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COMPLEX VARIABLES - II

Bilinear Transformation

A transformation defined by

$$W = \frac{az+b}{cz+d} \text{ where } a, b, c, d \text{ are}$$

real or complex constants such that $ad - bc \neq 0$ is called a bilinear transformation.

If a point z maps onto itself i.e., $W = z$, then, the point is called an invariant point or a fixed point of bilinear transformation.

Method to find Bilinear Transformation

Step 1: Given, W_1, W_2, W_3 corresponding to z_1, z_2, z_3, assume the bilinear transformation in the form $W = \dfrac{az+b}{cz+d}$

Step 2: Substitute the given set of points to obtain a set of 3 equations in 4 unknowns.

Step 3: Deduce a pair of equations in any 3 unknowns, and solve by the rule of cross multiplication.

Step 4: Using these values in W, we get the required Bilinear Transformation.

Problems

1. Find the Bilinear Transformation which maps the points $z = 1, i, -1$ onto $W = i, 0, -i$. Under this transformation, find the image of $|z| < 1$.

Sol:- Let $W = \dfrac{az+b}{cz+d}$

For $z = 1, W = i$

$$i = \frac{a+b}{c+d}$$

$$ic + id = a + b$$

$$a + b - ic - id = 0 \longrightarrow (1)$$

For $z = i, W = 0$

$$0 = \frac{ai+b}{ci+d}$$

$$0 = ai + b$$

$$ai + b = 0 \longrightarrow (2)$$

For $z = -1, W = -i$

$$-i = \frac{-a+b}{-c+d}$$

$$ci - id = -a + b$$

$$-a + b - ic + id = 0 \longrightarrow (3)$$

① + ③ gives

$$2b - 2ic = 0 \quad \text{or}$$

$$b - ic = 0 \longrightarrow ④$$

Solve ② and ④

$$ai + b + 0c = 0$$
$$0a + b - ic = 0$$

RCM

$$\begin{array}{ccccc} a & b & c & & \\ i & 1 & 0 & i & 1 \\ 0 & 1 & -i & 0 & 1 \end{array}$$

$$\frac{a}{-i-0} = \frac{b}{0+i^2} = \frac{c}{i-0} = k$$

$$\frac{a}{-i} = \frac{b}{i^2} = \frac{c}{i} = k$$

$$\boxed{a = -ik \ , \ b = -k \ , \ c = ik}$$

$$a + b - ic - id = 0$$

$$id = a + b - ic$$
$$= -ik - k + k$$
$$id = -ik$$
$$\boxed{d = -k}$$

$$\therefore W = \frac{az+b}{cz+d}$$

$$= \frac{-ikz - k}{ikz - k}$$

$$W = \frac{K(-iz-1)}{K(iz-1)}$$

$$W = \frac{-iz-1}{iz-1}$$

$$= \frac{iz+1}{1-iz}$$

$$\boxed{W = \frac{1+iz}{1-iz}}$$

$$W(1-iz) = 1+iz$$

$$W - Wiz = 1+iz$$

$$W - 1 = iz + Wiz$$

$$W-1 = iz(1+W)$$

$$z = \frac{W-1}{i(1+W)}$$

$$z = \frac{i(1-W)}{1+W}$$

$|z| < 1$

Given,

$|z| < 1$

$$\Rightarrow \left| \frac{i(1-W)}{1+W} \right| < 1$$

$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$|i| = \sqrt{0^2+1^2} = 1$

$W = f(z) = u+iv$

$$\Rightarrow |i|\,|1-W| < |1+W|$$

$$\Rightarrow |1-W| < |1+W|$$

$$\Rightarrow |1-(u+iv)| < |1+(u+iv)|$$

$$\Rightarrow |(1-u)-iv| < |(1+u)+iv|$$

$$\Rightarrow \sqrt{(1-u)^2+v^2} < \sqrt{(1+u)^2+v^2}$$

$$\Rightarrow (1-u)^2+v^2 < (1+u)^2+v^2$$

$\Rightarrow 1+u^2-2u+v^2 < 1+u^2+2u+v^2$

$\Rightarrow 0 < 4u$

$\Rightarrow 0 < u$

$\Rightarrow \boxed{u > 0}$

2. Find the bilinear transformation which maps $z = -1, 0, 1$ onto $W = 0, i, 3i$

Sol:- Let $W = \dfrac{az+b}{cz+d}$

For $z=-1$, $W=0$

$$0 = \frac{-a+b}{-c+d}$$

$\Rightarrow -a+b = 0 \longrightarrow (1)$

For $z=0$, $W=i$

$$i = \frac{b}{d}$$

$b - id = 0 \longrightarrow (2)$

For $z=1$, $W=3i$

$$3i = \frac{a+b}{c+d}$$

$3i(c+d) = a+b$

$a+b-i(3c+3d) = 0 \longrightarrow (3)$

Solve ① and ②

$-a + b + 0d = 0$

$0a + b - id = 0$

$$\begin{array}{cccccc} a & b & d & & & \\ -1 & 1 & 0 & -1 & 1 \\ 0 & 1 & -i & 0 & 1 \end{array}$$

