Probability Theory - 2

Random variables - Introduction.

In a random experiment, if a real variable is associated with every outcome then it is called a random variable (or) stochastic variable.

Random variables are usually denoted by

$X, Y, Z, \ldots$

The set of all real numbers of a random variable X is called the range of X.

Example:- 1) while tossing a coin, suppose that the value '1' is associated for the outcome 'head' and '0' for the outcome 'tail'. when we've the sample space

$S = \{H, T\}$

and if X is the random variable

then $X(H) = 1$ and $X(T) = 0$

$\therefore$ Range of $X = \{0, 1\}$

Example 2) Suppose a coin is tossed twice, we shall associate two different random variab-les x, y as follows.

where we've the sample space,

$S = \{HH, HT, TH, TT\}$

X = Number of 'heads' in the outcome.

The association of the elements in S to X is as follows.

out come	HH	HT	TH	TT
Random variable X	2	1	1	0

Range of $X = \{0, 1, 2\}$

Discrete and continuous random variables

Defn:- If a random variable takes finite or countably infinite number of values then it is called a discrete random variable.

Here countably infinite means a sequence of real numbers. It is evident that a discrete random variable will have finite (or) countably infinite range.

Ex:- 1) Tossing a coin and observing the outcome.

2) Tossing coins and observing the no of heads turning up.

3) Throwing a 'die' and observing the no on the face.

Defⁿ:- If a random variable takes non countable infinite number of values then it is called a non-discrete (or) continuous random variable. Equivalently we can say that, if the range of random variable X is an interval of real no's then X is a continuous random variable. A continuous random variable can assume any value in the interval of real no's.

Example 1) weight of articles.

2) Length of nails produced by a machine.

3) observing the pointer on a speedometer / voltmeter.

4) Conducting a survey on the life of electric bulbs.

Discrete probability distribution - Definitions.

if for each value x_i of a discrete random variable X, we assign a real number $p(x_i)$ such that

1) $p(x_i) \geq 0$ 2) $\sum_i p(x_i) = 1$

then the function $p(x)$ is called a probability function.

If the probability that x take the values x_i is p_i, then

$$p(X = x_i) = p_i \text{ (or) } p(x_i)$$

The set of values $[x_i, p(x_i)]$ is called a discrete (finite) probability distribution of the discrete random variable X. The function $p(x)$ is called the probability density function (p.d.f) (or) the probability mass function (p.m.f)

The distribution function $f(x)$ defined by

$$f(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i),$$

x being an integer is called the commulative distribution function (c.d.f)

The mean and variance of the discrete probability distribution is defined as follows.

1) Mean $(\mu) = \sum_i x_i \cdot p(x_i)$

2) Variance $(V) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

3) Standard deviation $(\sigma) = \sqrt{V}$

Note :- Variance can also be put in the form

$$V = \sum_i x_i^2 p(x_i) - \left[\sum_i x_i p(x_i)\right]^2$$

worked problems.

1) A coin is tossed twice. A random variable X represent the no of heads turning up. Find the discrete probability distribution for X. Also find its mean and variance.

>> $S = \{HH, HT, TH, TT\}$

The association of the elements of S to the random variable X are respectively 2, 1, 1, 0

ie $X = \{0, 1, 2\}$

Now $p(HH) = \frac{1}{4}$, $p(HT) = \frac{1}{4}$, $p(TH) = \frac{1}{4}$

$p(TT) = \frac{1}{4}$

$p(X=0$, ie no head$) = p(TT) = \frac{1}{4}$

$p(X=1$, ie one head$) = p(HT \cup TH) = p(HT) + p(TH)$

$= \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

$p(X=2$, ie 2 heads$) = p(HH) = \frac{1}{4}$

The discrete probability distribution for X is as follows.

$X = x_i$	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

we observe that $p(x_i) > 0$ and $\sum p(x_i) = 1$

Now Mean $= \mu = \sum_i x_i \cdot p(x_i)$

$$\mu = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}$$

$$= 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$\boxed{M = 1}$$

Variance $V = \sum_i (x_i - M)^2 \cdot p(x_i)$

$$V = (x_1 - M)^2 \cdot p(x_1) + (x_2 - M)^2 \cdot p(x_2) + (x_3 - M)^2 \cdot p(x_3)$$

$$= (0-1)^2 \frac{1}{4} + (1-1)^2 \cdot \frac{1}{2} + (2-1)^2 \cdot \frac{1}{4}$$

$$= \frac{1}{4} + 0 + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Thus Mean $= 1$ and Variance $= 1/2$.

2) S.T the foll^ng distribution represents a a discrete probability distribution. Find the mean and variance.

x	10	20	30	40
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

>> we observe that $p(x) > 0$ for all x
and $\sum p(x) = 1$

Mean $(\mu) = \sum_i x_i \cdot p(x_i)$

$$\mu = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3) + x_4 \cdot p(x_4)$$

$$= 10 \cdot \frac{1}{8} + 20 \cdot \frac{3}{8} + 30 \cdot \frac{3}{8} + 40 \cdot \frac{1}{8}$$

$$= \frac{10}{8} + \frac{60}{8} + \frac{90}{8} + \frac{40}{8}$$

$$= \frac{10 + 60 + 90 + 40}{8} = \frac{200}{8} = 25$$

Variance $(V) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = (x_1 - \mu)^2 \cdot p(x_1) + (x_2 - \mu)^2 \cdot p(x_2) + (x_3 - \mu)^2 \cdot p(x_3) + (x_4 - \mu)^2 \cdot p(x_4)$$

$$= (10 - 25)^2 \cdot \frac{1}{8} + (20 - 25)^2 \cdot \frac{3}{8} + (30 - 25)^2 \cdot \frac{3}{8} + (40 - 25)^2 \cdot \frac{1}{8}$$

$$= (-15)^2 \cdot \frac{1}{8} + (-5)^2 \cdot \frac{3}{8} + (5)^2 \cdot \frac{3}{8} + (15)^2 \cdot \frac{1}{8}$$

$$= \frac{225}{8} + \frac{25 \times 3}{8} + \frac{25 \times 3}{8} + \frac{225}{8}$$

$$= \frac{225}{8} + \frac{75}{8} + \frac{75}{8} + \frac{225}{8}$$

$$= \frac{600}{8} = 75$$

Thus $\boxed{\text{Mean} = 25}$ $\boxed{\text{Variance} = 75}$

Standard deviation $\sigma = \sqrt{V} = \sqrt{75} = 8.6603$.

3) Find the value of K such that the foll^ng distribution represents a finite probability distri-bution. Hence find the mean and standard deviation. Also find $P(x \le 1)$, $P(x > 1)$ and $P(-1 < x \le 2)$

x	-3	-2	-1	0	1	2	3
$p(x)$	K	2K	3K	4K	3K	2K	K

>> we must we. $p(x) \geq 0$ for all x and $\sum p(x) = 1$

The first cond^n is satisfied if

The first cond^n requires that,

if $K \geq 0$ and the 2nd cond^n requires that,

$K + 2K + 3K + 4K + 3K + 2K + K = 1$

$16K = 1 \Rightarrow \boxed{K = \frac{1}{16}}$

The discrete finite probability distribution is as follows.

x	x_1: -3	x_2: -2	x_3: -1	x_4: 0	x_5: 1	x_6: 2	x_7: 3
$p(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Mean $(\mu) = \sum_i x_i \cdot p(x_i)$

$\mu = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + \ldots\ldots + x_7 \cdot p(x_7)$

$= -3 \cdot \frac{1}{16} + (-2) \cdot \frac{2}{16} + (-1) \cdot \frac{3}{16} + 0 \cdot \frac{4}{16} + 1 \cdot \frac{3}{16} + 2 \cdot \frac{2}{16} + 3 \cdot \frac{1}{16}$

$= -\frac{3}{16} - \frac{4}{16} - \frac{3}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$

$= \frac{-3-4-3+3+4+3}{16} = \underline{\underline{0}}$

$$\text{Variance } (V) = \sum_i (x_i - m)^2 \cdot p(x_i)$$

$$V = (x_1 - m)^2 \cdot p(x_1) + (x_2 - m)^2 \cdot p(x_2) + \cdots + (x_7 - m)^2 \cdot p(x_7)$$

$$= (-3-0)^2 \cdot \frac{1}{16} + (-2-0)^2 \cdot \frac{2}{16} + (-1-0)^2 \cdot \frac{3}{16} + (0-0)^2 \cdot \frac{4}{16}$$

$$(1-0)^2 \cdot \frac{3}{16} + (2-0)^2 \cdot \frac{2}{16} + (3-0)^2 \cdot \frac{1}{16}$$

$$= \frac{9}{16} + \frac{8}{16} + \frac{3}{16} + \frac{3}{16} + \frac{8}{16} + \frac{9}{16}$$

$$= \frac{9+8+3+3+8+9}{16} = \frac{40}{16} = \underline{\underline{\frac{5}{2}}}$$

