$ $\quad$ $\frac{\partial v}{\partial y} = \cos x \cosh y$

$\frac{\partial v}{\partial x} = -\sin x \sinh y$ $\quad$ $\frac{-\partial u}{\partial y} = -(\sin x \sinh y)$

as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

u and v satisfies C-R equations

$\therefore$ $w = f(z)$ is analytic

So, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = (\cos x \cosh y) + i(-\sin x \sinh y)$

So, put $x = z$ and $y = 0$ $\quad$ $\cosh x = \frac{e^{x}+e^{-x}}{2}$
$\sinh x = \frac{e^{x}-e^{-x}}{2i}$

$f'(z) = \cos z\,(1) + i(-\sin z\,(0))$

$f'(z) = \cos z$

Q Show $f(z) = \log z$ is analytic and hence find $f'(z)$

SOL: Hence if $z = x + iy$ $\quad \log(x+iy)$

not seperatable
So, go to polar form

$$u + iv = \log(re^{i\theta})$$

$$= \log r + \log e^{i\theta}$$

$$u + iv = \log r + i\theta \log_e e$$

$$u + iv = \log r + i\theta$$

$$u = \log r \qquad v = \theta$$

So, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}$ $\qquad \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{1}{r}\cdot(1)$

$\dfrac{\partial v}{\partial r} = 0$ $\qquad -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = \dfrac{-1}{r}\cdot 0 = 0$

So, it satisfies C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

So, then $f(z)$ is analytic

So, $f'(z) = e^{-i\theta}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}\right)$

$$= e^{-i\theta}\left(\frac{1}{r} + i(0)\right)$$

$$= \frac{e^{-i\theta}}{r}$$

then put $r = 2$ and $\theta = 0$ $\qquad$ So, $\boxed{f'(z) = \dfrac{1}{2}}$

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Q Show that $f(z) = z^n$ is analytic and hence find $f'(z)$.

Sol: Given

$f(z) = z^n \qquad Z = x+iy$

$u+iv = (x+iy)^n$

not Seperatable So,

$u+iv = (re^{i\theta})^n$

$u+iv = r^n e^{in\theta}$

$= r^n(\cos n\theta + i\sin n\theta)$

$u+iv = r^n\cos n\theta + i\, r^n \sin n\theta$

$u = r^n \cos n\theta \qquad v = r^n \sin n\theta$

$\frac{\partial u}{\partial r} = \cos n\theta \; n \cdot r^{n-1} \qquad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{\cancel{r}} \cdot \cancel{r^n} \cdot \cos n\theta \cdot n$

$= n \cdot r^{n-1} \cdot \cos n\theta$

$\frac{\partial v}{\partial r} = \sin n\theta \; n \cdot r^{n-1} \qquad \frac{-1}{r}\frac{\partial u}{\partial \theta} = \frac{-1}{r} \cdot r^n \cdot -\sin n\theta \cdot n$

$= r^{n-1} \sin n\theta \cdot n$

So, By C-R equations

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$ } So, this is analytic

So, $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$= e^{-i\theta}\left(\cos n\theta \cdot n \cdot r^{n-1} + i \sin n\theta \cdot n \cdot r^{n-1}\right)$

So, $r = z$ and $\theta = 0$

$$f'(z) = 1\left(1 \cdot n \cdot z^{n-1} + i(0)\, n\, (z)^{n-1}\right)$$

$$\boxed{f'(z) = n \cdot z^{n-1}}$$

$\therefore$ Thus, we can find $f'(z)$.

Milne - Thomson method

Step-1 :- Given u or v find the partial derivatives with respect to x and y (r and θ) And use the result

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\left(f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\right)$$

Step-2 :- Use C-R equations so that $f'(z)$ can be written interms of the known function.

Step-3 :- Substitute for the partial derivatives and put $x = z$, $y = 0$ ($r = z, \theta = 0$).

Step-4 :- Integrate with respect to z to get $f(z)$.

Q. Find the analytic function $f(z)$ whose real part is $\log\sqrt{x^2+y^2}$. Hence find its imaginary part.

Sol: Given $u = \log\sqrt{x^2+y^2}$

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2}\cdot\cancel{2}x = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2}\cdot\cancel{2}y = \frac{y}{x^2+y^2}$$

So, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\cancel{f'(z) = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}}$$

By C-R equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

So, $\frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}$

So, $$f'(z) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

Then $x = z$ and $y = 0$

$$f'(z) = \frac{z}{z^2+0} + i\frac{0}{z^2+0^2}$$

$f'(z) = \frac{1}{z}$

Integrating

$$\int f'(z) = \int\frac{1}{z}\,dz$$

So, $\boxed{f(z) = \log z} + c$

$f(z) = u+iv$

$\log z + c = u+iv$

$\log(x+iy) + c = u+iv$

↓

So, not Seperatable so, go to polar form

$u+iv = \log(re^{i\theta}) + c$

$u+iv = \log r + \log e^{i\theta} + c$

$u+iv = \log r + i\theta + c$

So, $\left.\begin{array}{l} u = \log r + c \\ v = \theta \end{array}\right\}$ so, transfer to Cartisian

$r = \sqrt{x^2+y^2}$

So, $u = \log\left(\sqrt{x^2+y^2}\right) + c$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$v = \tan^{-1}\left(\frac{y}{x}\right)$

Q Find the analytic function $f(z)$ whose imaginary part is $e^x(x\sin y + y\cos y)$ Hence find its real part.