$$\frac{a}{-i-0} = \frac{b}{0-i} = \frac{d}{-1-0} = K$$

$$\frac{a}{-i} = \frac{b}{-i} = \frac{d}{-1} = K$$

$$\boxed{a = -iK,\ b = -iK,\ d = -K}$$

$b - id = 0$

$id = b$

$d = \frac{b}{i}$

$d = \frac{-iK}{i}$

$\boxed{d = -K}$

$\therefore W = \frac{az + b}{cz + d}$

$= \frac{-iKz - iK}{-Kz - K}$

$= \frac{-K(iz + i)}{-K(1+z)}$

$W = \frac{i + iz}{1+z}$

① $a + b - 3ic - 3id = 0$

$-iK - iK - 3ic + 3iK = 0$

$3ic = iK$

$3c = K$

$\boxed{c = \frac{K}{3}}$

$\therefore W = \frac{az + b}{cz + d}$

$W = \frac{-iKz - iK}{\frac{Kz}{3} - K}$

$W = \frac{-K(iz + i)}{K(\frac{z}{3} - 1)}$

$W = \frac{iz + i}{1 - \frac{z}{3}}$

$$\boxed{W = \frac{3i(z+1)}{(3-z)}}$$

3. Find the bilinear transformation which maps $z = 1, i, -1$ onto $W = 2, i, -2$.

Also find the invariant points of the transformation

Sol:- Let $W = \dfrac{az+b}{cz+d}$

For $z = 1$, $W = 2$

$$2 = \frac{a+b}{c+d}$$

$$2(c+d) = a+b$$

$$a+b-2c-2d = 0 \longrightarrow ①$$

For $z = i$, $W = i$

$$i = \frac{ai+b}{ci+d}$$

$$ci^2 + id = ai+b$$

$$ai+b = -c+id$$

$$ai+b+c-id = 0 \longrightarrow ②$$

For $z = -1$, $W = -2$

$$-2 = \frac{-a+b}{-c+d}$$

$$2c-2d = -a+b$$

$$-a+b-2c+2d = 0 \longrightarrow ③$$

$$a-b+2c-2d = 0 \longrightarrow ③$$

① − ② gives

$$(1-i)a - 3c + (i-2)d = 0 \longrightarrow ④$$

① + ③ gives

$$2a - 4d = 0$$

$$a - 2d = 0 \longrightarrow ⑤$$

Solve ④ and ⑤

$$(1-i)a - 3c + (i-2)d = 0$$
$$a + 0c - 2d = 0$$

a	c	d		
$1-i$	-3	$i-2$	$1-i$	-3
1	0	-2	1	0

$$\frac{a}{6-0} = \frac{c}{1(i-2)+2(1-i)} = \frac{d}{0+3} = k$$

$$\cancel{\frac{a}{6} = \frac{c}{ai-2a+2-2i} = \frac{d}{3a} = k}$$

$$\cancel{\frac{a}{6} = \frac{c}{i(a-2)-2a+2} = \frac{d}{3a} = k}$$

$$\frac{a}{6} = \frac{c}{i-2+2-2i} = \frac{d}{3} = k$$

$$\boxed{a = 6k,\ c = -ik,\ d = 3k}$$

$$a + b - 2c - 2d = 0$$

$$6k + b + 2ik - 6k = 0$$

$$\boxed{b = -2ik}$$

$$\therefore W = \frac{az+b}{cz+d}$$

$$W = \frac{6Kz - 2iK}{-iKz + 3K}$$

$$W = \frac{2K(3z - i)}{K(-iz + 3)}$$

$$\boxed{W = \frac{6z - 2i}{-iz + 3}}$$

To find the invariant points, put $W = z$

$$z = \frac{6z - 2i}{-iz + 3}$$

$$-iz^2 + 3z = 6z - 2i$$

$$-iz^2 = 3z - 2i$$

$$iz^2 + 3z - 2i = 0$$

$\div i$

$$z^2 + \frac{3z}{i} - 2 = 0$$

$P = -2$

$S = -3i$

$$z^2 - 3iz - 2 = 0$$

$$z = \frac{3i \pm \sqrt{9i^2 - 4(1)(-2)}}{2}$$

$$= \frac{3i \pm \sqrt{-9+8}}{2}$$

$$= \frac{3i \pm i}{2}$$

$\boxed{z = 2i, i}$ are the invariant points

4. Find the bilinear transformation that transforms the points $z_1 = 0$, $z_2 = -i$, $z_3 = -1$ onto the points $W_1 = i$, $W_2 = 1$, $W_3 = 0$

Sol:- Let $W = \dfrac{az+b}{cz+d}$

For $z_1 = 0$, $W_1 = i$

$$i = \frac{b}{d}$$

$$b - id = 0 \longrightarrow (1)$$

For $z_2 = -i$, $W_2 = 1$

$$1 = \frac{-ai+b}{-ci+d}$$

$$-ci + d = -ai + b$$

$$ai - b - ci + d = 0 \longrightarrow (2)$$

For $z_3 = -1$, $W_3 = 0$

$$0 = \frac{-a+b}{-c+d}$$

$$-a + b = 0$$

$$a - b = 0 \longrightarrow (3)$$

$$0a + b - id = 0 \longrightarrow (1)$$

$$a - b + 0d = 0 \longrightarrow (3)$$

Solving (1) and (3)

a	b	d		
0	1	$-i$	0	1
1	-1	0	1	-1

$$\frac{a}{0-i} = \frac{b}{-i} = \frac{d}{0-1} = K$$

$$\frac{a}{-i} = \frac{b}{-i} = \frac{d}{-1} = K$$

$$\boxed{a = -iK,\ b = -iK,\ d = -K}$$

$$ai - b - ci + d = 0$$

$$-i^2K + iK - ci - K = 0$$

$$iK - ci = 0$$

$$ci = iK$$

$$\boxed{c = K}$$

$$\therefore W = \frac{-iKz - iK}{Kz - K}$$

$$= \frac{K(-iz - i)}{K(z-1)}$$

$$W = \frac{i + iz}{1 - z}$$

$$\boxed{W = \frac{i(1+z)}{(1-z)}}$$

5. Find the bilinear transformation which maps the points $z = \infty, i, 0$ onto $W = -1, -i, 1$. Also find the fixed points of the transformation.

Sol:- Let $W = \dfrac{az+b}{cz+d}$

$$W = \frac{z\left(a + \frac{b}{z}\right)}{z\left(c + \frac{d}{z}\right)}$$

$$W = \frac{a + \frac{b}{z}}{c + \frac{d}{z}}$$

when $z = \infty$, $W = -1$

$$-1 = \frac{a}{c}$$

$$\Rightarrow a + c = 0 \longrightarrow (1)$$

when $z = i$, $W = -i$

$$-i = \frac{ia + b}{ic + d}$$

$$c - id = ia + b$$

$$ia + b - c + id = 0 \longrightarrow (2)$$

when $z = 0$, $W = 1$

$$1 = \frac{b}{d}$$

$$b - d = 0 \longrightarrow (3)$$

(1) + (2) gives

$$a(i+1) + b + id = 0$$

$$0a + b - d = 0$$

$$\begin{array}{ccccc} a & b & d & & \\ i+1 & 1 & i & i+1 & 1 \\ 0 & 1 & -1 & 0 & 1 \end{array}$$

$$\frac{a}{-1-i} = \frac{b}{i+1} = \frac{d}{i+1} = K$$

$$\boxed{a = (-1-i)K, \quad b = (i+1)K, \quad d = (i+1)K}$$

$$a = -c$$

$$c = -a$$

$$= -(-1-i)K$$

$$\boxed{c = (1+i)K}$$

$$\therefore W = \frac{az+b}{cz+d}$$

$$= \frac{-K(1+i)z + (1+i)K}{(1+i)Kz + (1+i)K}$$

$$W = (1+i)K\left[\frac{-z+1}{z+1}\right]$$

$$\boxed{W = \frac{1-z}{1+z}}$$

To find the fixed points, put $W = z$

$$z = \frac{1-z}{1+z}$$

$$z + z^2 - 1 + z = 0$$

$$z^2 + 2z - 1 = 0$$

$$z = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2}$$

$$z = \frac{-2 \pm \sqrt{8}}{2}$$

$$z = \frac{-2 \pm 2\sqrt{2}}{2} \Rightarrow z = \sqrt{2}-1, \ -\sqrt{2}-1$$

$$\boxed{z = -1 \pm \sqrt{2}}$$

6. Find the bilinear transformation which maps the points $z = 0, 1, \infty$ onto $W = -5, -1, -3$. Find the invariant points of the transformation. 2/4/14

Sol: Let $W = \dfrac{az+b}{cz+d}$

When $z = 0$, $W = -5$

$$-5 = \frac{b}{d}$$

$$b + 5d = 0 \longrightarrow (1)$$

When $z = 1$, $W = -1$

$$-1 = \frac{a+b}{c+d}$$

$$-c-d = a+b$$

$$a+b+c+d = 0 \longrightarrow (2)$$

~~When $z = \infty$, $W = -3$~~

~~$-3 =$~~

~~$W = z\left(\frac{a}{z}\right.$~~

$$W = \frac{z\left(a + \frac{b}{z}\right)}{z\left(c + \frac{d}{z}\right)} = \frac{a + \frac{b}{z}}{c + \frac{d}{z}}$$

When $z = \infty$, $W = -3$

$$-3 = \frac{a}{c}$$

$$a + 3c = 0 \longrightarrow (3)$$

② - ③ gives

$b - 2c + d = 0 \rightarrow$ ④

Solving ① and ④

$b - 2c + d = 0$

$b + 0c + 5d = 0$

b	c	d		
1	-2	1	1	-2
1	0	5	1	0

$$\frac{b}{-10-0} = \frac{c}{1-5} = \frac{d}{0+2} = K$$

$$\boxed{b = -10K, \quad c = -4K, \quad d = 2K}$$

$3c = -a$

$\cancel{c = \frac{-a}{3}}$

$\cancel{c =}$

$a = -3c$

$$\boxed{a = 12K}$$

$$\therefore W = \frac{az + b}{cz + d}$$

$$= \frac{12Kz - 10K}{-4Kz + 2K}$$

$$= \frac{K(12z - 10)}{K(-4z + 2)} = \frac{2(6z - 5)}{2(-2z + 1)}$$

$$W = \frac{6z - 5}{1 - 2z}$$

To find the invariant points, put $W = z$

$$z = \frac{6z-5}{1-2z}$$

$$z - 2z^2 = 6z - 5$$

$\cancel{-2z^2}$

$\infty = \frac{1}{0}$

$0 = \frac{1}{\infty}$

$$2z^2 + 5z - 5 = 0$$

$$z = \frac{-5 \pm \sqrt{25-4(2)(-5)}}{4}$$

$$= \frac{-5 \pm \sqrt{25+40}}{4}$$

$$\boxed{z = \frac{-5 \pm \sqrt{65}}{4}}$$

7. $z = 1, i, -1$ onto $W = 0, 1, \infty$, find W.