Thus $K = \frac{1}{16}$, Mean $= 0$, $S.D = \sqrt{5/2}$

Also $p(x \leq 1) = p(-3) + p(-2) + p(-1) + p(0) + p(1)$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$$

$$= \underline{\underline{\frac{13}{16}}}$$

$$p(x > 1) = p(2) + p(3)$$

$$= \frac{2}{16} + \frac{1}{16}$$

$$= \underline{\frac{3}{16}}$$

$$p(-1 < x \leq 2) = p(0) + p(1) + p(2)$$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16}$$

$$= \underline{\frac{9}{16}}$$

4) The p.d.f. of a variate x is given by the following table.

x	0	1	2	3	4	5	6
$p(x)$	K	3K	5K	7K	9K	11K	13K

For what value of k, this represents a valid probability distribution?
Also find $p(x \geq 5)$ and $p(3 < x \leq 6)$

>> The probability distribution is valid if
$p(x) \geq 0$ and $\Sigma p(x) = 1$

Hence we must have $k \geq 0$ and
$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$49k = 1 \Rightarrow k = \frac{1}{49}$

The discrete probability distribution is as follows

$$\begin{array}{r} 11 \\ 13 \\ \hline 24 \end{array}$$

x	0	1	2	3	4	5	6
$p(x)$	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

Also, $p(x \geq 5) = p(5) + p(6) = \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$

$p(3 < x \leq 6) = p(4) + p(5) + p(6)$

$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{33}{49}$

5) The probability distribution of a finite random variable X is given by the following table

x_i	-2	-1	0	1	2	3
$p(x_i)$	0.1	K	0.2	2K	0.3	K

Find the value of k, mean and variance.

>> we must have $p(x_i) \geq 0$ and $\Sigma p(x_i) = 1$ for a probability distribution.

$\Sigma p(x_i) = 1$ required $4k + 0.6 = 1$

$4k = 1 - 0.6$

$4k = 0.4$

$k = \frac{0.4}{4} = \underline{0.1}$

Prob distribution

x_i	-2	-1	0	1	2	3
$p(x_i)$	0.1	0.1	0.2	0.2	0.3	0.1

$\text{Mean } (M) = \sum_i x_i \cdot p(x_i)$

$M = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3) + x_4 \cdot p(x_4) + x_5 \cdot p(x_5) + x_6 \cdot p(x_6)$

$= -2 \cdot (0.1) + (-1)(0.1) + 0 \cdot (0.2) + 1 \cdot (0.2) + 2(0.3) + 3(0.1)$

$= \underline{0.8}$

$\text{Variance } (V) = \sum_i (x_i - M)^2 \cdot p(x_i)$

$V = (-2 - 0.8)^2 \cdot (0.1) + (-1 - 0.8)^2 (0.1) + (0 - 0.8)^2 0.2 + (3 - 0.8)^2 0.1$

$+ (1 - 0.8)^2 0.2 + (2 - 0.8)^2 0.3$

$= 2.16$

Thus $\underline{\text{Mean} = 0.8}$ and $\underline{\text{Variance} = 2.16}$

6) A random variable X has the foll^ng Probability function for various values of x

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Find k ii) Evaluate $p(x<6)$, $p(x\geq 6)$, $p(3<x\leq 6)$ Also find the probability distribution function of X.

>> we must have $p(x)\geq 0$ and $\Sigma p(x)=1$

The first cond^n is satisfied for $k\geq 0$ and we have to find k such that $\Sigma p(x)=1$

ie. $0+k+2k+2k+3k+k^2+2k^2+(7k^2+k)=1$

ie. $10k^2+9k-1=0$

(or) $(10k-1)(k+1)=0$

(or) $k=1/10$ and $k=-1$

if $k=-1$ the first condition fails and hence

$k\neq -1 \therefore k=\frac{1}{10}$

Hence we have the following table

x	0	1	2	3	4	5	6	7
$p(x)$	0	$1/10$	$1/5$	$1/5$	$3/10$	$1/100$	$1/50$	$17/100$

Now $p(x<6) = p(0)+p(1)+p(2)+p(3)+p(4)+p(5)$

$= 0+\frac{1}{10}+\frac{1}{5}+\frac{1}{5}+\frac{3}{10}+\frac{1}{100}$

$= 0.81$

$$p(x \geq 6) = p(6) + p(7)$$
$$= \frac{1}{50} + \frac{17}{100} = \frac{19}{100} = 0.19$$

$$p(3 < x \leq 6) = p(4) + p(5) + p(6)$$
$$= 3/10 + 1/100 + 1/50$$
$$= 33/100 = \underline{\underline{0.33}}$$

The probability distribution is as follows.

x	0	1	2	3	4	5	6	7
$p(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

The distribution function of X is

$f(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$ is also called cumulative distribution fuⁿ and the same is as follows.

x	0	1	2	3	4	5
$f(x)$	0	0+0.1 = 0.1	0.1+0.2 = 0.3	0.3+0.2 = 0.5	0.5+0.3 = 0.8	0.8+0.01 = 0.81

6	7
0.81 + 0.02 = 0.83	0.83 + 0.17 = 1

7) A random variable X take the values $-3, -2, -1, 0, 1, 2, 3$ such that $p(x=0) = p(x<0)$ and $p(x=-3) = p(x=-2) = p(x=-1) = p(x=1) = p(x=2) = p(x=3)$. Find the probability distribution.

>> Let the distribution $[x, p(x)]$ be as follows

x	-3	-2	-1	0	1	2	3
p(x)	P_1	P_2	P_3	P_4	P_5	P_6	P_7

By data, $p(x=0) = p(x<0)$

$\Rightarrow p(x=0) = p(x=-1) + p(x=-2) + p(x=-3)$ ——— ①

ie $P_4 = P_3 + P_2 + P_1$

Also by data

$P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = P_7$ ——— ②

Further we must have, ——— ③

$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = 1$

using ② in ① we get $3P_1 = P_4$

using ② in ③ $6P_1 + P_4 = 1$

$\Rightarrow 6P_1 + 3P_1 = 1 \quad \because P_4 = 3P_1$

$9P_1 = 1 \Rightarrow \boxed{P_1 = 1/9}$ and $3P_1 = P_4 \Rightarrow P_4 = \frac{3}{9} = \frac{1}{3}$ $\boxed{P_4 = \frac{1}{3}}$

Thus the discrete / finite probability distribution of

x	-3	-2	-1	0	1	2	3
p(x)	1/9	1/9	1/9	1/3	1/9	1/9	1/9

8) From a sealed box containing a dozen (12) apples it was found that 3 apples are perished. obtain the probability distribution of the no of perished apples when 2 apples are drawn at random. Also find the mean and variance?

>> Let x be the number of perished apples. Since 2 were drawn we have $x = 0, 1, 2$, then 2 out 12 can be selected in ${}^{12}C_2$ ways. remaining 9 apple are good apples. and 3 are perished apples, Hence we have

$p(x=0)$ = probability of getting 0 perished apples $= \dfrac{{}^3C_0 \cdot {}^9C_2}{{}^{12}C_2} = \dfrac{6}{11}$

$p(x=1)$ = probability of getting 1 perished apples $= \dfrac{{}^3C_1 \cdot {}^9C_1}{{}^{12}C_2} = \dfrac{9}{22}$

$p(x=2)$ = probability of getting 2 perished apples $= \dfrac{{}^3C_2 \cdot {}^9C_0}{{}^{12}C_2} = \dfrac{1}{22}$

Thus the discrete / finite probability distribution is

$X = x_i$	0	1	2
$p(x) = p_i$	$6/11$	$9/22$	$1/22$

Mean $(\mu) = \sum_i x_i \cdot p(x_i)$

$\mu = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3)$

$= 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} + 2 \cdot \frac{1}{22}$

$= \frac{9}{22} + \frac{2}{22} = \frac{11}{22} = \frac{1}{2} = 0.5$

Variance $(v) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$v = \left(0 - \frac{1}{2}\right)^2 \cdot \frac{6}{11} + \left(1 - \frac{1}{2}\right)^2 \cdot \frac{9}{22} + \left(2 - \frac{1}{2}\right)^2 \cdot \frac{1}{22}$

$= \left(-\frac{1}{2}\right)^2 \cdot \frac{6}{11} + \left(\frac{1}{2}\right)^2 \frac{9}{22} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{22}$

$= \frac{1}{4} \cdot \frac{6}{11} + \frac{1}{4} \cdot \frac{9}{22} + \frac{9}{4} \cdot \frac{1}{22}$

$= \frac{6}{44} + \frac{9}{88} + \frac{9}{88}$

$= \frac{12+9+9}{88} = \frac{\cancel{30}^{15}}{\cancel{88}_{44}} = \frac{15}{44}$

~~thu~~ $S.D = (\sigma) = \sqrt{v} = \sqrt{15/44}$

Thus Mean $= \frac{1}{2}$

Variance $= \frac{15}{44}$

$S.D = \sqrt{15/44}$

9) If the random variable X take the values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, find the probability distribution fu^n of X.