SOL: Given $v = xe^x \sin y + y \cdot e^x \cos y$

So, $\frac{\partial v}{\partial x} = \sin y\left(x \cdot e^x \cdot 1 + 1 \cdot e^x\right) + 1 \cdot e^x \cdot y\cos y$

$\frac{\partial v}{\partial y} = xe^x \cos y + e^x\left(y \cdot (-\sin y) + \cos y \cdot 1\right)$

So, $\dfrac{\partial v}{\partial x} = xe^x \sin y + e^x \sin y + y \cdot e^x \cos y$

$\dfrac{\partial v}{\partial y} = xe^x \cos y + e^x \cos y - ye^x \sin y$

So, $f'(z) = \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}$

So,

By C-R equations

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ $\qquad$ $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$= xe^x \cos y + e^x \cos y - ye^x \sin y$ $\qquad$ $= -\left(\dfrac{\partial u}{\partial y}\right)$

$$f'(z) = \left(xe^x \cos y + e^x \cos y - ye^x \sin y\right) + i\left(xe^x \sin y + e^x \sin y + ye^x \cos y\right)$$

So, $x = z$ and $y = 0$

$$f'(z) = \left(ze^z(1) + e^z(1) - 0\right) + i\left(ze^z(0) + 0 + 0\right)$$

$$f'(z) = ze^z + e^z$$
$$= e^z(z+1)$$

Integrating on Both sides

$$f(z) = \int \overset{\text{II}}{e^z} \overset{\text{I}}{(z+1)}\, dz$$
$$= (z+1) \cdot e^z - \int (1) \cdot e^z \, dz$$
$$= e^z \cdot z + \cancel{e^z} - \cancel{e^z} + c = ze^z + c$$

So,

$$f(z) = ze^z + c$$

$$u + iv = (x+iy)\, e^{x+iy} + c$$

$$u+iv = x \cdot e^x \cdot e^{iy} + (iy\ e^x \cdot e^{iy}) + c$$

$$u+iv = xe^x(\cos y + i\sin y)$$

$$+ iy\, e^x (\cos y + i \sin y) + c$$

$$= xe^x \cos y + i\, xe^x \sin y$$

$$+ i\, ye^x \cos y - ye^x \sin y + c$$

So,

$$u = xe^x \cos y - y \sin y\, e^x$$

$$= e^x (x\cos y - y \sin y)$$

$$v = xe^x \sin y + ye^x \cos y$$

$$= e^x (x \sin y + y \cos y) + c$$

Q Find the analytic function given that

$$u = e^{-x}\{(x^2-y^2)\cos y + 2xy \sin y\}$$

Sol: given

$$u = e^{-x} x^2 \cos y - e^{-x} y^2 \cos y + 2xe^{-x} y \sin y$$

So,

$$\frac{\partial u}{\partial x} = \left(x^2 \cdot e^{-x}(-1) + 2x \cdot e^{-x}\right)\cos y$$
$$- e^{-x}(-1)\, y^2 \cos y$$
$$+ 2\left(x \cdot e^{-x}(-1) + 1 \cdot e^{-x}\right) y \sin y$$

$$= -x^2 e^{-x}\cos y + 2x e^{-x}\cos y$$
$$+ e^{-x} y^2 \cos y$$
$$- 2x\, e^{-x}\, y \sin y + e^{-x}\, y \sin y$$

So,

$$\frac{\partial u}{\partial y} = e^{-x} x^2(-\sin y) - e^{-x}\left(y^2(-\sin y) + 2y\cos y\right)$$
$$+ 2x\, e^{-x}\left(y \cdot \cos y + 1 \cdot \sin y\right)$$
$$= -x^2 e^{-x}\sin y + e^{-x} y^2 \sin y - 2y e^{-x}\cos y$$
$$+ 2x e^{-x} y \cos y + 2x e^{-x}\sin y$$

So, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ $\quad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

So, $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

So, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$$= x^2 e^{-x}\sin y - e^{-x} y^2 \sin y + 2y e^{-x}\cos y$$
$$-2x e^{-x} y \cos y - 2x e^{-x}\sin y$$

So,

$$f'(z) = \begin{pmatrix} -x^2 e^{-x}\cos y + 2x e^{-x}\cos y \\ + e^{-x} y^2 \cos y - 2x\, e^{-x} y \sin y \\ + e^{-x}\, y \sin y \end{pmatrix} + i\left(\frac{\partial v}{\partial x}\right)$$

$x=2 \quad y=0$

$f'(z) = -z^2 e^{-z} + 2z e^{-z} + i(0)$

So,

$$f'(z) = (-z^2 e^{-z} + 2z e^{-z})$$

$$= e^{-z}(2z - z^2)$$

So, Integrating on Both sides

$$f(z) = (2z - z^2)\frac{e^{-z}}{-1} - (2-2z)\cdot\frac{e^{-z}}{+1} + (-2)\cdot\frac{e^{-z}}{-1}$$

$$f(z) = (2z - z^2)\frac{e^{-z}}{-1} - (2-2z)e^{-z} + 2 e^{-z}$$

$$= e^{-z}(\cancel{-2z} + z^2 - \cancel{2} + \cancel{2z} + \cancel{2})$$

$$= e^{-z}(z^2 \cancel{-2z} - 0)$$

$$= e^{-z}(z^2 - 0) + C$$

$$= z^2 e^{-z} + C$$

$$u + iv = z^2 e^{-z}$$

$$= (x+iy)^2 e^{-(x+iy)}$$

$$= (x^2 - y^2 + 2ixy)\, e$$

19/02/20

Q Find the analytic function such that the real part is $\frac{x^4-y^4-2x}{x^2+y^2}$. $\longrightarrow$ By Question we follow Cartision form maximum

Sol: Given $u = \frac{x^4-y^4-2x}{x^2+y^2}$

$$u = \frac{(x^2)^2-(y^2)^2-2x}{x^2+y^2}$$

$$u = \frac{(x^2+y^2)(x^2-y^2)-2x}{x^2+y^2}$$

$$u = x^2-y^2 - \frac{2x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = 2x - 2\cdot\left(\frac{1\cdot(x^2+y^2)-x(2x)}{(x^2+y^2)^2}\right)$$

$$= 2x - 2\left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right)$$

$$= 2x - 2\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

$$\frac{\partial u}{\partial y} = -2y - 2\cdot x\cdot\frac{-1}{x^2+y^2}\cdot 2y$$

$$= -2y + \frac{4xy}{x^2+y^2}$$

So, we know C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

we know $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

So, $\frac{\partial v}{\partial x} = -\left(\frac{\partial u}{\partial y}\right) = 2y - \frac{4xy}{x^2+y^2}$