Sol:- Let $W = \frac{az+b}{cz+d}$

when $z = 1$, $W = 0$

$\cancel{0 = z}$ $\quad 0 = \frac{a+b}{c+d}$

$a + b = 0 \longrightarrow (1)$

when $z = i$, $W = 1$

$$1 = \frac{ai+b}{ci+d}$$

$$ci + d = ai + b$$

$$ai + b - ci - d = 0 \longrightarrow (2)$$

$\cancel{W = \frac{z(a+\frac{b}{z})}{z(c+\frac{d}{z})}}$ $= \cancel{\frac{a+\frac{b}{z}}{\ldots}}$

$$\infty = \frac{-a+b}{-c+d}$$

$$\frac{1}{\infty} = \frac{-c+d}{-a+b}$$

$$0 = \frac{-c+d}{-a+b}$$

$$0 = -c+d$$

$$-c+d=0 \longrightarrow (3)$$

$\boxed{\frac{1}{0} = \infty}$ $\boxed{\frac{1}{\infty} = 0}$

(1) - (2) gives

$$a(1-i)+ci+d=0 \longrightarrow (4)$$

Solving (3) and (4)

a	c	d		
$1-i$	i	1	$1-i$	i
0	-1	1	0	-1

$$\frac{a}{i+1} = \frac{c}{0-1+i} = \frac{d}{-1+i} = K$$

$$\boxed{a = K(i+1),\ c = K(i-1),\ d = K(i-1)}$$

$$b = -a$$

$$\boxed{b = -K(i+1)}$$

$$\therefore W = \frac{az+b}{cz+d}$$

$$= \frac{K(i+1)z - K(i+1)}{K(i-1)z + K(i-1)} = \frac{(i+1)z-(i+1)}{(i-1)z+i-1}$$

$$W = \frac{(1+i)(z-1)}{(i-1)(z+1)}$$

$$W = \frac{(1+i)(z-1)}{(i-1)(z+1)} \times \left(\frac{i+1}{i+1}\right)$$

$$W = \frac{1-1+2i}{-2} \quad \frac{z-1}{z+1}$$

$$W = -i\left(\frac{z-1}{z+1}\right)$$

$$\boxed{W = i\left(\frac{1-z}{1+z}\right)}$$

8. Find the bilinear transformation of $z = -1, i, 1$, $W = 1, i, -1$.

Sol:- Let $W = \dfrac{az+b}{cz+d}$

When $z=-1$, $W=1$

$$1 = \frac{-a+b}{-c+d}$$

$$-a+b+c-d = 0 \longrightarrow (1)$$

When $z = i$, $W = i$

$$i = \frac{ai+b}{ci+d}$$

$$-c+id = ai+b$$

$$ai+b+c-id = 0 \longrightarrow (2)$$

When $z=1$, $W=-1$

$$-1 = \frac{a+b}{c+d}$$

$$-c-d = a+b$$

$$a+b+c+d = 0 \longrightarrow (3)$$

① + ③ gives

$2b+2c=0$

$\Rightarrow b+c=0 \rightarrow$ ④

③ × i gives

$ai+bi+ci+di=0 \rightarrow$ ⑤

② - ⑤ gives

$b(1-i)+c(1-i)-2id=0 \rightarrow$ ⑥

Solve ④ and ⑥

b	c	d		
1	1	0	1	1
$1-i$	$1-i$	$-2i$	$1-i$	$1-i$

$$\frac{b}{-2i}=\frac{c}{+2i}=\frac{d}{1-i-(1-i)}=K$$

$$\frac{b}{-2i}=\frac{c}{2i}=\frac{d}{0}=K$$

$$\boxed{b=-2iK,\quad c=2iK,\quad d=0}$$

$a=-b-c-d$

$=2iK-2iK$

$\boxed{a=0}$

$$\therefore\ W=\frac{az+b}{cz+d}$$

$$W=\frac{b}{\cancel{2iKz}}=\frac{-2iK}{2iKz}$$

$$\boxed{W=\frac{-1}{z}}$$

CONFORMAL TRANSFORMATION

A transformation that preserves the angle between the curves both in magnitude and direction is called a Conformal Transformation.

1. Discussion of $W = e^z$

Given, $W = e^z$

$$u+iv = e^{x+iy}$$

$$u+iv = e^x \cdot e^{iy}$$

$$u+iv = e^x(\cos y + i\sin y)$$

$$\therefore \left.\begin{array}{l} u = e^x \cos y \\ v = e^x \sin y \end{array}\right\} \longrightarrow (1)$$

Case(i)

Let $x = c_1$

$$\therefore (1) \Rightarrow \quad u = e^{c_1}\cos y$$

$$v = e^{c_1}\sin y$$

Square & add to eliminate y

$$\therefore u^2+v^2 = (e^{c_1})^2$$

This represents a circle in the W plane with centre at the origin and radius as e^{c_1}

$\therefore$ The straight line parallel to the Y-axis in the z-plane is mapped to a circle in the W-plane

Case (ii)

Let $y = C_2$

$\therefore$ (1) $\Rightarrow u = e^{x} \cos C_2$

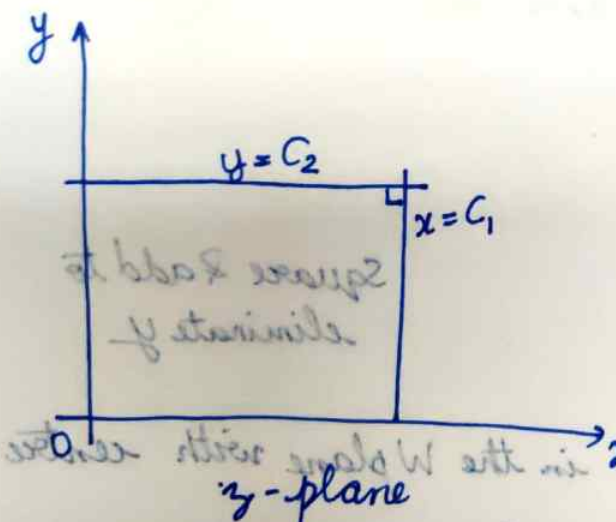
(2) $\Rightarrow v = e^{x} \sin C_2$

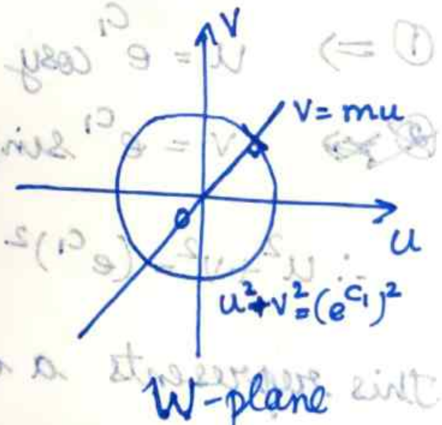
$$\frac{v}{u} = \frac{\sin C_2}{\cos C_2} = \tan C_2 \quad \text{[Eliminate } x\text{]}$$

$v = mu \qquad m = \tan C_2$

This represents a straight line passing through the origin in the W-plane

$\therefore$ The straight line parallel to the X-axis in the z plane is mapped to a straight line in the W plane.