>> Let the distribution $[X, P(X)]$ be as follows.

X	1	2	3	4
$P(X)$	P_1	P_2	P_3	P_4

By data

$$2P_1 = 3P_2 = P_3 = 5P_4 \quad \text{------} (1)$$

also we have $P_1 + P_2 + P_3 + P_4 = 1$ --- (2)

from (1) $2P_1 = 3P_2$; $2P_1 = P_3$, $2P_1 = 5P_4$

$\Rightarrow \boxed{P_2 = 2/3\, P_1}$, $\boxed{P_3 = 2P_1}$, $\boxed{P_4 = \frac{2}{5}P_1}$

Hence (2) $\Rightarrow$

$$P_1 + \frac{2}{3}P_1 + 2P_1 + \frac{2}{5}P_1 = 1$$

$$3P_1 + \frac{2}{3}P_1 + \frac{2}{5}P_1 = 1$$

$$\frac{(3\times15)P_1 + 2P_1\times5 + (2\times3)P_1}{15} = 1$$

$$\frac{45P_1 + 10P_1 + 6P_1}{15} = 1$$

$$\frac{61P_1}{15} = 1$$

$$\Rightarrow \boxed{P_1 = \frac{15}{61}}$$

$P_2 = \frac{2}{3}\cdot P_1$

$P_2 = \frac{2}{\cancel{3}}\cdot\frac{\cancel{15}^{5}}{61} = \frac{10}{61}$

$\boxed{P_2 = \frac{10}{61}}$

$P_3 = 2\cdot P_1 = \frac{2.15}{61} = \frac{30}{61}$

$P_4 = \frac{2}{5}P_1 = \frac{2}{\cancel{5}}\cdot\frac{\cancel{15}^{3}}{61} = \frac{6}{61}$

$\boxed{P_3 = \frac{30}{61}}$

$\boxed{P_4 = \frac{6}{61}}$

The probability distribution $p(x)$ and the cumulative distribution $f(x) = \sum_{i=1}^{x} p(x_i)$ is as follows,

$X = x_i$	1	2	3	4
$p(x)$	15/61	10/61	30/61	6/61
$f(x)$	15/61	25/61	55/61	$61/61 = 1$

10) A random variable X has $p(x) = 2^{-x}$ where $x = 1, 2, 3, \ldots$ Show that $p(x)$ is a probability function. Also find $p(X \text{ even})$, $p(X \text{ being divisible by } 3)$ and $p(X \geq 5)$

>> $p(x) = 2^{-x} = \frac{1}{2^x}$ Evidently $p(x) > 0$ for all x.

$$\sum_i p(x_i) = \sum_{x=1}^{\infty} \frac{1}{2^x} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$$

RHS is a geometric series of the form $a + ar + ar^2 + \ldots$ whose sum to infinity is $\frac{a}{1-r}$

where we have $a = \frac{1}{2} = r$

$$\therefore \sum_i p(x_i) = \frac{a}{1-r}$$

$$= \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

Hence $p(x) = 2^{-x}$ is a probability function

Case i) $p(x \text{ even}) = \sum_{x=2,4,6\ldots}^{\infty} p(x)$

$$= \sum_{2,4,6}^{\infty} \frac{1}{2^x}$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$$

$$= \frac{1/2^2}{1-1/2^2} = \frac{1/4}{\frac{4-1}{4}} = \frac{1/4}{3/4} = \frac{1}{3}$$

Thus $\boxed{p(x \text{ even}) = \frac{1}{3}}$

Case ii) $p(x \text{ divisible by } 3) = \sum_{x=3,6,9\ldots}^{\infty} p(x)$

$$= \sum_{3,6,9\ldots}^{\infty} \frac{1}{2^x}$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots$$

$$= \frac{1/2^3}{1-1/2^3} = \frac{1/8}{\frac{8-1}{8}} = \frac{\frac{1}{8}}{7/8} = \frac{1}{7}$$

Thus $\boxed{p(x \text{ divisible by } 3) = 1/7}$

Case iii) $p(x \geq 5) = 1 - p(x < 5)$

$$= 1 - \sum_{i=1}^{4} p(x_i)$$

$$= 1 - \sum_{i=1}^{4} \frac{1}{2^x}$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right]$$

$$= 1 - \frac{15}{16} = \frac{1}{16}$$

Thus $\boxed{p(x \geq 5) = \frac{1}{16}}$

11) if X is a discrete random variable taking values 1, 2, 3, ---- with $p(x) = \frac{1}{2}\left(\frac{2}{3}\right)^x$, find p(X being an odd number) by first establishing that p(x) is a probability function.

>> $p(x) = \frac{1}{2}\left(\frac{2}{3}\right)^x$

$$\sum_i p(x_i) = \sum_{x=1}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^x$$

$$= \frac{1}{2}\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \text{----}\right]$$

$$= \frac{1}{2} \cdot \frac{2/3}{1-2/3} = \frac{1}{\cancel{2}} \cdot \frac{\frac{\cancel{2}}{\cancel{3}}}{1/\cancel{3}} = 1$$

$\therefore$ p(x) is a probability function.

Next $p(X \text{ being an odd no.}) = \sum_{x=1,3,5-}^{\infty} p(x)$

$$= \sum_{1,3,5-\,-}^{\infty} \frac{1}{2} \cdot \left(\frac{2}{3}\right)^x$$

$$= \frac{1}{2}\left[\frac{2}{3} + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^5 + \text{---}\right]$$

$$= \frac{1}{2}\left[a + ar + ar^2 + \text{--}\right] \quad a = \frac{2}{3},\ r = \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} \cdot \frac{a}{1-r} = \frac{1}{2} \cdot \frac{2/3}{1-4/9} = \frac{1}{2} \cdot \frac{2/3}{\frac{5}{9}}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{9}^{3}}{5} = \frac{3}{5}$$

Thus $p(X \text{ being an odd no.}) = \frac{3}{5}$

12) X is a discrete random variable having $p(x)$ defined as follows. ⑪

$$p(x) = \begin{cases} x/15 & \text{if } 1 \le x \le 5 \\ 0 & \text{if } x > 5 \end{cases}$$

show that $p(x)$ is a probability function.

Find i) $P(X = 1 \text{ or } 2)$

ii) $P\left(\left(\frac{1}{2} < X < \frac{5}{2}\right) / X > 1\right)$

>> The probability distribution is as follows

x	1	2	3	4	5	6, 7, ...
$p(x)$	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$	0

we've $p(x) \geq 0$ and $\sum_i p(x_i) = 1$

$\therefore \frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} \Rightarrow$

$$\frac{1+2+3+4+5}{15} = \frac{15}{15} = 1$$

$\therefore p(x)$ is a probability function.

Now $P(X = 1 \text{ or } 2) = P(X=1) + P(X=2)$

$$= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$$

Thus $\boxed{P(X = 1 \text{ or } 2) = \frac{1}{5}}$

Bernoulli's Theorem

Stmt:- The probability of x successes in n trials is equal to $nc_x p^x q^{n-x}$ where p is the probability of success and q is the probability of failure.

Proof:- Since p is the probability of success, the probability of x successes is

$p \cdot p \cdot p \cdots x$ times $= p^x$

Also x successes imply $(n-x)$ failures and since q is the probability of failure, the probability of $(n-x)$ failures is q^{n-x}.