Then

$$f'(z) = \left\{2x - 2\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)\right\} + i\left\{2y - \frac{4xy}{x^2+y^2}\right\}$$

So, $x = z \quad y = 0$

$$f'(z) = 2z - 2\left(\frac{-z^2}{(z^2)^2}\right) + i\{0 - 0\}$$

$$= 2z - 2\left(-\frac{\cancel{z^2}}{\cancel{z^4}}\right)$$

$$= 2z + \frac{2}{z^2}$$

So, Integrating w.r.t z

$$f(z) = \cancel{2}\cdot\frac{z^2}{\cancel{2}} + 2\cdot\frac{z^{-1}}{-1}$$

$$= z^2 - \frac{2}{z} + C$$

So, $u + iv = z^2 - \frac{2}{z}$

So, $z = re^{i\theta}$

$$u+iv = (re^{i\theta})^2 - \frac{2}{re^{i\theta}}$$

$$= r^2(\cos\theta + i\sin\theta)^2 - \frac{2}{r}(\cos\theta - i\sin\theta)$$

$$= r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta) - \frac{2}{r}(\cos\theta - i\sin\theta)$$

So difficult to Simplify so, go to Cartision form

$$u+iv = (x+iy)^2 - \frac{2}{x+iy} \times \frac{x-iy}{x-iy} + c$$

$$= (x^2-y^2+2ixy) - \frac{2(x-iy)}{x^2+y^2} + c$$

$$= \frac{x^2-y^2-2x}{x^2+y^2} + i\left(2xy + \frac{2y}{x^2+y^2}\right) + c$$

So, the imaginary part $V = 2xy + \frac{2y}{x^2+y^2}$

and $u = \frac{x^2-y^2-2x}{x^2+y^2} + c$

Note:-

Complex potential:- The analytic function for which ϕ is the real part and ψ is the imaginary part is Called Complex potential for the flow.

Q

Sol

i.e: $w = \phi + i\psi \longrightarrow$ Same as $u+iv$.

ϕ = ~~Complex~~ Velocity Potential

ψ = Stream function

Q = For the certain 2d flow of an incompressible fluid. Find the Complex potential. given that velocity potential is $x^2 - y^2 + \frac{x}{x^2+y^2}$

Sol: Given $\phi = x^2 - y^2 + \frac{x}{x^2+y^2} = u$

So, By partial differentiation.

$$\frac{\partial u}{\partial x} = 2x + \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

So, $$\frac{\partial u}{\partial y} = -2y + x \cdot \frac{-1}{x^2+y^2} \cdot 2y$$

$$= -2y - \frac{2xy}{x^2+y^2}$$

So, By C-R equations $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\left(-2y - \frac{2xy}{x^2+y^2}\right)$$

$$= 2y + \frac{2xy}{x^2+y^2}$$

So,

$$f'(z) = \left(2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right) + i\left(2y + \frac{2xy}{x^2+y^2}\right)$$

So, $x = z$ and $y = 0$

$$f'(z) = \left(2z + \frac{-z^2}{z^4}\right) + i(0)$$

$$= 2z - \frac{1}{z^2}$$

So, integrating on Both sides

$$f(z) = \cancel{2}\cdot\frac{z^2}{\cancel{2}} - \frac{z^{-1}}{-1}$$

$$= z^2 + \frac{1}{z} + C$$ endhere

So, $z = x+iy$ $f(z) = u+iv$

$$u+iv = (x+iy)^2 + \frac{1}{(x+iy)}\frac{(x-iy)}{x-iy} + C$$

$$= (x^2+i^2y^2+2ixy) + \frac{x-iy}{x^2+y^2} + C$$

$$= (x^2-y^2+2ixy) + \frac{x-iy}{x^2+y^2} + C$$

So, comparing real and imaginary parts

$$u = x^2 - y^2 + \frac{x}{x^2+y^2} + c$$

$$v = 2xy - \frac{y}{x^2+y^2}$$

So, the velocity potential $\phi = x^2 - y^2 + \frac{x}{x^2+y^2} + c$

stream function $\psi = 2xy - \frac{y}{x^2+y^2}$

So, the Complex potential

$$\omega = \phi + i\psi$$

$$= \left(x^2 - y^2 + \frac{x}{x^2+y^2} + c\right) + i\left(2xy - \frac{y}{x^2+y^2}\right)$$

So, the Complex potentioh

→ is $f(z) = z^2 + \frac{1}{z} = \omega$

Q For a Certain two dimensional flow the velocity potential is $\phi = 3x^2y - y^3$. Find the complex potential

Sol: Given $\phi = 3x^2y - y^3 = u$

So, $u = 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = 6xy \qquad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

So, By C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = \frac{-\partial u}{\partial x}$

So, $\frac{\partial v}{\partial x} = 3y^2 - 3x^2$

So, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = 6xy + i(3y^2 - 3x^2)$

So, $x = z \quad y = 0 \quad f'(z) = -i\,3z^2$

So, then integrating

$f(z) = -i \cdot \cancel{3} \cdot \frac{z^3}{\cancel{3}} + c$

$f(z) = -iz^3 + c$

Q Show that $u = \sin x \cdot \cosh y + 2\cos x \cdot \sinh y + x^2 - y^2 + 4xy$ is harmonic. Hence find $f(z)$.

Sol:

Note:- Refer 1st consequence if it is harmonic then it should satisfy Laplace's equation

So, $\frac{\partial u}{\partial x} = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$ (1)

$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y - 2\cos x \sinh y + 2$

$$\frac{\partial u}{\partial y} = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y + 2\cos x \sinh y - 2 \quad \text{— (2)}$$

So, By Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Adding (1) + (2)

So,

$$(-\sin x \cosh y - 2\cos x \sin y + 2) + \sin x \cosh y + 2\cos x \sin y - 2 = 0$$

So, this is harmonic.