2. Discussion of $W = z^2$ 4/4/14

Given, $W = z^2$

$$u + iv = (x+iy)^2$$

$$u + iv = x^2 - y^2 + 2xiy$$

$$\therefore \left.\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}\right\} \longrightarrow ①$$

Case(i)

Let $x = c_1$

$$\therefore ① \Rightarrow \quad u = c_1^2 - y^2$$

$$v = 2c_1 y$$

$$\Rightarrow \frac{v}{2c_1} = y$$

$$\therefore u = c_1^2 - \frac{v^2}{4c_1^2}$$

$$u - c_1^2 = \frac{-v^2}{4c_1^2}$$

$$v^2 = -4c_1^2 (u - c_1^2)$$

Std form $(y-k)^2 = 4a(x-h)$

This represents a parabola with vertex at,

$$V = (h, k) = (c_1^2, 0)$$

and focus at,

$$S = (a+h, k) = (-c_1^2 + c_1^2, 0) = (0,0)$$

$\therefore$ The straight line parallel to the y-axis in the z-plane is mapped to a parabola in the W-plane

Similarly,

$x = -c_1$ is also transformed to the same parabola

Case (ii)

Let $y = C_2$

$\therefore ① \Rightarrow u = x^2 - C_2^2$

$v = 2x C_2$

$\Rightarrow x = \frac{v}{2C_2}$

$u = \frac{v^2}{4C_2^2} - C_2^2$

$u + C_2^2 = \frac{v^2}{4C_2^2}$

$v^2 = 4C_2^2(u + C_2^2)$

Std form $(y-k)^2 = 4a(x-h)$

This represents a parabola with vertex at

$V = (h, k) = (-C_2^2, 0)$

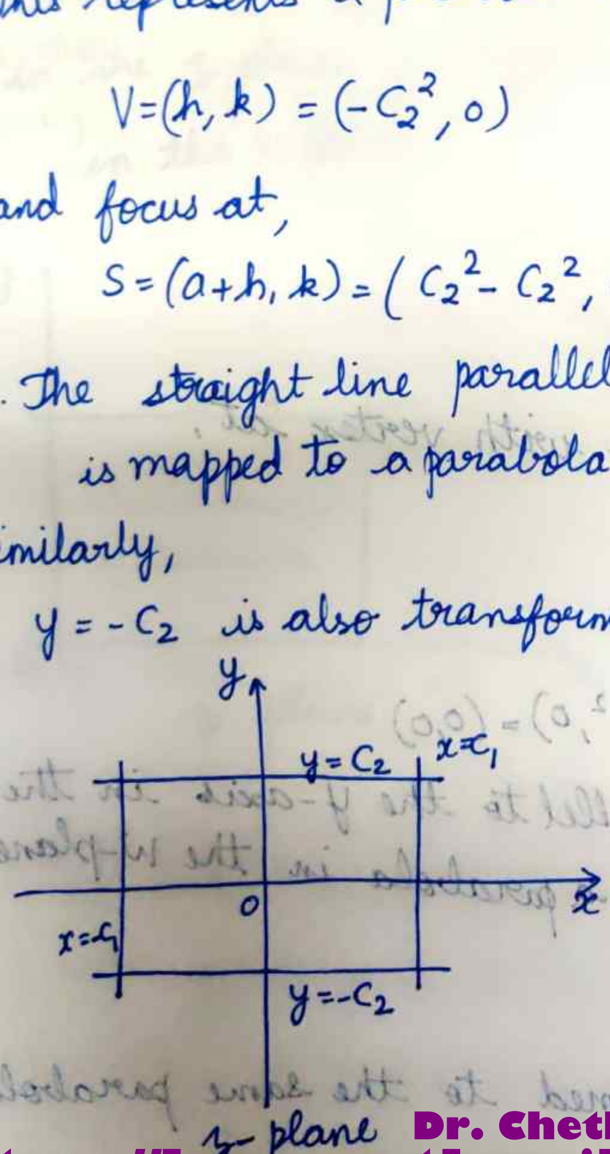
and focus at,

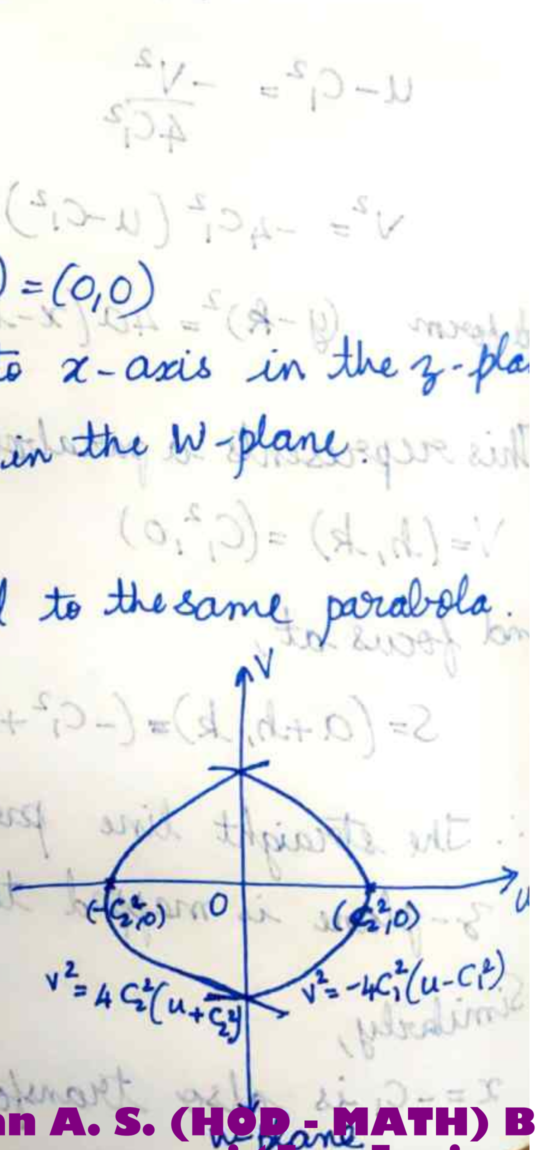
$S = (a+h, k) = (C_2^2 - C_2^2, 0) = (0,0)$

$\therefore$ The straight line parallel to x-axis in the z-pla[ne] is mapped to a parabola in the W-plane.

Similarly,

$y = -C_2$ is also transformed to the same parabola.





3. Discussion of $W = z + \frac{a^2}{z}$ $\quad W = z + \frac{1}{z}$

Given, $W = z + \frac{a^2}{z}$

$$u + iv = re^{i\theta} + \frac{a^2}{re^{i\theta}} \qquad \text{[Polar form]}$$

$$u + iv = r(\cos\theta + i\sin\theta) \cancel{+ \frac{a^2}{r(\cos}}$$

$$+ \frac{a^2}{r}(\cos\theta - i\sin\theta)$$

$$\therefore u = r\cos\theta + \frac{a^2}{r}\cos\theta = \cos\theta\left(r + \frac{a^2}{r}\right)$$

$$v = r\sin\theta - \frac{a^2}{r}\sin\theta = \sin\theta\left(r - \frac{a^2}{r}\right) \quad \rightarrow (1)$$

Case (i)

Let $r = c_1$

$$\therefore (1) \Rightarrow u = \cos\theta\left(c_1 + \frac{a^2}{c_1}\right)$$

$$v = \sin\theta\left(c_1 - \frac{a^2}{c_1}\right)$$

$$\frac{u}{c_1 + \frac{a^2}{c_1}} = \cos\theta$$

$$\frac{v}{c_1 - \frac{a^2}{c_1}} = \sin\theta$$

Square and add

$$\frac{u^2}{\left(c_1 + \frac{a^2}{c_1}\right)^2} + \frac{v^2}{\left(c_1 - \frac{a^2}{c_1}\right)^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ellipse with

$\therefore$ The circle in the z-plane is mapped to an ellipse in the W-plane

Case(ii)