By the multiplication rule the probability of the simultaneous happening is $p^x q^{n-x}$. But x successes in n trials can occur in nc_x ways and all these cases are favourable to the event.

Hence by the addition rule, the probability of x successes out of n trials is given by

$p^x q^{n-x} + p^x q^{n-x} + \cdots nc_x$ times $= nc_x p^x q^{n-x}$

This proves Bernoulli's theorem.

Binomial Distribution.

If p is the probability of success and q is the Probability of failure, the probability of x successes out of n trials is given by

$$p(x) = {}^nc_x \, p^x q^{n-x}$$

we form the following probability distribution of $[x, p(x)]$ where $x = 0, 1, 2, \ldots n$

x	0	1	2	---	n
$p(x)$	q^n	${}^nc_1 \, q^{n-1} p$	${}^nc_2 \, q^{n-2} p^2$	--	p^n

It may be observed that the value of $p(x)$ for different values $x = 0, 1, 2, \ldots n$ are the successive terms in the binomial expansion of $(q+p)^n$ and accordingly this distribution is called the Binomial Distribution (or)

Bernoulli Distribution.

$$\Sigma p(x) = q^n + {}^nc_1 \, q^{n-1} p + {}^nc_2 \, q^{n-2} p^2 + \text{---} + p^n = (p+q)^n = 1$$

Hence $p(x)$ is a probability function

Mean and Standard Deviation of the Binomial Distribution.

$$\text{Mean } (\mu) = \sum_{x=0}^{n} x\, p(x) \qquad \because {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$x! = x(x-1)!$$

$$\mu = \sum_{x=0}^{n} x \cdot {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{\cancel{x} \cdot n(n-1)!}{\cancel{x}(x-1)!\,(n-x)!} p \cdot p^{x-1} \cdot q^{n-x}$$

$$= np \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!} p^{x-1} q^{[(n-1)-(x-1)]}$$

$$= np \sum_{x=1}^{n} {}^{(n-1)}C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(q+p)^{n-1}$$

$$= np\,(1)^n$$

$$= np$$

Thus $\boxed{\text{Mean } (\mu) = np}$

$$\text{Variance } (V) = \sum_{x=0}^{n} x^2 p(x) - \mu^2 \quad \text{——— (1)}$$

Now $\displaystyle\sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} [x(x+1)+x]\, p(x)$

$$= \sum_{x=0}^{n} x(x-1)\cdot p(x) + \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\cdot {}^{n}C_{x}\, p^{x} q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1)\cdot \frac{n!}{x!\,(n-x)!}\, p^{x} q^{n-x} + np$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)}\cdot \frac{n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!\,(n-x)!}\, p^{2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=0}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!}\, p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!}\, p^{x-2} q^{[(n-2)-(x-2)]} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{(n-2)}C_{(x-2)}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

$$(1) \Rightarrow V = n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p)$$

$$= np(q)$$

$\because p+q=1$

$p = 1-q$

(or)

$q = 1-p$

$$\boxed{(V) = npq}$$

$$\boxed{S.D(\sigma) = \sqrt{V} = \sqrt{npq}}$$

Thus we've the Binomial distribution Mean$(\mu) = np$ and $S.D(\sigma) = \sqrt{npq}$

Poisson Distribution.

Poisson distribution is regarded as the limiting form of the binomial distribution when n is very large $(n \to \infty)$ and p the probability of success is very small $(p \to 0)$ so that np tends to a fixed finite constant say m.

we have in the case of binomial distribution, the Probability of x successes out of n trials.

$$P(x) = {}^nC_x \, p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2) \cdots (n-(x-1))}{x!} p^x q^{n-x}$$

$$= \frac{n \cdot n\left(1-\frac{1}{n}\right) \cdot n\left(1-\frac{2}{n}\right) \cdots n\left(1-\frac{(x-1)}{n}\right)}{x!} p^x q^{n-x}$$

$$= \frac{n^x \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{(x-1)}{n}\right)}{x!} \cdot p^x q^{n-x}$$

$$= \frac{(np)^x \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{(x-1)}{n}\right) q^n}{x! \, q^x}$$

But $(np)^x = m$

$$q^n = (1-p)^n = \left(1-\frac{m}{n}\right)^n = \left\{\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}\right\}^{-m}$$

Denoting $-\frac{m}{n} = k$ we have

$q^n = \{(1+k)^{1/k}\}^{-m} \rightarrow e^{-m}$ as $n \rightarrow \infty$ (or) $k \rightarrow 0$

[Note : $\lim_{k \rightarrow 0} (1+k)^{1/k} = e$]

Further $q^x = (1-p)^x \rightarrow 1$ for a fixed x as $p \rightarrow 0$

Also the factors $\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right) --- \left(1-\frac{x-1}{n}\right)$

will all tend to 1 as $n \rightarrow \infty$

Thus we get

$$p(x) = \frac{m^x e^{-m}}{x!}$$

This is known as the poisson distribution of the random variable.

$p(x)$ is also called Poisson probability function and x is called a poisson variate.

The distribution of Probabilities for $x = 0, 1, 2, 3, ---$ is as follows

x	0	1	2	3	---
$p(x)$	e^{-m}	$\frac{m e^{-m}}{1!}$	$\frac{m^2 e^{-m}}{2!}$	$\frac{m^3 e^{-m}}{3!}$	

we have $p(x) \geq 0$ and

$$\sum_{x=0}^{\infty} p(x) = e^{-m} + \frac{m e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} + ---$$

$$= e^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + ---\right\}$$

$$= e^{-m} \cdot e^{m} = e^{-m+m} = e^0 = e^0 = 1$$

$\therefore \sum p(x) = 1$

Hence $p(x)$ is a probability function

Mean and Standard Deviation of the Poisson Distribution

$$\text{Mean } (M) = \sum_{x=0}^{\infty} x \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{m^x e^{-m}}{x!}$$

$$= \sum_{x=0}^{\infty} \cancel{x} \cdot \frac{m \cdot m^{x-1}}{\cancel{x}(x-1)!} e^{-m}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

put $x = 1, 2, 3 \ldots$
expanding summation

$$= m e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right]$$

$$= m e^{-m} \cdot e^{m} = m e^{0} = m$$

$$\therefore \boxed{\text{Mean } (M) = m}$$

$$\text{Variance } (V) = \sum_{x=0}^{\infty} x^2 p(x) - M^2 \quad \text{———} (1)$$

$$\text{Now } \sum x^2 p(x) = \sum [x(x-1) + x] \cdot p(x)$$

$$= \sum [x(x-1) + x] \cdot \frac{m^x e^{-m}}{x!}$$

$$= \sum x(x-1) \cdot \frac{m^x e^{-m}}{x!} + \sum x \cdot \frac{m^x e^{-m}}{x!}$$

$$= \sum \cancel{x(x-1)} \cdot \frac{m^x e^{-m}}{\cancel{x(x-1)}(x-2)!} + \sum x \cdot \frac{m^x e^{-m}}{x(x-1)!}$$

$$= \sum_{0}^{\infty} \frac{m^x e^{-m}}{(x-2)!} + \sum_{0}^{\infty} \frac{m^x e^{-m}}{(x-1)!}$$

$$= \sum_{2}^{\infty} \frac{m^2 m^{x-2} e^{-m}}{(x-2)!} + \sum_{1}^{\infty} \frac{m\, m^{x-1} e^{-m}}{(x-1)!}$$

$$= m^2 e^{-m} \sum_{2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$\sum x^2 p(x) = m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$= m^2 e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots \right] + m$$

$$= m^2 e^{-m} . e^{m} + m$$

$$= m^2 + m$$

(1) $\Rightarrow$

Variance $= m^2 + m - (m)^2$

$$\boxed{V = m}$$

$\therefore$ $S.D (\sigma) = \sqrt{V} = \sqrt{m}$

Thus we have for the poisson distribution

$\boxed{\text{Mean } (\mu) = m}$ and

$\boxed{S.D (\sigma) = \sqrt{m}}$

further we can say that the mean and variance are equal for the Poisson distribution.

Problems

1) Find the binomial probability distribution which has mean 2 and variance 4/3.