By CR equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = (\cos x \cosh y - 2\sin x \sinh y + 2x + 4y) + i(-\sin x \sinh y - 2\cos x \cosh y + 2y - 4x)$$

Then $x = z \quad y = 0$

$$f'(z) = (\cos z + 2z) + i(-2\cos z - 4z)$$

$$f'(z) = (1-2i)(\cos z + 2z)$$

So, integrating

$$f(z) = (1-2i)\,(\sin z + z^2) + C$$

Note:- A function ϕ is Said to be a harmonic function if it Satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Q Show that $v = \left(r - \frac{k^2}{r}\right)\sin\theta$ is harmonic. Find the analytic function $f(z)$ and also its real part.

Sol: given $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$$\frac{\partial v}{\partial r} = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$$

$$\frac{\partial^2 v}{\partial r^2} = \left(\frac{-2}{r^3}k^2\right)\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \left(r - \frac{k^2}{r}\right)\cos\theta$$

$$\frac{\partial^2 v}{\partial \theta^2} = \left(r - \frac{k^2}{r}\right)(-\sin\theta)$$

So, verifying the Laplace equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$

$$= \left(\frac{-2}{r^3}k^2\right)\sin\theta + \frac{1}{r}\cdot\left(1+\frac{k^2}{r^2}\right)\sin\theta$$

$$+ \frac{1}{r^2}\left(r - \frac{k^2}{r}\right)(-\sin\theta)$$

$$= \cancel{\frac{-2k^2\sin\theta}{r^2}} + \cancel{\frac{\sin\theta}{r}} + \cancel{\frac{k^2\sin\theta}{r^3}}$$

$$+ \cancel{\left(\frac{-\sin\theta}{r}\right)} + \cancel{\frac{k^2\sin\theta}{r^3}}$$

$$= 0$$

So, this is harmonic

Then $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

So, $\frac{\partial v}{\partial r} = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$

$$\frac{\partial v}{\partial \theta} = \left(r - \frac{k^2}{r}\right)\cos\theta$$

So, By C-R equation

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$= \frac{+1}{r}\left(r - \frac{k^2}{r}\right)\cos\theta$$

So, $f'(z) = e^{-i\theta}\left(\frac{1}{r}\left(r - \frac{k^2}{r}\right)\cos\theta\right.$

$$\left. + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right)$$

put $r = 2$ and $\theta = 0$

$$f'(z) = \frac{1}{2}\left(2 - \frac{k^2}{2}\right) + i(0)$$

$$f'(z) = 1 - \frac{k^2}{z^2}$$

So, integrating on Both sides

$$f(z) = z - k^2 \cdot \frac{z^{-1}}{-1} + c$$

$$f(z) = z + \frac{k^2}{z} + c$$

So, $z = re^{i\theta}$

$$f(z) = re^{i\theta} + \frac{k^2}{re^{i\theta}} + c$$

$$= r(\cos\theta + i\sin\theta) + \frac{k^2}{r}(\cos\theta - i\sin\theta) + c$$

So,

$$f(z) = \left(\left(r + \frac{k^2}{r}\right)\cos\theta + c\right) + i\left(r\sin\theta - \frac{k^2}{r}\sin\theta\right)$$

So, $u = \left(r + \frac{k^2}{r}\right)\cos\theta + c \longrightarrow$ real part

$$v = \left(r - \frac{k^2}{r}\right)\sin\theta$$

20/02/20

Q Show that $u = \frac{1}{r^2}\cos 2\theta$ is harmonic. Hence find the analytic function $f(z)$.

Sol: given that $u = \frac{1}{r^2}\cos 2\theta$

$$\frac{\partial u}{\partial r} = \frac{-2}{r^3}\cos 2\theta$$

$$\frac{\partial^2 u}{\partial r^2} = -2 \times -3 \frac{1}{r^4}\cos 2\theta$$

$$= \frac{6}{r^4}\cos 2\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{1}{r^2}(-\sin 2\theta)\cdot 2$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r^2}\cdot(-\cos 2\theta)\,4$$

So, Verify Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \cancel{\frac{6}{r^4}\cos 2\theta} + \cancel{\frac{1}{r}\cdot\frac{-2}{r^3}\cos 2\theta} + \cancel{\frac{1}{r^2}}\,\cancel{\frac{1}{r^2}}\,4(-\cos 2\theta)$$

$$= 0$$

So, we will find

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

So, By C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{-1}{r} \times \frac{1}{r^2}\, 2\,(-\sin 2\theta)$$

$$= \frac{2\sin 2\theta}{r^3}$$

So,

$$f'(z) = e^{-i\theta}\left(\frac{-2}{r^3}\cos 2\theta + i\,\frac{2}{r^3}\sin 2\theta\right)$$

Substitute $r = z \quad \theta = 0$

$$f'(z) = \frac{-2}{z^3}(1) + i(0)$$

$$f'(z) = \frac{-2}{z^3}$$

So, integrating on Both sides

$$f(z) = -\cancel{2}\cdot\frac{z^{-2}}{-\cancel{2}}$$

$$f(z) = \frac{1}{z^2} + C$$

So, $z = re^{i\theta}$

So,

$$f(z) = \frac{1}{r^2(e^{i\theta})^2} + C$$

$$f(z) = \frac{1}{r^2}(\cos 2\theta + i\sin 2\theta) + c$$

So, the real part

$$u = \frac{1}{r^2}\cos 2\theta + c$$

$$v = \frac{1}{r^2}\sin 2\theta$$

no need

Q Find the analytic function $f(z)$ given that

$$u - v = e^x(\cos y - \sin y)$$

Sol: Differentiate partially w.r.t x partially

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$$

Keep this same ①

Diff partially w.r.t y

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x(-\sin y - \cos y)$$

input

Use C-R equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$\frac{-\partial v}{\partial x} - \cancel{\frac{\partial u}{\partial x}} = e^x(-\sin y - \cos y) \quad — ②$$

$$\cancel{\frac{\partial u}{\partial x}} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y) \quad \to ①$$

$$-2\frac{\partial v}{\partial x} = e^x(-\sin y - \sin y)$$

$$\frac{\partial v}{\partial x} = \frac{e^x}{2}\left(\sin y + \sin y\right) \quad \text{— in ①}$$

$$= \frac{e^x}{\cancel{2}} \times \cancel{2}\sin y = e^x \sin y$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x\left(\cos y - \sin y\right)$$

$$\frac{\partial u}{\partial x} = e^x \cos y - \cancel{e^x \sin y} + \cancel{e^x \sin y}$$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