Let $\theta = C_2$

$\therefore$ ① $\Rightarrow$ $u = \left(r + \frac{a^2}{r}\right)\cos C_2$

$$v = \left(r - \frac{a^2}{r}\right)\sin C_2$$

$$\frac{u}{\cos C_2} = r + \frac{a^2}{r}$$

$$\frac{v}{\sin C_2} = r - \frac{a^2}{r}$$

Squaring the above equations and subtracting we get,

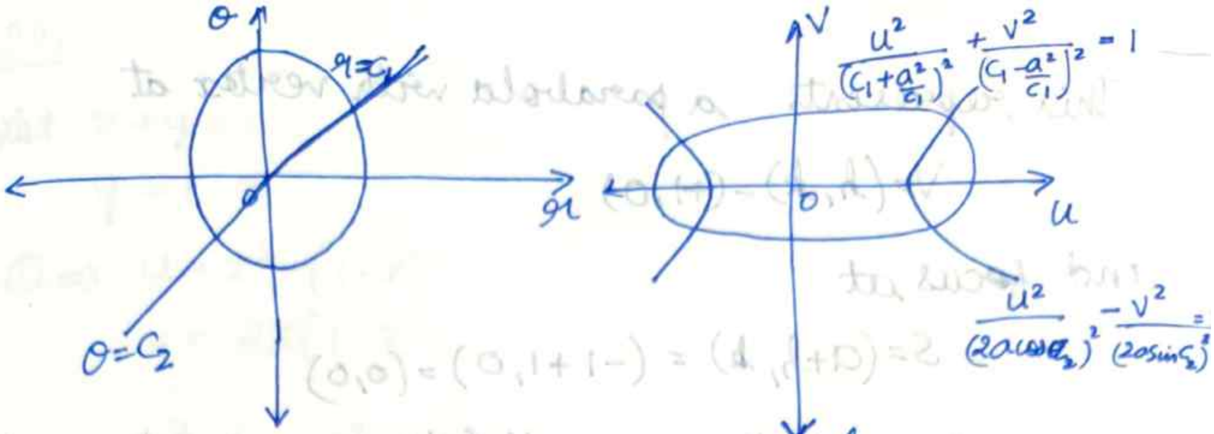
$$\frac{u^2}{\cos^2 C_2} - \frac{v^2}{\sin^2 C_2} = \left(r + \frac{a^2}{r}\right)^2 - \left(r - \frac{a^2}{r}\right)^2$$

$$= r^2 + \frac{a^4}{r^2} + 2a^2 - \left(r^2 + \frac{a^4}{r^2} - 2a^2\right)$$

$$\frac{u^2}{\cos^2 C_2} - \frac{v^2}{\sin^2 C_2} = 4a^2$$

$$\frac{u^2}{(2a\cos C_2)^2} - \frac{v^2}{(2a\sin C_2)^2} = 1$$

$\therefore$ The line $\theta = C_2$ in the z-plane is mapped to hyperbola in the W-plane



Problems

1. Find the region in the W-plane bounded by the lines $x=1, y=1$, $x+y=1$ under the transformation $W=z^2$

Sol:- Given,

$$W=z^2$$

$$u+iv=(x+iy)^2$$

$$u+iv = x^2-y^2+2xyi$$

$$\therefore \left.\begin{aligned} u &= x^2-y^2 \\ v &= 2xy \end{aligned}\right\} \longrightarrow (1)$$

Case(i)

Let $x=1$

$$\therefore (1) \Rightarrow u=1-y^2$$

$$v=2y$$

$$\Rightarrow \frac{v}{2}=y$$

$$\therefore u = 1-\frac{v^2}{4}$$

$$-\frac{v^2}{4} = u-1$$

$$v^2=-4(u-1)$$

Std form $(y-k)^2=4a(x-h)$

This represents a parabola with vertex at

$$V=(h,k)=(+1,0)$$

and focus at

$$S=(a+h,k)=(-1+1,0)=(0,0)$$

$\therefore$ The straight line parallel to y-axis ~~at~~ in the z-plane is mapped to a parabola in the W-plane

Case(ii)

Let $y=1$

$\therefore ① \Rightarrow \quad u = x^2 - 1$

$v = 2x$

$\Rightarrow x = \frac{v}{2}$

$\therefore u = \frac{v^2}{4} - 1$

$\frac{v^2}{4} = +u + 1$

$v^2 = +4(u+1)$

Std form $(y-k)^2 = 4a(x-h)$

This represents a parabola with vertex at

$$V=(h,k)=(-1,0)$$

and focus at,

$$S=(a+h,k)=(+1-1,0)=(0,0)$$

$\therefore$ The straight line parallel to x-axis in the z-plane is mapped to a parabola in the W-plane.

Case (iii)

Let $x+y=1$

$y=1-x$

$\therefore (1) \Rightarrow u = x^2-(1-x)^2$

$v = 2x(1-x)$

$u = x^2-(1+x^2-2x)$

$u = 2x-1$

$u+1 = 2x$

$x = \frac{u+1}{2}$

$\therefore v = 2\left(\frac{u+1}{2}\right)\left(1-\frac{u+1}{2}\right)$

$v = u+1\left(\frac{2-u-1}{2}\right)$

$v = \frac{(u+1)(1-u)}{2}$

$v = \frac{(1+u)(1-u)}{2}$

~~$2v = (1-u)^2$~~

$2v = 1-u^2$

$u^2 = 1-2v = -2v+1$

$u^2 = -2\left(v-\frac{1}{2}\right)$

Std form $(x-h)^2 = 4a(y-k)$

This represents a parabola with vertex at

$V = (h, k) = \left(0, \frac{1}{2}\right)$

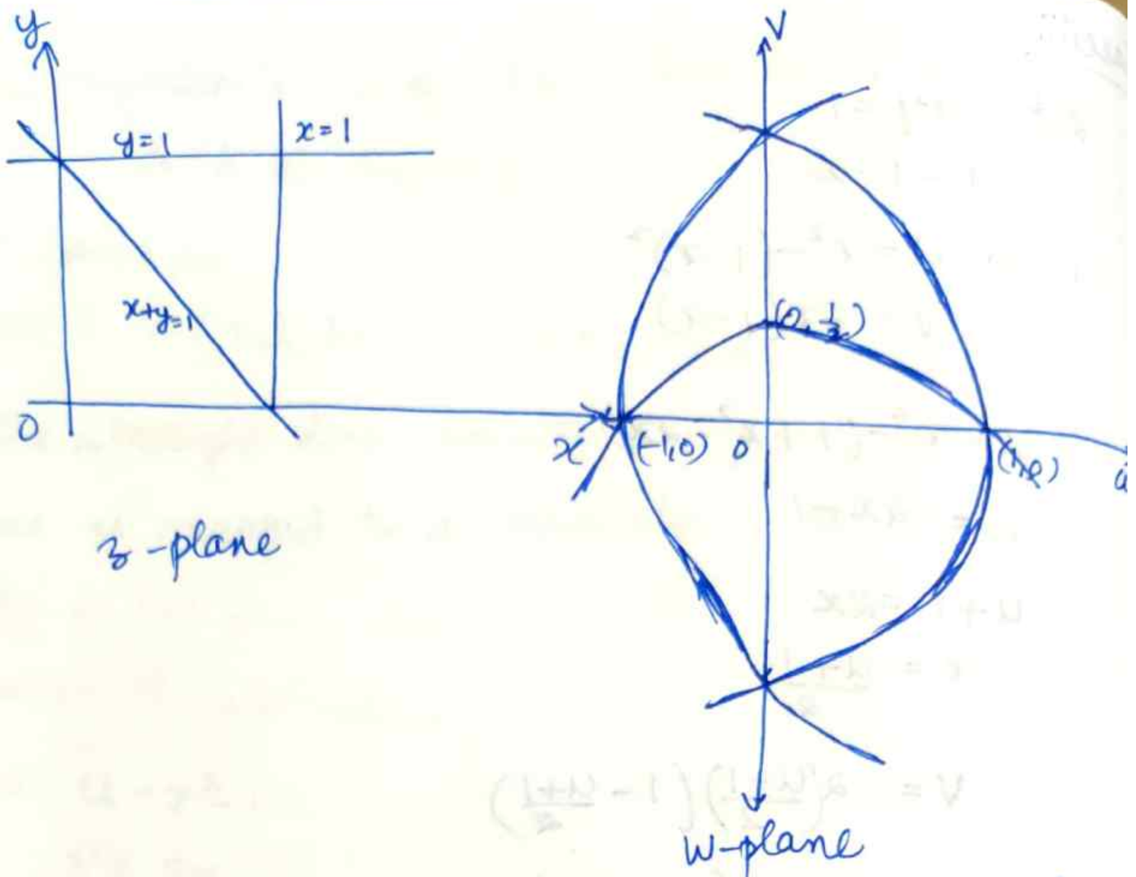
$4a = -2$

$a = \frac{-1}{2}$

and focus at,

$S = (h, a+k) = \left(0, -\frac{1}{2}+\frac{1}{2}\right) = (0, 0)$

$\therefore$ The line $x+y=1$ in the z-plane is mapped to a parabola in the w-plane



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2. Under the transformation, $W = z^2$, find the images of

(i) $x - y = 1$, $x^2 - y^2 = 1$

(ii) The image of the square region bounded by the lines $x = 1$, ~~and~~ $x = 2$, $y = 1$, $y = 2$.