>> WKT the binomial distribution

mean = np and Variance = npq

Hence $np = 2$ and $npq = \frac{4}{3}$

$\Rightarrow npq = \frac{4}{3} \Rightarrow 2q = \frac{4}{3} \Rightarrow q = \frac{\cancel{4}^{2}}{3 \cdot \cancel{2}} = \frac{2}{3}$

$\boxed{q = \frac{2}{3}}$ WKT $p+q=1$

$p = 1-q$

$p = 1 - \frac{2}{3} = \frac{3-2}{3} = \frac{1}{3}$

$\boxed{p = \frac{1}{3}}$

and $np = 2$

$n(\frac{1}{3}) = 2$

$\boxed{n = 6}$

The binomial probability fu^n $p(x) = {}^{n}c_{x}\, p^{x} q^{n-x}$ becomes $p(x) = {}^{6}c_{x} (1/3)^{x} (\frac{2}{3})^{6-x}$

The distribution of Probabilities is as follows. ($x \rightarrow$ starts with 0 . . . . to ends with ∞)

B.D/P.D $x = 0, 1, 2 \dots \infty$

x	0	1	2
$p(x)$	${}^{6}c_{0} (\frac{1}{3})^{0} (\frac{2}{3})^{6}$	${}^{6}c_{1} (\frac{1}{3})^{1} (\frac{2}{3})^{5}$	${}^{6}c_{2} (\frac{1}{3})^{2} (\frac{2}{3})^{4}$

3	4	5	6
${}^{6}c_{3} (\frac{1}{3})^{3} (\frac{2}{3})^{3}$	${}^{6}c_{4} (\frac{1}{3})^{4} (\frac{2}{3})^{2}$	${}^{6}c_{5} (\frac{1}{3})^{5} (\frac{2}{3})$	${}^{6}c_{6} (\frac{1}{3})^{6} (\frac{2}{3})^{0}$

2) when a coin is tossed 4 times, find the probability of getting

i) exactly one head

ii) atmost 3 heads

iii) atleast 2 heads

>> $p = p(H) = \frac{1}{2} = 0.5$, $q = 0.5$, $\underline{n=4}$

Let $x \rightarrow$ be the prob. of heads

wkT the probability of x successes out of n trials is given by $p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

i) prob (exactly one head) is $p(x=1)$

ie $p(1 \text{ head}) = p(x=1)$

$= {}^{4}C_{1} (0.5)^{1} (0.5)^{3}$

$= 4 (0.5)^{4} = \underline{\underline{0.25}}$

Atmost (max) $\rightarrow \leq$

Atleast (min) $\rightarrow \geq$

more than $\rightarrow >$

less than $\rightarrow <$

2 or more $\rightarrow x \geq 2$

1 or more $\rightarrow x \geq 1$

ii) p (at most 3 heads)

ie $p(x \leq 3) = p(x=0) + p(x=1) + p(x=2) + p(x=3)$

$p(x \leq 3) = {}^{4}C_{0} (0.5)^{0} (0.5)^{4} + {}^{4}C_{1} (0.5)^{1} (0.5)^{3}$

$+ {}^{4}C_{2} (0.5)^{2} (0.5)^{2} + {}^{4}C_{3} (0.5)^{3} (0.5)^{1}$

$= {}^{4}C_{0} (0.5)^{4} + {}^{4}C_{1} (0.5)^{4} + {}^{4}C_{2} (0.5)^{4} + {}^{4}C_{3} (0.5)^{4}$

$= (0.5)^{4} \left[{}^{4}C_{0} + {}^{4}C_{1} + {}^{4}C_{2} + {}^{4}C_{3} \right]$

$= (0.5)^{4} [1 + 4 + 6 + 4]$

$= (0.5)^{4} [15]$

$= \cancel{2.625}$ $\underline{\underline{0.9375}}$

iii) p(atleast 2 heads)

$$p(x \geq 2) = p(2) + p(3) + p(4)$$

$$= p(x=2) + p(x=3) + p(x=4)$$

$$= 4C_2 (0.5)^2 (0.5)^2 + 4C_3 (0.5)^3 (0.5)^1 + 4C_4 (0.5)^4 (0.5)^0$$

$$= 4C_2 (0.5)^4 + 4C_3 (0.5)^4 + 4C_4 (0.5)^4 (1)$$

$$= (0.5)^4 [4C_2 + 4C_3 + 4C_4]$$

$$= (0.5)^4 \cdot [6 + 4 + 1]$$

$$= (0.5)^4 [11]$$

$$= 0.6875$$

(OR)

$$p(x \geq 2) = 1 - p(x < 2)$$

$$= 1 - [p(x=0) + p(x=1)]$$

$$= 1 - [4C_0 (0.5)^0 (0.5)^4 + 4C_1 (0.5)^1 (0.5)^3]$$

$$= 1 - [4C_0 (1) \cdot (0.5)^4 + 4C_1 (0.5)^4]$$

$$= 1 - (0.5)^4 [4C_0 + 4C_1]$$

$$= 1 - (0.5)^4 [1 + 4]$$

$$= 1 - (0.5)^4 (5)$$

$$= 1 - 0.3125$$

$$= 0.6875$$

Q3) The Probability that a pen manufactured by a factory be defective is $\frac{1}{10}$. If 12 such pens are manufactured, what is the probability that i) exactly 2 are defective
ii) atleast 2 are defective
iii) none of them are defective.

>> Let $x \to$ be defective.

Probability of a defective pen is $p = \frac{1}{10}$

Probability of a non-defective pen $= q = 1-p$

$$= 1 - \frac{1}{10} = \frac{10-1}{10} = \frac{9}{10} = 0.9$$

we have $p(x) = {}^{n}c_{x}\, p^{x} q^{n-x}$ where $n = 12$

$\boxed{p = 0.1}$ $\boxed{q = 0.9}$ $\boxed{n = 12}$

i) prob. (exactly two defective)

$$p(x=2) = {}^{12}c_2 (0.1)^2 (0.9)^{10} = \underline{0.2301}$$

ii) prob(atleast 2 defective)

$$p(x \geq 2) = p(x=2) + p(x=3) + \text{——} + p(x=12)$$

(or)

$$= 1 - p(x<2)$$

$$= 1 - [p(x=0) + p(x=1)]$$

$$= 1 - [{}^{12}c_0 (0.1)^0 (0.9)^{12} + {}^{12}c_1 (0.1)^1 (0.9)^{11}]$$

$$= 1 - 0.6590$$

$$= \underline{0.341}$$

iii) prob.(no of defective)

$$p(x=0) = {}^{12}c_0 \, (0.1)^0 (0.9)^{12} = \underline{0.2824}$$

4) In a consignment of electric lamps 5% are defective. if a random sample of 8 lamps are inspected what is the probability that one or more lamps are defective?

Let $x \rightarrow$ be the defective lamps

$\gg$ probability of a defective lamp $= p = \frac{5}{100} = 0.05$

$\boxed{p = 0.05}$ and $q = 1-p = 1-0.05 = 0.95$

$\boxed{q = 0.95}$ $\boxed{n=8}$

we have $p(x) = {}^{n}c_x \, p^x q^{n-x}$

prob (1 or more lamps are defective) $+ p(x=8)$

ie $p(x \geq 1) = p(x=1) + p(x=2) + \text{---} - + p(x=8)$

(or)

$$p(x \geq 1) = 1 - p(x<1)$$
$$= 1 - p(x=0)$$
$$= 1 - {}^{8}c_0 \, (0.05)^0 (0.95)^{8-0}$$
$$= 1 - {}^{8}c_0 \, (0.95)^8$$
$$= 1 - 1.(0.95)^8$$
$$= 1 - 0.6634$$
$$= \underline{0.3366}$$

Thus the required probability is $\underline{0.3366}$.

5) The probability that a person aged 60 years will live upto 70 is 0.65. what is the probability that out of 10 persons aged 60 atleast 7 of them will live upto 70.