So, then $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$f'(z) = e^x \cos y + i\, e^x \sin y$$

$x = z \quad y = 0 \quad$ so,

$$f'(z) = e^z(1) + i(0)$$

integrating on Both sides

$$\boxed{f(z) = e^z + C}$$

Q Find the analytic function given that

$$u - v = (x-y)\left(x^2 + 4xy + y^2\right)$$

SOL:

given

$$u - v = (x-y)\left(x^2 + 4xy + y^2\right)$$

$$u - v = x^3 + 4x^2y + xy^2 - yx^2 - 4xy^2 - y^3$$

$$u - v = x^3 - y^3 + 4x^2y - 4xy^2 + xy^2 - yx^2$$

$$u - v = x^3 - y^3 + 3x^2y - 3xy^2$$

Partially diff w.r.t x

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \text{ — const } (1)$$

partially diff w.r.t y

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -3y^2 + 3x^2 - 6xy \quad — (2)$$

So, By C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$\frac{-\partial v}{\partial x} - \cancel{\frac{\partial u}{\partial x}} = -3y^2 + 3x^2 - \cancel{6xy} \quad — (2)$$

$$\cancel{\frac{\partial u}{\partial x}} - \frac{\partial v}{\partial x} = 3x^2 + \cancel{6xy} - 3y^2 \quad — (1)$$

$$-2\frac{\partial v}{\partial x} = 6x^2 - 6y^2$$

$$-\cancel{2}^{1}\frac{\partial v}{\partial x} = \cancel{6}^{3}(x^2 - y^2)$$

$$\boxed{\frac{\partial v}{\partial x} = 3(y^2 - x^2)} \quad — (1)$$

So, in (1)

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2$$

$$\frac{\partial u}{\partial x} - 3y^2 + \cancel{3x^2} = \cancel{3x^2} + 6xy - \cancel{3y^2}$$

$$\boxed{\frac{\partial u}{\partial x} = 6xy}$$

So, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ correct substitute correctly

So, $f'(z) = i(3y^2 - 3x^2) + (6xy)$

$x = z \quad y = 0$

$$f'(z) = -i3 \cdot z^2$$

So, integrate on Both sides

$$f(z) = -\cancel{3}i\frac{z^3}{\cancel{3}} + c$$

$$\boxed{f(z) = -iz^3 + c}$$

Q Find the analytic function $f(z)$ given that $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

Sol:

Given $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

So, Partially diff w.r.t r

$$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta) \quad -(1)$$

So, partiall diff w.r.t θ

$$\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} = \frac{1}{r^2}(-\sin 2\theta \cdot 2 - \cos 2\theta \cdot 2)$$

$$= \frac{-2}{r^2}(\sin 2\theta + \cos 2\theta)$$

So, By C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$-r\frac{\partial v}{\partial r} + r\frac{\partial u}{\partial r} = \frac{-2}{r^2}(\sin 2\theta + \cos 2\theta) \rightarrow (2)$$

$$-\cancel{\frac{\partial v}{\partial r}} + \frac{\partial u}{\partial r} = \frac{-2}{r^3}(\sin 2\theta + \cos 2\theta)$$

$$\frac{\partial u}{\partial r} + \cancel{\frac{\partial v}{\partial r}} = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta)$$

$$\cancel{2}\frac{\partial u}{\partial r} = \frac{-\cancel{2}}{r^3}(2\cos 2\theta$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{-2\cos 2\theta}{r^3}} \text{ in } (1)$$

$$\frac{-2}{r^3}\cancel{\cos 2\theta} + \frac{\partial v}{\partial r} = -\frac{2}{r^3}\cancel{\cos 2\theta} + \frac{2}{r^3}\sin 2\theta$$

$$\boxed{\frac{\partial v}{\partial r} = \frac{2}{r^3}\sin 2\theta}$$

So,

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$f'(z) = e^{-i\theta}\left(\frac{-2\cos 2\theta}{r^3} + i\left(\frac{2}{r^3}\sin 2\theta\right)\right)$$

$\theta = 0 \quad r = z$

$$f'(z) = \frac{-2}{z^3}(1) + i(0)$$

So, integrate on Both sides

$$f(z) = -\cancel{2}\cdot\frac{z^{-2}}{-\cancel{2}} + C$$

$$\boxed{f(z) = \frac{1}{z^2} + C}$$

imp Q

Construction of Analytic functions By finding The Harmonic Conjugate

Step-1: Given u or v Find the partial derivatives w.r.t x and y.

Step-2: Substitute for the known partial derivatives in C-R equations and Solve By Direct integration to get v or u.

Step-3: put $x = z$ and $y = 0$ to get $f(z)$ interms of z.

$$\text{MT}: \quad u \text{ or } v \downarrow f(z) \downarrow v \; u$$

$$\text{Harmonic conju}: \quad u \text{ or } v \;\to\; \text{find } u \;/\; v \;\downarrow\; f(z)$$

Q Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find its Harmonic Conjugate. Also find the Corresponding analytic function $f(z)$.