CAUCHY'S THEOREM

Statement: If $f(z)$ is analytic at all points inside and on a simple closed curve, C, then

$$\int_C f(z)\,dz = 0$$

Proof:

Let $f(z)=u+iv$ be analytic

$\Rightarrow$ u and v satisfies C-R equations

i.e., $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

We have, $z = x+iy$

$dz = dx + idy$

Consider,

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dx$$

Using Green's Theorem, we get

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0 + (0)i$$

$$\therefore \boxed{\int_C f(z)\,dz = 0}$$

CONSEQUENCES of Cauchy's Theorem

1. If $f(z)$ is analytic in a region R and if A and B are any two points in it, then, $\int_A^B f(z)\,dz$ is independent of the path joining A and B.

2. If C_1 and C_2 are 2 simple closed curves, such that C_2 lies entirely within C_1 and if $f(z)$ is analytic on C_1, C_2 and in the region bounded by C_1 and C_2, then,

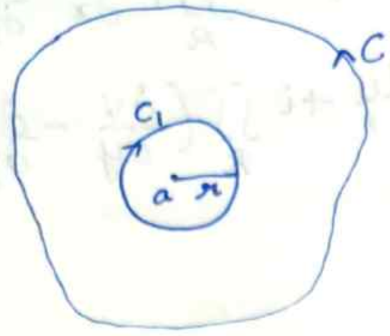
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

CAUCHY'S INTEGRAL FORMULA

Statement: If $f(z)$ is analytic inside and on a simple closed curve C, and if a is any point within C, then,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Proof:-



Since a is a point within C, we enclose it by a circle C_1 with centre at a and radius r, such that, C_1 lies entirely within C.

Clearly,

$\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the angular region between C and C_1.

As a consequence of Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

The equation of C_1 is

$$|z-a| = r$$

$$\Rightarrow z-a = re^{i\theta}$$

$$\Rightarrow z = a+re^{i\theta}, \quad 0 \le \theta \le 2\pi$$

$$\Rightarrow dz = ire^{i\theta}\,d\theta$$

Eqn of circle in polar form is $|z-a| = r$ ($0°$ to $360°$)

Centre, $z=a$

Radius $= r$

$$\therefore \int_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

As $r \to 0$,

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta$$

$$\int_C \frac{f(z)}{z-a}\,dz = i\,f(a)\,[2\pi - 0]$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i \cdot f(a)$$

$$\therefore \boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$

GENERALISED CAUCHY'S INTEGRAL FORMULA

Statement: If $f(z)$ is analytic, inside and on a closed curve C and a is any point within C, then,

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Proof:-

By Cauchy's Integral Formula, we have,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Differentiate wrt a

$$f'(a) = \frac{1}{2\pi i} \int_C -\frac{f(z)}{(z-a)^2} (-1)\, dz$$

$$f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz$$

Differentiate wrt a

$$f''(a) = \frac{1}{2\pi i} \int_C \frac{-2f(z)}{(z-a)^3} (-1)\, dz$$

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3}\, dz$$

Differentiating the above equation $(n-2)$ times wrt a we get,

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz}$$

Problems

1. By using Cauchy's Integral Formula, eva[illegible]

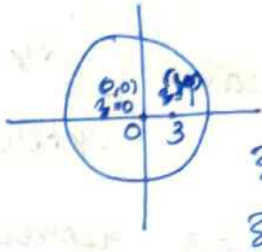
$\int_C \frac{1}{z(z-1)}\,dz$ where C is the circle, $|z| = 3$

Sol:- $|z| = 3$ represents a circle with centre at the origin and radius $= 3$.

Consider,

$z(z-1) = 0$

$z = 0, z = 1$



$z = 0\ (0,0)$

$z = 1 (1,0)$

~~Let $f(z) = 1$~~

Both the points $z=0$ and $z=1$ lies inside the circle $|z| = 3$

$\therefore f(z) = 1$

Consider,

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$\frac{1}{z(z-1)} = \frac{A(z-1)+Bz}{z(z-1)}$$

$1 = A(z-1) + Bz$

Put $z = 1$, Put $z = 0$

$\boxed{1 = B}$ $\quad$ $1 = -A$

$\boxed{A = -1}$

$$\therefore \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{1}{z(z-1)}\,dz = \int_C f(z)\left\{\frac{-1}{z}+\frac{1}{z-1}\right\}dz$$

$$= -\int_C \frac{f(z)}{z}\,dz + \int_C \frac{f(z)}{z-1}\,dz$$

$$= -2\pi i\, f(0) + 2\pi i\, f(1)$$ [By Cauchy's Integral Formula]

$$= -2\pi i + 2\pi i$$

$$= \underline{\underline{0}}$$

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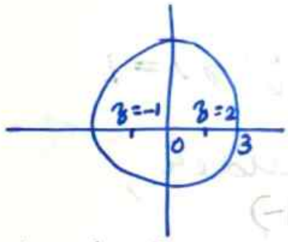
2. Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz$ where $C : |z| = 3$

Sol:- $|z| = 3$ represents a circle with centre at the origin and radius $= 3$.

Consider,

$(z+1)(z-2) = 0$

$z = -1, 2$ $\boxed{z=-1}$ $\boxed{z=2}$



(-1,0)

$z=-1$ and $z=2$ lies inside the circle $|z| = 3$. (2,0)

$\therefore f(z) = e^{2z}$

Consider,

$$\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$$

$1 = A(z-2) + B(z+1)$

$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

Put $z = +2$, Put $z = -1$

$1 = +3B$ $\boxed{-1/3 = A}$

$\boxed{B = +1/3}$

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{-1/3}{z+1} + \frac{1/3}{z-2}$$

$$\int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = \int_C f(z)\left\{\frac{-1/3}{z+1} + \frac{1/3}{z-2}\right\} dz$$

$$= -\frac{1}{3}\oint_C \frac{f(z)}{z+1}\,dz + \frac{1}{3}\int_C \frac{f(z)}{z-2}\,dz$$

$$= +\frac{1}{3}\left\{-2\pi i\, f(-1) + 2\pi i\, f(2)\right\}$$

$$= \frac{2\pi i}{3}\left\{-e^{-2} + e^{4}\right\}$$ [By Cauchy's Integral Formula]

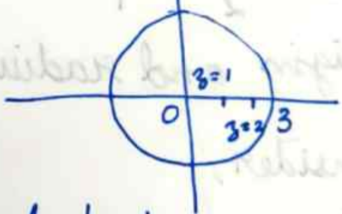
3. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ where C is a curve, $|z| = 3$.

Sol:- $|z| = 3$ represents a circle with centre at the origin and radius $= 3$

Consider,

$(z-1)(z-2) = 0$

$\boxed{z=1}, \boxed{z=2}$



$z=1$ and $z=2$ lies inside the circle $|z| = 3$. (1,0) (2,0)

$\therefore f(z) = \sin \pi z^2 + \cos \pi z^2$

Consider,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$1 = A(z-2) + B(z-1)$

Put $z = 2$, Put $z = 1$

$\boxed{1 = B}$ $\boxed{-1 = A}$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C f(z) \left\{ \frac{-1}{z-1} + \frac{1}{z-2} \right\} dz$$

$$= -\int_C \frac{f(z)}{z-1} dz + \int_C \frac{f(z)}{z-2} dz$$

$$= -2\pi i \, f(-1) + 2\pi i \, f(2)$$

$= \cancel{\sin \pi}$

$$= 2\pi i + 2\pi i$$

$$= \underline{\underline{4\pi i}}$$

4. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$ where C is the curve,

$|z| = \frac{3}{2}$

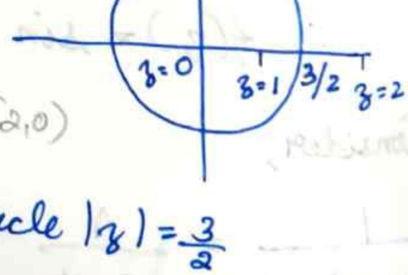
Sol:- $|z| = \frac{3}{2}$ represents a circle with centre at the origin and radius $= \frac{3}{2}$.