>> Let $x \rightarrow$ be the no of persons aged 60 years living upto 70 years.

by data $p = 0.65$ $q = 1-p = 1-0.65 = 0.35$

total no of persons $n = 10$

$\therefore \boxed{p = 0.65}$ $\boxed{q = 0.35}$ $\boxed{n = 10}$

consider $p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$$\boxed{p(x) = {}^{10}C_{x}\,(0.65)^{x}\,(0.35)^{10-x}}$$

To find $p(x \geq 7) = p(7) + p(8) + p(9) + p(10)$

$= p(x=7) + p(x=8) + p(x=9) + p(x=10)$

$= {}^{10}C_{7}(0.65)^{7}(0.35)^{3} + {}^{10}C_{8}(0.65)^{8}(0.35)^{2} + {}^{10}C_{9}(0.65)^{9}(0.35)^{1} + {}^{10}C_{10}(0.65)^{10}(0.35)^{0}$

$= {}^{10}C_{7}(0.65)^{7}(0.35)^{3} + {}^{10}C_{8}(0.65)^{8}(0.35)^{2} + {}^{10}C_{9}(0.65)^{9}(0.35) + {}^{10}C_{10}(0.65)^{10}$

$= \underline{0.5138}$ (OR)

$p(x \geq 7) = 1 - p(x < 7)$

$= 1 - [p(x=0) + p(x=1) + \cdots + p(x=6)]$

$= \underline{0.5138}$

6) The no of telephone lines busy at an instant of time is a binomial variate with probability 0.1 that a line is busy. if 10 lines are chosen at random, what is the proba-bility that

i) no line is busy
ii) all lines are busy
iii) atleast one line is busy
iv) atmost 2 lines are busy.

>> Let x denote the no of telephone lines busy

so by the data $\boxed{p=0.1}$ $\quad q=1-p=1-0.1=0.9$

$\boxed{q=0.9}$

$\boxed{n=10}$

we have $p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$$\boxed{p(x) = {}^{10}C_{x}\,(0.1)^{x}(0.9)^{n-x}}$$

i) prob(that no line is busy)

$$p(x=0) = {}^{10}C_{0}\,(0.1)^{0}\,(0.9)^{10-0}$$
$$= 1\,(1)\cdot(0.9)^{10}$$
$$= (0.9)^{10} = \underline{\underline{0.3487}}$$

ii) prob(that all lines are busy)

$$p(x=10) = {}^{10}C_{10}\,(0.1)^{10}(0.9)^{0}$$
$$= 1\,(0.1)^{10}(1)$$
$$= \underline{\underline{(0.1)^{10}}}$$

iii) Prob that atleast one line is busy.

$= 1 -$ prob of no line is busy

$= 1 - p(x=0)$

$= 1 - 0.3487$

$= \underline{\underline{0.6513}}$.

iv) Prob that atmost 2 lines are busy

$$p(x \le 2) = p(x=0) + p(x=1) + p(x=2)$$

$$= (0.9)^{10} + 10_{C_1}(0.1)^1(0.9)^9 + 10_{C_2}(0.1)^2(0.9)^8$$

$$= \underline{\underline{0.9298}}$$

(or)

$$p(x \le 2) = 1 - p(x > 2)$$

$$= 1 - [p[x=3) + p(x=4) + \text{———} + p(x=10)]$$

$$= \underline{\underline{0.9298}}$$

7) In a quiz contest of answering 'yes' (or) 'No' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked? Also find the probability of the same if there are 4 options for a correct answer.

>> Let x denote the correct answer

in the first case,

$p = \frac{1}{2}$, and $q = \frac{1}{2}$ and $\underline{\underline{n = 10}}$.

$$p(x) = {}^{n}C_x \, p^x q^{n-x}$$

$$\boxed{p(x) = {}^{10}C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{10-x}}$$

we have to find $P(x \geq 6)$

$$p(x \geq 6) = p(6) + p(7) + p(8) + p(9) + p(10)$$

$$= {}^{10}C_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^4 + {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^3 + {}^{10}C_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^2 + {}^{10}C_9 \left(\tfrac{1}{2}\right)^9 \left(\tfrac{1}{2}\right) + {}^{10}C_{10} \left(\tfrac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}} \left[{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right]$$

$$= \frac{1}{2^{10}} \left[210 + 120 + 45 + 10 + 1 \right]$$

$$= \frac{386}{1024} = 0.377$$

Thus $p(x \geq 6) = 0.377$

In the second case when there are 4 options

$p = \frac{1}{4}$, $q = 3/4$, $n = 10$

$$\boxed{p(x) = {}^{10}C_x \left(\tfrac{1}{4}\right)^x \left(3/4\right)^{10-x}}$$

Hence $p(x \geq 6) = p(6) + p(7) + p(8) + p(9) + p(10)$

$$= \frac{1}{4^{10}} \left[3^4 \, {}^{10}C_6 + 3^3 \, {}^{10}C_7 + 3^2 \, {}^{10}C_8 + 3 \cdot {}^{10}C_9 + {}^{10}C_{10} \right]$$

$$= \frac{1}{4^{10}} \left[81 \times 210 + 27 \times 120 + 9 \times 45 + 3 \times 10 + 1 \right]$$

$$= 0.019$$

Thus $p(x \geq 6) = 0.019$

8) In Sampling a large number of parts manufactured by a company the mean no of defectives in samples of 20 is 2 out of 1000 such samples how many would be expected to contain atleast 3 defective parts.

$\Rightarrow$ Let $x \to$ defective part

Mean $(\mu) = 2$ and $n = 20$

$np = 2 \Rightarrow 20 \cdot p = 2$

$\Rightarrow p = 2/20 = 1/10$

$p = \frac{1}{10} = 0.1 \quad q = 1 - p = 0.9$

$\therefore \boxed{p = 0.1} \quad \boxed{q = 0.9} \quad \boxed{n = 20}$

$p(x) = {}^nC_x \, p^x q^{n-x}$

$$\boxed{p(x) = {}^{20}C_x (0.1)^x (0.9)^{20-x}}$$

Probability of atleast 3 defective parts

$p(x \geq 3) = p(3) + p(4) + p(5) + \cdots + p(20)$

$= 1 - p(x < 3)$

$= 1 - [p(x=0) + p(x=1) + p(x=2)]$

$= 1 - [{}^{20}C_0 (0.1)^0 (0.9)^{20} + {}^{20}C_1 (0.1)^1 (0.9)^{19} + {}^{20}C_2 (0.1)^2 (0.9)^{18}]$

$= 1 - [(0.9)^{20} + 20 (0.1)(0.9)^{19} + 190 (0.1)^2 (0.9)^{18}]$

$= \underline{0.323}$

Thus the no of defectives in 1000 samples

is $1000 \times 0.323 = \underline{323}$.

https://hemanthrajhemu.github.io

9) If the mean and standard deviation of the no of correctly answered questions in a test given to 4096 students are 2.5 and $\sqrt{1.875}$. Find an estimate of the no of candidates answering correctly (i) 8 or more questions ii) 2 or less iii) 5 questions. (21)

>> we have mean $(\mu) = np$ and $S.D(\sigma) = \sqrt{npq}$ for a binomial distribution.

By data $np = 2.5$ and $\sqrt{npq} = \sqrt{1.875}$

(or)

$npq = 1.875$

Hence we have

$2.5(q) = 1.875$

$q = \frac{1.875}{2.5} = 0.75$

$p = 1 - q = 1 - 0.75 = 0.25$

Since $np = 2.5 \Rightarrow n(0.25) = 2.5$

$n = 2.5/0.25 = 10$

Thus $\boxed{p = 0.25 = \frac{1}{4}}$ $\boxed{q = \frac{3}{4}}$ $\boxed{n = 10}$

Let $x \to$ denote the no of correctly answered questions.

$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x} = {}^{10}C_{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{10-x}$

$p(x) = \frac{1}{4^{10}} \left[{}^{10}C_{x} (3)^{10-x} \right] \longrightarrow$ 1 Student

Since the estimate is needed for 4096 students we have $f(x) = 4096 \times p(x)$

$f(x) = \frac{4096}{4^{10}} \left[{}^{10}C_{x}\, 3^{10-x} \right]$

$$f(x) = \frac{2^{12}}{2^{20}}\left[{}^{10}C_x (3)^{10-x}\right]$$
$$= \frac{1}{2^8}\left[{}^{10}C_x (3)^{10-x}\right]$$
$$f(x) = \frac{1}{256}\left[{}^{10}C_x (3)^{10-x}\right]$$

i) we have to find $f(8)+f(9)+f(10)$
$$= \frac{1}{256}\left[{}^{10}C_8 (3)^{10-8} + {}^{10}C_9 (3)^{10-9} + {}^{10}C_{10} (3)^{10-10}\right]$$
$$= \frac{1}{256}\left[{}^{10}C_8 (3)^2 + {}^{10}C_9 (3)^1 + 1.\right]$$
$$= \frac{1}{256}\left[45\times 9 + 10\times 3+1\right] = \frac{436}{256} = 1.6992 \simeq \underline{\underline{2}}$$