Sol: given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\frac{\partial u}{\partial x} = 3x^2 \cancel{- 3y^2} + 6x$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \cancel{- 6y} + 6$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6$$

so, By Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$$

$$= 0 \longrightarrow \text{So, this is harmonic.}$$

By C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -1 \qquad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \quad -2$$

from eq 1

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$$

integrate w.r.t y on Both Sides c/o PDE

$$v = \int (3x^2 - 3y^2 + 6x)\, dy$$

$$v = 3x^2y - y^3 + 6xy + f(x) \quad \text{--- (3)}$$

from eq 2

$$\frac{\partial v}{\partial x} = 6xy + 6y$$

integrate w.r.t x

$$v = \int (6xy + 6y)\, dx$$

$$v = 3x^2y + 6yx + g(y) \quad \text{--- (4)}$$

Comparing 3 and 4 we get

$$f(x) = 0 \quad g(y) = -y^3$$

So,

$$\boxed{v = 3x^2y + 6xy - y^3}$$

So, the analytic function is

$$f(z) = u + iv$$

$$f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1)$$
$$+ i\,(3x^2y + 6yx - y^3)$$

put $x = 2$ and $y = 0$

$$f(z) = (z^3 + 3z^2 + 1) + i(0)$$

So, $\boxed{f(z) = (z^3 + 3z^2 + 1)}$

Q Find the Constant a Such that the function $u = \cos ax \cdot \cosh y$ is harmonic. Hence find its Harmonic Conjugate.

Sol: given $u = \cos ax \cosh y$

$$\frac{\partial u}{\partial x} = (-\sin ax) \cdot a \cosh y$$

$$\frac{\partial^2 u}{\partial x^2} = (-\cos ax)\, a^2 \cosh y$$

$$\frac{\partial u}{\partial y} = \cos ax\, \cancel{\cosh y}\, \sinh y$$

$$\frac{\partial^2 u}{\partial y^2} = \cos ax \cosh y$$

So, it is harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So, $a = 1$

$$(-\cos ax\, a^2 + \cos ax) = 0 \qquad \boxed{a = 1}$$

$$\cos ax\,(1 - a^2) = 0$$

So, $u = \cos x \cosh y$

So, then By C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \Big\} \times$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

So,

$$\frac{\partial v}{\partial y} = a(-\sin ax)\cosh y \qquad a=1$$

$$= -\sin x \cosh y$$

integrate w.r.t y

$$V = \int -(\sin x \cosh y)\,dy$$

$$V = -\sin x \overset{\sin}{\cancel{\cosh}}\, y + f(x)$$

$$\frac{\partial v}{\partial x} = -(\cos x \overset{\sinh y}{\cancel{\cosh y}})$$

$\int \cosh y = \sinh y$

$\int \sinh y = \cosh y$

integrate w.r.t x

$$V = \int -(\cos x \overset{\sinh y}{\cancel{\cosh y}})\,dx$$

$$V = -\overset{(+\sin x)}{\cos x}\ \overset{\sinh y}{\overset{\cancel{\cosh y}}{\sinh y}} + g(y)$$

$$V = -\sin x \underset{\sinh y}{\cancel{\cosh y}} + g(y)$$

$e^{ax} + e^{-ax}$

$e^{ax} - e^{ax}$

So, By comparing $f(x) = g(y) = 0$

Then $V = -\sin x \cosh y$

Then, $f(z) = u + iv$

$= \cos x \cosh y + i(-\sin x \sinh y)$

25/02/20 $\longrightarrow$ absent to class

Q Show that $u = e^x(x\cos y - y\sin y)$ is harmonic. Find the harmonic Conjugate. Also determine the corresponding analytic function.

Sol: Given $u = e^x(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x(\cos y) + e^x\cos y$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + y\sin y)$$

So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$= \cancel{xe^x\cos y} - \cancel{ye^x\sin y} + \cancel{2e^x\cos y}$

$- \cancel{xe^x\cos y} - \cancel{e^x\cos y} - \cancel{e^x\cos y} + \cancel{e^x y\sin y}$

$= 0$

Hence 'u' is harmonic as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

By C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y - y\sin y + \cos y)$$

So, integrating on Both Sides w.r.t y

$$v = \int \{e^x \cos y + xe^x \cos y - \underbrace{ye^x \sin y}_{\text{Bernouli}}\} dy \quad \text{c/o p.d.e}$$

$$v = e^x(\sin y) + xe^x \sin y - e^x\left\{y \cdot \frac{-\cos y}{1} - 1 \cdot \frac{-\sin y}{1}\right\}$$

$$v = \cancel{e^x \sin y} + xe^x \sin y + e^x y\cos y - \cancel{e^x \sin y}$$

$$v = e^x(x\sin y + y\cos y) + f(x) \quad \text{--- } (1)$$

$$\frac{\partial v}{\partial x} = -e^x(-x\sin y - \sin y - y\cos y)$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + \sin y + y\cos y)$$

So, integrating on Both Sides w.r.t x

c/o P.D.E

$$v = \int \{\underbrace{xe^x \sin y}_{\text{Bernouli}} + e^x \sin y + e^x y\cos y\} dx$$

$$v = \sin y\left\{x \cdot \frac{e^x}{1} - 1 \cdot \frac{e^x}{1}\right\} + e^x \sin y + e^x y\cos y$$

$V = xe^x \sin y - \cancel{e^x \sin y} + \cancel{e^x \sin y} + e^x y \cos y$

so,

$V = (xe^x \sin y + e^x y \cos y) + g(y)$

$V = e^x(x \sin y + y \cos y) + g(y)$ —— 2

By comparing 1 and 2

$f(x) = g(y) = 0$ So,

$f(z) = u + iv$

$f(z) = e^x(x \cos y - y \sin y) + i(e^x(x \sin y + y \cos y))$

put $x = z$ $y = 0$

$\boxed{f(z) = e^z(z)}$ → This is the corresponding analytic function.

Q In a two dimensional flow if the Velocity potential $\phi = e^{-x} \cos y + xy$. Find the Stream function.