Consider,

$z(z-1)(z-2) = 0$

$\boxed{z=0}$ $\boxed{z=1}$ $\boxed{z=2}$

(0,0) (1,0) (2,0)



$z=0$ and $z=1$ lies inside the circle $|z| = \frac{3}{2}$

$$\therefore f(z) = \frac{4-3z}{z-2}$$

Consider, $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

$$1 = A(z-1) + Bz$$

Put $z=0$, $\quad$ Put $z=1$

$\boxed{A=-1}$ $\quad$ $\boxed{1=B}$

$$\therefore \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_C \frac{f(z)}{z(z-1)}\,dz$$

$$= \int_C f(z)\left\{\frac{-1}{z} + \frac{1}{z-1}\right\}$$

$$= -\int_C \frac{f(z)}{z} + \int_C \frac{f(z)}{z-1}$$

$$= -2\pi i\, f(0) + 2\pi i\, f(1)$$

$$= 4\pi i - 2\pi i$$

$$= \underline{2\pi i}$$

5. Evaluate $\int_C \frac{z}{(z^2+1)(z^2-9)}\,dz$ where c is the curve (i) $|z|=2$ (ii) $|z-2|=2$

Sol:- $|z|=2$ represents a circle with centre at the origin and radius $=2$

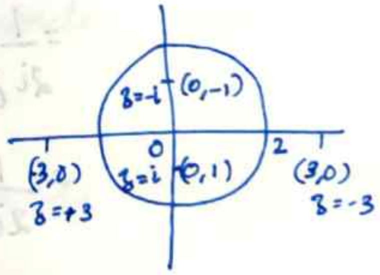
Consider,

$(z^2+1)(z^2-9)=0$

$(z^2+1)(z+3)(z-3)=0$

$\Rightarrow z^2=-1$, $\boxed{z=3}$, $\boxed{z=-3}$

$\boxed{z=\pm i}$



$\boxed{z=\pm i}$, $\boxed{z=\pm 3}$

$\pm i$
(0, 1) $\quad$ (3,0)
(0, -1) $\quad$ (-3,0)

$z=\pm i$ lies inside the circle and $z=\pm 3$ lies outside the circle.

$$\therefore f(z) = \frac{z}{z^2-9}$$

Consider,

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$$

$$1 = A(z+i) + B(z-i)$$

Put $z=-i$, Put $z=i$

$1 = -2iB$ $\qquad$ $1 = 2Ai$

$B = \frac{-1}{2i}$ $\qquad$ $A = \frac{1}{2i}$

$\boxed{B = \frac{i}{2}}$ $\qquad$ $\boxed{A = \frac{-i}{2}}$

$$4 \quad \frac{1}{z^2+1} = \frac{\frac{1}{2i}}{z-i} + \frac{\frac{-1}{2i}}{z+i}$$

$$\frac{1}{z^2+1} = \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{z+i}\right\}$$

$$\therefore \int_C \frac{z}{(z^2+1)(z^2-9)}\,dz = \int_C f(z)\left\{\frac{1}{z^2+1}\right\}dz$$

$$= \frac{1}{2i}\int_C f(z)\left\{\frac{1}{z-i} - \frac{1}{z+i}\right\}dz$$

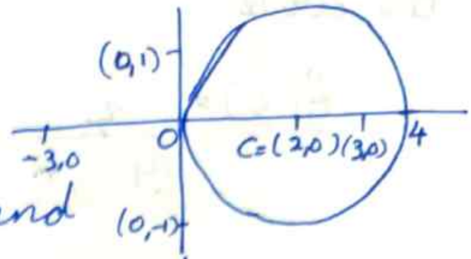
$$= \frac{1}{2i}\left\{\int_C \frac{f(z)}{z-i} - \int_C \frac{f(z)}{z+i}\right\}dz$$

$$= \frac{1}{2i}\left\{2\pi i f(i) - 2\pi i f(-i)\right\}$$

$$= \frac{1}{2i}\left\{\frac{2\pi i^2}{-10} - \frac{2\pi i(-i)}{-10}\right\}$$

$$= \frac{1}{2i}\left\{\frac{2\pi}{10} + \frac{2\pi}{+10}\right\} = \frac{1}{2i}\left\{\frac{\pi}{5} + \frac{\pi}{5}\right\} = \frac{\pi}{5i}$$

(ii) $|z-2| = 2$ represents a circle with centre at $z = 2$ and radius $= 2$.



$\boxed{z = \pm i}$, $\boxed{z = \pm 3}$

$z = 3$ lies inside the circle and $z = \pm i$ and $z = -3$ lies outside the circle.

$$\therefore f(z) = \frac{z}{(z^2+1)(z+3)}$$

Consider,

$$\frac{1}{z-3}$$

$$\int_C \frac{z}{(z^2+1)(z^2-9)} dz = \int_C \frac{f(z)}{z-3} dz$$

$$= 2\pi i f(3)$$

$$= \frac{2\pi i}{20}$$

$$= \frac{\pi i}{10}$$

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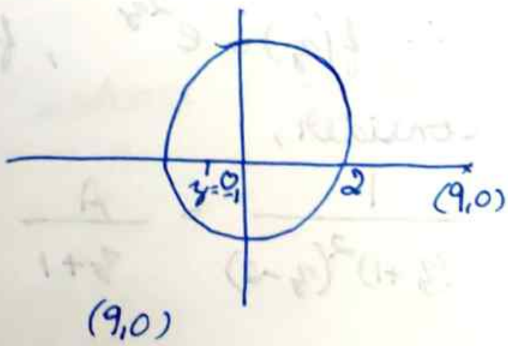
6. By using Cauchy's Integral formula, evaluate $\int_C \frac{z}{(9-z)^2(z+1)} dz$ where C is a circle, $|z| = 2$.

Sol:- $|z| = 2$ represents a circle with centre at origin and radius $= 2$.



Consider,

$(9-z)^2(z+1) = 0$

$(9-z)^2 = 0$ or $z+1 = 0$

$(9-z)^2 = 0$, $z = -1$ $\Rightarrow z = 9, 9$

(9,0)

$\boxed{z = 9, z = -1}$

$z = -1$ lies inside the circle and $z = 9$ lies outside the circle.

$$\therefore f(z) = \frac{z}{(9-z)^2}$$

Consider,

$$\int_C \frac{z}{(9-z)^2(z+1)}\,dz = \int_C \frac{f(z)}{z+1}\,dz$$

$$= 2\pi i\, f(-1)$$

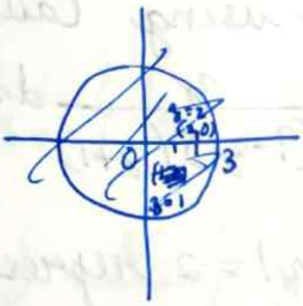
$$= 2\pi i\left(\frac{-1}{100}\right)$$

$$= \frac{-\pi i}{50}$$

7. Evaluate $\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz$ where C is a curve, $|z| = 3$

Sol:- $|z| = 3$ represents a circle with centre at origin and radius $= 3$.

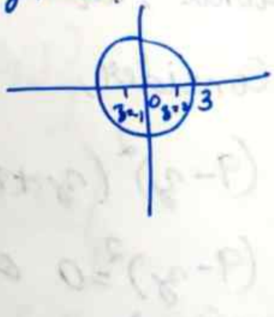
Consider,



$(z+1)^2(z-2) = 0$

$z = -1$ ~~$z=-1$~~ $z = 2$

$z = -1$ and $z = 2$ lies inside the circle, $|z| = 3$.