No. of students correctly answering 8 or more questions is $\underline{2}$.

ii) we have to find $f(2)+f(1)+f(0)$
$$= \frac{1}{256}\left[{}^{10}C_2 (3)^8 + {}^{10}C_1 3^9 + 3^{10}\right]$$
$$= \frac{3^8}{256}\left[{}^{10}C_2 + {}^{10}C_1 \cdot 3 + 3^2\right]$$
$$= \frac{3^8}{256}(45+30+9) = 2152.8 \simeq$$
$$= 2152.8281$$
$$= 2152.8 \simeq \underline{\underline{2153}}$$

No. of students correctly answering 2 or less than 2 questions is $\underline{\underline{2153}}$

iii) we have to find $f(5)$
$$= \frac{1}{256}\left[{}^{10}C_5 \cdot 3^5\right] = 239.2 \simeq \underline{\underline{239}}$$

No. of students correctly answering 5 questions is $\underline{\underline{239}}$

10) An air line knows that 5% of the people making reservations on a certain flight will not turn up. consequently their policy is to sell 52 tickets for a flight that can only hold 50 people., what is the probability that there will be a seat for every passenger who turns up? (22)

>> The probability (p) that a passenger will not turn up is

$p = 0.05$ and $q = 0.95$

Let x denote the no of passangers who will not turn up.

$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$ where $n = 52$

$\therefore p(x) = {}^{52}C_{x} (0.05)^{x} (0.95)^{52-x}$

A seat is assumed for every passenger who turns up if the no of passengers who fails to turn up is more than or equal to 2.

Hence we have to find $p(x \geq 2)$

$$p(x \geq 2) = 1 - p(x < 2)$$
$$= 1 - [p(x=0) + p(x=1)]$$
$$= 1 - [{}^{52}C_{0} (0.05)^{0} (0.95)^{52} + {}^{52}C_{1} (0.05)^{1} (0.95)^{51}]$$
$$= 1 - [(0.95)^{52} + 52 (0.05)^{1} (0.95)^{51}]$$
$$= 1 - (0.95)^{51} [(0.95) + 52 (0.05)]$$
$$= 1 - 0.2595 = \underline{0.7405}$$

Probability that a seat is available for every passenger is $\underline{0.7405}$

11) In 800 families with 5 children each how many families would be expected to have
i) 3 boys ii) 5 girls iii) either 2 or 3 boys
iv) atmost 2 girls by assuming probabilities for boys and girls to be equal.

>> p = prob of having a boy $= 1/2$

q = prob of having a girl $= 1/2$

Let x denote the no of boys in the family

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad \text{where } n=5$$

ie $$p(x) = {}^{5}C_{x} (1/2)^{x} (1/2)^{5-x}$$

$$= \frac{1}{2^5}\, {}^{5}C_{x} = \frac{{}^{5}C_{x}}{32}$$

$$\frac{2 \times 2}{2} \times \frac{2 \times 2}{4} \times \frac{2 \times 2}{4}$$
$$8 \times 4 = 32$$

Since we have to find the expected no in respect of 800 families we have

$$f(x) = 800\, p(x)$$
$$= 800\, \frac{{}^{5}C_{x}}{32} = 25 \cdot {}^{5}C_{x} = f(x)$$

Thus $f(x) = 25\, {}^{5}C_{x}$

i) we have to $f(3)$

$$f(3) = 25 \cdot {}^{5}C_{3} = 25 \cdot 10 = 250$$

Expected no of families with 3 boys is 250

ii) we have to find $f(0)$

$f(0) = 25\ {}^5C_0 = 25 \times 1 = 25$

Expected no. of families with 5 girls is 25

iii) we have to find $f(2) + f(3)$

$= 25\ {}^5C_2 + 25 \cdot {}^5C_3$

$= 25 \times 10 + 25 \times 10$

$= 250 + 250 = 500$

Expected no. of families with 2 or 3 boys is 500

iv) Atmost 2 girls means that families can have 5 boys and 0 girls (or) 4 boys and 1 girl (or) 3 boys and 2 girls.

Hence we have to find $f(5) + f(4) + f(3)$

$= 25 \cdot {}^5C_5 + 25 \cdot {}^5C_4 + 25 \cdot {}^5C_3$

$= 25 \cdot (1 + 5 + 10)$

$= 25\ (16)$

$= 400$

Expected no. of families with atmost 2 girls is 400.

(7) A class of 100 students contains 10 bright students. Five students from the class are picked at random. Find the probabilities that

i) none of the picked is a bright student.

ii) all the picked are bright students.

>> since 10 out of 100 students are bright. The Probability that a student picked is bright ie $p = \frac{10}{100} = \frac{1}{10} = 0.1$

$q = 0.9$ $\quad x \to$ bright students

$n = 5$

$\therefore p(x) = n_{C_x} p^x q^{n-x}$

$p(x) = 5_{C_x} (0.1)^x (0.9)^{5-x}$

i) p (none of p. b. s) = $p(x=0)$

$p(x=0) = 5_{C_0} (0.1)^0 (0.9)^{5-0}$

$= (0.9)^5 = 0.5905$.

ii) $p(x=5) = 5_{C_5} (0.1)^5 (0.9)^0$

$= 1 (0.1)^5 (1)$

$= (0.1)^5$

$= 1 \times 10^{-5}$

$= 0.00001$.

(4) The probability that a man aged 60 will live to be 70 is 0.65. what is the prob that out of 10 men, now aged 60

i) exactly 9 will live to be 70.

ii) atmost 9 will live to be 70.

iii) atleast 7 will live to be 70?

Let $x \rightarrow$ living

$p = 0.65$ and $q = 1-p = 1-0.65 = 0.35$, $\underline{n=10}$

$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$p(x) = {}^{10}C_{x} (0.65)^{x} (0.35)^{10-x}$

i) $p(x=9) = {}^{10}C_{9} (0.65)^{9} (0.35)^{10-9}$

$= {}^{10}C_{9} (0.65)^{9} (0.35)^{1}$

$= 10 \cdot \quad ''$

$= \underline{0.0725}$

ii) $p(x \le 9) = p(x=9) + p(x=10)$

$= {}^{10}C_{9} (0.65)^{9} (0.35) + {}^{10}C_{10} (0.65)^{10} (0.35)^{0}$

$= 0.0725 + (0.65)^{10}$

$p(x \le 9) = 1 - p(x > 9)$

$= 1 - p(x=10)$

$= 1 - (0.65)^{10}$

$= 1 - 0.01346$

$= \underline{0.9865}$

iii) $p(x \geq 7) = p(x=7) + p(x=8) + p(x=9)$
$+ p(x=10)$

$= \underline{\underline{0.5139}}$.

i) Fit a poisson distribution for the foll^ng data and calculate the theoritical frequencies:

x	0	1	2	3	4
f	122	60	15	2	1

>> we shall compute the mean (μ) of the given distribution

$$\mu = \frac{\Sigma fx}{\Sigma f} = \frac{0\times122 + 1\times60 + 2\times15 + 3\times2 + 4\times1}{122+60+15+2+1}$$

$$= \frac{60+30+6+4}{200} = 0.5$$

we have mean (μ) = m for the poisson D

The poisson distribution is

$$p(x) = \frac{m^x e^{-m}}{x!} \quad \text{and let } f(x) = 200p(x)$$

ie $f(x) = 200 \cdot \frac{(0.5)^x e^{-0.5}}{x!}$ But $e^{-0.5} \simeq 0.6065$

$$f(x) = \frac{200\times0.6065\,(0.5)^x}{x!}$$

$$f(x) = \frac{121.3\cdot(0.5)^x}{x!}$$

putting $x = 0, 1, 2, 3, 4$ in $f(x)$ we obtain the theoretical frequencies. They are as follows