Sol: Given $\phi = e^{-x} \cos y + xy = u$

Given $u = e^{-x} \cos y + xy$

$\frac{\partial u}{\partial x} = -e^{-x} \cos y + y$ $\frac{\partial u}{\partial y} = -e^{-x} \sin y + x$

$\frac{\partial^2 u}{\partial x^2} = e^{-x} \cos y$ X
not required

By C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

so, $\dfrac{\partial v}{\partial y} = -e^{-x}\cos y + y$

so, integrate Both Sides w.r.t 'y'

$$V = \int \{-e^{-x}\cos y + y\}\, dy$$

$$V = -e^{-x}\sin y + \frac{y^2}{2} + f(x) \quad \text{—— (1)}$$

$$\frac{\partial v}{\partial x} = -\left(-e^{-x}\sin y + x\right)$$

$$= e^{-x}\sin y - x$$

So, integrate Both sides w.r.t 'x'

$$V = \int \{e^{-x}\sin y - x\}\, dx$$

$$V = -e^{-x}\sin y - \frac{x^2}{2} + g(y) \quad \text{—— (2)}$$

so, By Comparing (1) & (2)

$$f(x) = \frac{-x^2}{2} \qquad g(y) = \frac{y^2}{2}$$

$$V = -e^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2}$$

∴ The Stream function is

$$\boxed{\psi = -e^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2}}$$

Q. Show that $u = \left(r + \frac{1}{r}\right)\cos\theta$ is harmonic. Find its harmonic conjugate and also the corresponding analytic function $f(z)$?

Sol: Given $u = \left(r + \frac{1}{r}\right)\cos\theta$

$$\frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta \qquad \frac{\partial^2 u}{\partial r^2} = -\left(\frac{-2}{r^3}\right)\cos\theta$$

$$\frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta \Rightarrow \frac{\partial^2 u}{\partial r^2} = \left(\frac{2}{r^3}\right)\cos\theta$$

$$\therefore \frac{\partial u}{\partial \theta} = \left(r + \frac{1}{r}\right)(-\sin\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \left(r + \frac{1}{r}\right)(-\cos\theta)$$

So, we know harmonic condition

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{1}{r^2}\right)\cos\theta + \frac{1}{r^2}\left(r + \frac{1}{r}\right)(-\cos\theta)$$

$$= \cancel{\frac{2}{r^3}}\cos\theta + \cancel{\frac{1}{r}}\cos\theta - \cancel{\frac{1}{r^3}}\cos\theta + \cancel{\frac{-1}{r}}\cos\theta - \cancel{\frac{1}{r^3}}\cos\theta$$

$= 0$ So, this u is harmonic.

By C-R equations.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = r \frac{\partial u}{\partial r}$$

$$= r\left(1 - \frac{1}{r^2}\right)\cos\theta$$

integrating on Both sides w.r.t θ c/o p.D.t

$$v = \int r\left(1 - \frac{1}{r^2}\right) \cos\theta \, d\theta$$

$$v = \left(r - \frac{1}{r}\right) \sin\theta + f(r) \quad \text{--- (1)}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$= -\frac{1}{r}\left(r + \frac{1}{r}\right)(-\sin\theta)$$

$$= \left(1 + \frac{1}{r^2}\right) \sin\theta$$

integrating on Both sides w.r.t r c/o p.D.t

$$v = \int \left(1 + \frac{1}{r^2}\right) \sin\theta \, dr$$

$$= \left(r + \frac{r^{-1}}{-1}\right) \sin\theta + g(\theta)$$

$$v = \left(r - \frac{1}{r}\right) \sin\theta + g(\theta) \quad \text{--- (2)}$$

By comparing (1) and (2) $f(r) = g(\theta) = 0$

So,
$$V = \left(r - \frac{1}{r}\right) \sin\theta$$

So, $f(z) = u + iv$

$$= \left(r + \frac{1}{r}\right) \cos\theta + i\left(r - \frac{1}{r}\right) \sin\theta$$

$r = 2 \quad \theta = 0$

so, $f(z) = \left(z + \frac{1}{z}\right) + i(0)$

$$\boxed{f(z) = z + \frac{1}{z}}$$ Check once.

Q In a two dimensional flow the stream function $\psi = \frac{-y}{x^2+y^2}$ find the velocity potential ?

Sol: Given $\psi = \frac{-y}{x^2+y^2} = v$

So, $v = \frac{-y}{x^2+y^2}$ Convert it to polar form

$$v = \frac{-r\sin\theta}{r^2}$$

$$v = \frac{-\sin\theta}{r}$$

$$\frac{\partial v}{\partial r} = \frac{\sin\theta}{r^2} \qquad \frac{\partial v}{\partial \theta} = \frac{-\cos\theta}{r}$$

By C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{-\cos\theta}{r}$$

c/o p.D.E

$$\frac{\partial u}{\partial r} = \frac{-\cos\theta}{r^2}$$ So, integrating w.r.t 'r' on Both sides

$$u = \int -\frac{\cos\theta}{r^2}\, dr$$

$$u = -\cos\theta \cdot \frac{-1}{r} + f(\theta)$$

$$u = \frac{\cos\theta}{r} + f(\theta) \quad — ①$$

we know

$$\frac{\partial u}{\partial \theta} = -r \cdot \frac{\partial v}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = -\cancel{r} \cdot \frac{\sin\theta}{r^{\cancel{2}}}$$

c/o p.d.t

$$\frac{\partial u}{\partial \theta} = -\frac{\sin\theta}{r}$$

integrating Both sides w.r.t θ

$$u = \int \frac{-\sin\theta}{r} \, d\theta$$

$$u = \cos\theta \cdot \frac{1}{r} + g(r) \quad — ②$$

So, By Comparing ① and ② $f(\theta) = g(r) = 0$

So,

$$u = \frac{\cos\theta}{r}$$

So, the Velocity potential

$$\boxed{\phi = \frac{\cos\theta}{r}}$$

Check once.

Q Show that $v = \left(r - \frac{k^2}{r}\right)\sin\theta$ is harmonic. Hence find its harmonic conjugate and also $f(z)$:

Sol: Given $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$$\frac{\partial v}{\partial r} = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$$

$$\frac{\partial^2 v}{\partial r^2} = \left(0 - \frac{2k^2}{r^3}\right) \sin\theta$$

$$\frac{\partial v}{\partial \theta} = \left(r - \frac{k^2}{r}\right) \cos\theta$$

$$\frac{\partial^2 v}{\partial \theta^2} = \left(r - \frac{k^2}{r}\right)(-\sin\theta)$$

we know the harmonic relation.