$\therefore f(z) = e^{2z}$, $f'(z) = 2e^{2z}$

Consider,

$$\frac{1}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$$

$$1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$$

Put $z=-1$, Put $z = 2$ Put $z=0$

$1 = -3B$ $\quad$ $1 = 9C$ $\quad$ $1 = -2A - 2B + C$

$\boxed{B = \frac{-1}{3}}$ $\quad$ $\boxed{C = \frac{1}{9}}$ $\quad$ $2A = C - 2B - 1$

$$2A = \frac{1}{9} + \frac{2}{3} - 1 = \frac{-2}{9}$$

$$\boxed{A = \frac{-1}{9}}$$

$$\therefore \frac{1}{(z+1)^2(z-2)} = \frac{\frac{-1}{9}}{z+1} + \frac{\frac{-1}{3}}{(z+1)^2} + \frac{\frac{1}{9}}{z-2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \int_C f(z)\left\{\frac{-\frac{1}{9}}{z+1} - \frac{\frac{1}{3}}{(z+1)^2} + \frac{\frac{1}{9}}{z-2}\right\} dz$$

$$= \frac{-1}{9}\int_C \frac{f(z)}{z+1}\,dz - \frac{1}{3}\int_C \frac{f(z)}{(z+1)^2}\,dz + \frac{1}{9}\int_C \frac{f(z)}{z-2}\,dz$$

$$= \frac{-1}{9}\,2\pi i\, f(-1) - \frac{1}{3}\,\frac{2\pi i}{1!}\, f'(-1) + \frac{1}{9}\,2\pi i\, f(2)$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$= \frac{2\pi i}{3}\left\{\frac{-1}{3}e^{-2} - 2e^{-2} + \frac{1}{3}e^{4}\right\}$$

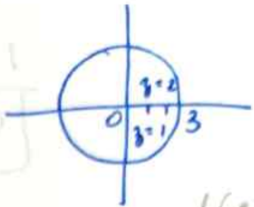
$$= \frac{2\pi i}{3}\left(\frac{-7}{3}e^{-2} + \frac{e^4}{3}\right)$$

$$= \frac{2\pi i}{9}\left(e^4 - 7e^{-2}\right)$$

8. Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\,dz$ where C is the curve,

(i) $|z| = 3$ $\quad$ (ii) $|z| = \frac{3}{2}$

Sol:- (i) $|z| = 3$ represents a circle with centre at origin and radius $= 3$



Consider,

$$(z-1)^2(z-2) = 0$$

$$\boxed{z = 1,\ z = 2}$$

$f(x) = \sin x^2$

$f'(x) = \cos x^2$

$z=1$ and $z=2$ lies inside the circle $|z| = 3$.

$\therefore\ f(z) = \sin \pi z^2 + \cos \pi z^2$, $f'(z) = \cos \pi z^2 \cdot 2\pi z - \sin \pi z^2 (2\pi z)$

$= 2\pi z\left(\cos \pi z^2 - \sin \pi z^2\right)$

Consider.

$$\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2} \qquad \Rightarrow f'(1) = -2\pi$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

Put $z = 1$, Put $z = 2$, Put $z = 0$

$1 = -B$, $\boxed{1 = C}$, $1 = 2A - 2B + C$

$\boxed{B = -1}$

$2A = 1 + 2B - C$

$= 1 - 2 - 1$

$2A = -2$

$\boxed{A = -1}$

$$\therefore\ \frac{1}{(z-1)^2(z-2)} = \frac{-1}{z-1} + \frac{-1}{(z-1)^2} + \frac{1}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\,dz = \int_C f(z)\left\{\frac{-1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{z-2}\right\}$$

$$= -\int_C \frac{f(z)\,dz}{z-1} - \int_C \frac{f(z)\,dz}{(z-1)^2} + \int_C \frac{f(z)\,dz}{z-2}$$

$$= -2\pi i\, f(1) - \frac{2\pi i}{1!} f'(1) + 2\pi i\, f(2)$$

$$= -2\pi i(-1) - 2\pi i(-2\pi) + 2\pi i$$

$$= 2\pi i + 4\pi^2 i + 2\pi i$$

$$= 4\pi i + 4\pi^2 i$$

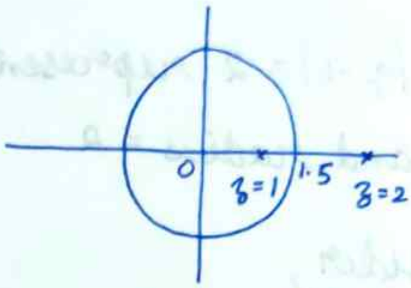
$$= 4\pi i(1+\pi)$$

(ii) $|z| = \frac{3}{2}$ represents a circle with centre at origin and radius $= \frac{3}{2}$

Consider,

$$(z-1)^2(z-2) = 0$$

$$\boxed{z = 1,\ z = 2}$$



$z=1$ lies inside the circle, $z=2$ lies outside the circle.

$$\therefore f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}$$

$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\,dz = \int_C \frac{f(z)}{(z-1)^2}\,dz$$

$$= \frac{2\pi i}{1!} f'(1)$$

$$f'(z) = \frac{(z-2)\left(\cos\pi z^2 \cdot 2\pi z + (-\sin\pi z^2).2\pi z\right) - (\sin\pi z^2 + \cos\pi z^2)}{(z-2)^2}$$

$$f'(z) = \frac{(z-2)(2\pi z)(\cos\pi z^2 - \sin\pi z^2) - (\sin\pi z^2 + \cos\pi z^2)}{(z-2)^2}$$

$$f'(1) = \frac{(-1)\,2\pi(-1) - (-1)}{1} = +2\pi - 1$$

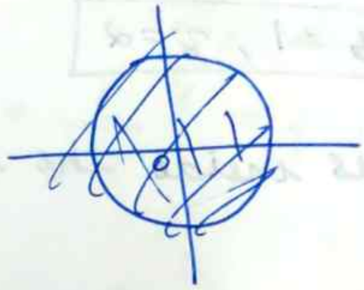
$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz = 2\pi i\,(2\pi - 1)$$
$$= \underline{\underline{4\pi^2 i - 2\pi i}}$$

9. Evaluate $\displaystyle\int_C \frac{z-1}{(z+1)^2 (z-2)} dz$ where C is the curve,

$|z-i| = 2$

<u>Sol</u>:- $|z-i| = 2$ represents a circle with centre at $(0,1)$ $[z=i]$ and radius $= 2$.

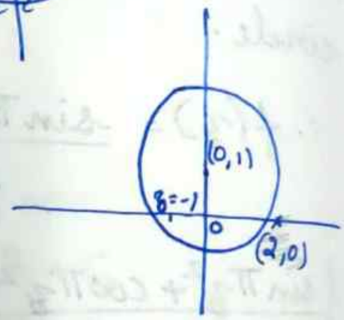
Consider,



$(z+1)^2 (z-2) = 0$

$\boxed{z = -1,\ z = 2}$

$z = -1$ lies inside the circle and $z = 2$ lies outside



$C = (0,1)$ C - Centre

$P_1 = (-1,0)$

$P_2 = (2,0)$

$CP_1 = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} < r$

$CP_2 = \sqrt{2^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5} > r$

$\therefore$ $z = -1$ lies inside the circle and $z = 2$ lies outside the circle $|z-i| = 2$.

$$\therefore f(z) = \frac{z-1}{z-2}$$

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$$\therefore \int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = \int_C \frac{f(z)}{(z+1)^2}\,dz$$

$$= \frac{2\pi i}{1!} f'(-1)$$

$$f'(z) = \frac{(z-2)-(z-1)}{(z-2)^2} = \frac{-1}{(z-2)^2}$$

$$f'(-1) = \frac{-1}{+9} = \frac{-1}{9}$$

$$\therefore \int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = -\frac{2\pi i}{9}$$

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