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$

$$= \left(0 - \frac{2k^2}{r^3}\right) \sin\theta + \frac{1}{r}\cdot\left(1 + \frac{k^2}{r^2}\right) \sin\theta$$

$$+ \frac{1}{r^2}\left(r - \frac{k^2}{r}\right)(-\sin\theta)$$

$$= \cancel{\sin\theta} - \cancel{\frac{2k^2}{r^3}\sin\theta} + \cancel{\frac{\sin\theta}{r}} + \cancel{\frac{k^2}{r^3}\sin\theta} - \cancel{\frac{\sin\theta}{r}}$$

$$+ \frac{k^2}{r^3}\cancel{\sin\theta}$$

$= 0$ So, v is harmonic

By C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\left(r - \frac{k^2}{r}\right)\cos\theta$$

$$= \left(1 - \frac{k^2}{r^2}\right)\cos\theta$$

integrating Both Sides w.r.t r c/o PDE

$$u = \left(r + \frac{k^2}{r}\right)\cos\theta + f(\theta) \quad - (1)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$= -r\left(1+\frac{k^2}{r^2}\right)\sin\theta$$

$$= \left(-r-\frac{k^2}{r}\right)\sin\theta$$

integrating on Both sides w.r.t θ I/o P.D.E

$$u = \int\left\{-r-\frac{k^2}{r}\right\}\sin\theta \, d\theta$$

$$= -\left(r+\frac{k^2}{r}\right) -\cos\theta + g(r)$$

$$= \left(r+\frac{k^2}{r}\right)\cos\theta + g(r) \quad \text{--- (2)}$$

By comparing equations (1) and (2)

$f(\theta) = g(r) = 0$ so,

$$u = \left(r+\frac{k^2}{r}\right)\cos\theta$$

so, $f(z) = u + iv$

$$= \left(r+\frac{k^2}{r}\right)\cos\theta + i\left(r-\frac{k^2}{r}\right)\sin\theta$$

$r = z$ and $\theta = 0$

$$f(z) = \left(z+\frac{k^2}{z}\right)(1) + i(0)$$

$$\boxed{f(z) = z+\frac{k^2}{z}}$$ Check once.

derivations

Q if $f(z) = u + iv$ is an analytic function then prove that

i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\ f(z)|^2 = 2|f'(z)|^2$

ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Sol: Given $f(z) = u+iv$ is analytic

u and v satisfies C-R equations

i.e: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

$Re\ f(z) = u$

$|Re\ f(z)| = \sqrt{u^2} = u$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\ f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2$$

$$= \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2)$$

$$= \frac{\partial}{\partial x}\left\{2u \cdot \frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{2u\ \frac{\partial u}{\partial y}\right\}$$

$$= 2\cdot\left\{u\cdot\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right\} + 2\left\{u\cdot\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right\}$$

$$= 2u\left\{\cancel{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}^{\,0}\right\} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2 \quad (1)$$

as $f(z)$ is analytic $\hookrightarrow$ Harmonic Condition

But we know

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

So, tun we got

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\ f(z)|^2 = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$= 2\left\{|f'(z)|\right\}^2$$

Hence prooved.

(ii) given $\rightarrow$ L.H.S given

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2$$

$f(z) = u + iv$

$|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2+v^2$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sqrt{u^2+v^2}\right)^2$$

$$= \frac{\partial^2}{\partial x^2}(u^2+v^2) + \frac{\partial^2}{\partial y^2}(u^2+v^2)$$

~~Let $u^2+v^2 = \phi^2$ — (1)~~

~~Differentiate w.r.t 'x'~~

$$\cancel{2\phi\frac{\partial\phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}}$$

$$= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)$$

$$= 2\left\{u\cdot\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\cdot\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$+ 2\left\{u\cdot\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\cdot\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

By C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$= 2u\left\{\cancel{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}\right\} \to 0$$

$$+ 2v\left\{\cancel{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}}\right\} \to 0 \quad + 4\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

Harmonic so,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Hence prooved.

26/02/20

Q If $f(z)$ is a regular function of z then show that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2$

$$= |f'(z)|^2$$

SOL: we know

$f(z) = u + iv$ is analytic

u and v Satisfies C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$|f(z)| = \sqrt{u^2+v^2} = \phi$$

$$\phi^2 = u^2+v^2 \quad \text{— (1)}$$

Differentiate w.r.t x

$$2\phi \frac{\partial \phi}{\partial x} = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x}$$

Squaring on Both sides

$$\phi^2 \left(\frac{\partial \phi}{\partial x}\right)^2 = u^2 \left(\frac{\partial u}{\partial x}\right)^2 + v^2 \left(\frac{\partial v}{\partial x}\right)^2 + 2uv \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \quad \text{— (2)}$$

So, Differentiate w.r.t 'y'

$$\cancel{2}\phi \frac{\partial \phi}{\partial y} = \cancel{2}u \frac{\partial u}{\partial y} + \cancel{2}v \frac{\partial v}{\partial y}$$

Squaring On Both Sides

$$\phi^2 \left(\frac{\partial \phi}{\partial y}\right)^2 = u^2 \left(\frac{\partial u}{\partial y}\right)^2 + v^2 \left(\frac{\partial v}{\partial y}\right)^2 + 2uv \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \quad \text{— (3)}$$

So, By Using C-R equations

(3) in terms of x

$$\phi^2\left(\frac{\partial\phi}{\partial y}\right)^2 = u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 + 2uv\left(\frac{-\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)$$

→ So, By By eq 4

Adding 4 + 2

$$\cancel{\phi^2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\} = \cancel{(u^2+v^2)}\left(\frac{\partial u}{\partial x}\right)^2 + \cancel{(u^2+v^2)}\left(\frac{\partial v}{\partial x}\right)^2$$

$\phi^2 = u^2+v^2$

$$f(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f(z)| = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

So, Hence $\phi = \sqrt{u^2+v^2} = |f(z)|$

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

CReq derivatives → cartisian
→ polar

Milne Thomson

Harmonic Conjugate

3 derivations.