$f(0) = \frac{121.3\,(0.5)^0}{0!} = 121.3$

$f(1) = \frac{121.3\,(0.5)^1}{1!} = 60.65$

$f(2) = \frac{121.3\,(0.5)^2}{2!} = 15.1625$

$f(3) = \frac{121.3\,(0.5)^3}{3!} = 2.527$

$f(4) = \frac{121.3\,(0.5)^4}{4!} = 0.3159$

2) The no of accidents per day (x) as recorded in a textile industry over a period of 400 days is given. Fit a poisson distribution for the data and calculate the theoretical frequencies

x	0	1	2	3	4	5
f	173	168	37	18	3	1

>> we have for the poisson distribution,

$$\text{Mean }(M) = m = \frac{\sum f(x)}{\sum f} = \frac{0 + 1\times168 + 2\times37 + 3\times18 + 4\times3 + 5\times1}{173+168+37+18+3+1}$$

$$= \frac{168+74+54+12+5}{400}$$

$$= 0.7825$$

The poisson distribution is $p(x) = \dfrac{m^x e^{-m}}{x!}$

Let $f(x) = 400 \cdot P(x)$

$$= 400 \cdot \frac{(0.7825)^x e^{-0.7825}}{x!}$$

$$= \frac{400 \times 0.4573 \; (0.7825)^x}{x!}$$

$$= \frac{182.9 \; (0.7825)^x}{x!}$$

Theoretical frequencies are got by substituting $x = 0, 1, 2, 3, 4, 5$ in $f(x)$ and they are as follows

$$f(0) = 182.9 \frac{(0.7825)^0}{0!} = 182.9 \simeq 183$$

$$f(1) = 182.9 \frac{(0.7825)^1}{1!} = 143.11925 \simeq 143$$

$$f(2) = 182.9 \frac{(0.7825)^2}{2!} = 55.995 \simeq 56$$

$$f(3) = 182.9 \frac{(0.7825)^3}{3!} = 14.6054 \simeq 15$$

$$f(4) = 182.9 \frac{(0.7825)^4}{4!} = 2.8571 \simeq 3$$

$$f(5) = 182.9 \frac{(0.7885)^5}{5!} = 0.4471 \simeq 0$$

3) In a certain factory turning out razor blades there is a small probability of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate no of packets containing i) no defective ii) one defective iii) two defective blades in a consignment of 10,000 packets.

>> p = probability of a defective blade $= \frac{1}{500} = 0.002$

In a packet of 10, the mean no of defective blades is

$$m = np = 10 \times 0.002 = 0.02$$

poisson distribution is $p(x) = \frac{m^x e^{-m}}{x!}$

$$p(x) = \frac{(0.02)^x e^{-0.02}}{x!}$$

Let $f(x) = 10,000\, p(x)$

$$= 10,000 \frac{(0.02)^x e^{-0.02}}{x!}$$

$$= \frac{10,000 \times 0.9802\, (0.02)^x}{x!}$$

$$= \frac{9802\, (0.02)^x}{x!}$$

i) prob of no defective

$$f(0) = \frac{9802\, (0.02)^0}{0!} = 9802$$

ii) prob of one defective

$$f(1) = \frac{9802\, (0.02)^1}{1!} = 196.04 \simeq 196$$

iii) prob of two defective

$$f(2) = \frac{9802\, (0.02)^2}{2!} = 1.9604 \simeq 2$$

4) The no of accidents in a year to taxi drivers in a city follows a Poisson d with mean 3. out of 1000 taxi drivers find approxima-tely the no of the drivers with

i) no accident in a year

ii) more than 3 accidents in a year.

→ By data, mean $(M) = 3$ and we have for the poisson distribution $\mu = m = 3$

The poisson distribution is given by

$$p(x) = \frac{m^x e^{-m}}{x!} = \frac{e^{-3} m^x}{x!}$$

Let $f(x) = 1000\, p(x) = 1000 \frac{e^{-3} m^x}{x!}$

$$f(x) = 1000 \times 0.04979 \frac{(+3)^x}{x!}$$

$$= 49.79 \times \frac{3^x}{x!}$$

$$f(x) = 50 \frac{3^x}{x!}$$

i) No. of drivers with no accident in a year $= f(0) = 50 \frac{3^0}{0!} = 50$

ii) prob of more than 3 accident in a year

$$p(x > 3) = 1 - p(x \leq 3)$$

$$= 1 - [f(x=0) + f(1) + f(2) + f(3)]$$

$$= 1 - \left[50 + 50 \cdot \frac{3^1}{1!} + 50 \cdot \frac{3^2}{2!} + 50 \cdot \frac{3^3}{3!} \right]$$

$$= 1 - [50 + 150 + 225 + 225)$$

$$= 1 - 650$$

5) 2% of the fuses manufactured by a firm are found to be defective. Find the prob that a box containing 200 fuses contains
i) no defective fuses
ii) 3 or more defective fuses.

→ p = prob of a defective defective fuse

$= 2/100 = 0.02$

∴ mean no of defectives $\mu = m = np$

$= 200 \times 0.02 = 4$

The poisson distribution is given by

$$p(x) = \frac{m^x e^{-m}}{x!} = \frac{4^x e^{-4}}{x!}$$

$$p(x) = 0.0183 \frac{4^x}{x!}$$

i) prob of no defective fuse

$$p(0) = 0.0183$$

ii) prob of 3 or more defective fuses

$$p(x \geq 3) = 1 - p(x < 3)$$

$$= 1 - [p(x=0) + p(x=1) + p(x=2)]$$

$$= 1 - \left[0.0183 + 0.0183 \frac{4^1}{1!} + 0.0183 \cdot \frac{4^2}{2!}\right]$$

$$= 1 - 0.0183\left[1 + 4 + \frac{16}{2}\right]$$

$$= 1 - 0.0183[1 + 4 + 8]$$

$$= \underline{\underline{0.7621}}$$

6) if the prob of a bad reaction from a certain infection is 0.001, determine the chance that out of 2000 individuals, more than two will get a bad reaction. (28)

prob of a bad reaction from a certain infection is $p = 0.001$

$\therefore$ mean $= m = np = 2000 \times 0.001 = 2$

$\therefore p(x) = \frac{m^x e^{-m}}{x!} = \frac{2^x e^{-2}}{x!} = 0.1353 \frac{2^x}{x!}$

$p(x) = 0.1353 \frac{2^x}{x!}$

prob (more than two) :

$p(x>2) = 1 - p(x \leq 2)$

$= 1 - [p(0) + p(1) + p(2)]$

$= 1 - 0.1353 \left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right]$ $\quad 2^4 = \frac{4}{2}$

$= 1 - 0.1353 [1 + 2 + 2]$

$= 1 - 0.1353 [5]$

$= 1 - 0.6765$

$= \underline{\underline{0.3235}}$

7) A communication channel receives independent pulses at the rate of 12 pulses per microsecond. The prob of transmission error is 0.001 for each micro second. compute the probabilities of

i) no error during a micro second.

ii) one error per micro second

iii) atleast one error per micro second

iv) two errors

v) atmost two errors.

>> mean no of errors in one micro second

$$\mu = np = m = 12 \times 0.001 = 0.012$$

The P.D $\quad p(x) = \dfrac{m^x e^{-m}}{x!}$

$$p(x) = \frac{(0.012)^x \, e^{-0.012}}{x!} = 0.9881 \, \frac{(0.012)^x}{x!}$$

i) $p(0) = 0.9881$ ($x=0$)

ii) $p(x=1) = 0.9881 \times 0.012 = 0.01186$

iii) $p(x \geq 1) = 1 - p(x \ngeq 1)$

$= 1 - p(x=0)$

$= 1 - 0.9881 = 0.0119$

iv) $p(x=2) = 0.9881 \dfrac{(0.012)^2}{2!} = 7.11432 \times 10^{-5}$

$= 0.000071$

v) $p(x \leq 2) = p(x=0) + p(x=1) + p(x=2)$

$= 0.9881 + 0.01186 + 0.000071$

$= \underline{\underline{1}}$

8) A shop has 4 disel generator sets which it hires every day. The demand for a gen set on an average is a poisson variate with value 5/2. obtain the prob that on a particular day

i) there was no demand

ii) a demand had to be refused.

>> By data m = mean demand for a generator

$= 5/2 = 2.5$

P.D $\quad p(x) = \frac{m^x e^{-m}}{x!} = \frac{e^{-2.5}(2.5)^x}{x!}$

$$p(x) = 0.0821 \frac{(2.5)^x}{x!}$$

i) No demand for a generator

$p(0) = 0.0821$

ii) if a demand had to be refused, there should have been a demand for more than 4 generator

we've to find $p(x>4)$

$p(x>4) = 1 - p(x \le 4)$

$= 1 - [p(0) + p(1) + p(2) + p(3) + p(4)]$

$= 1 - 0.0821 \left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1!} + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!} \right]$

$= 1 - 0.0821 [1 + 2.5 + 3.125 + 2.6041 + 1.6276]$

$= 1 - 0.8913$

$= 0.1087$

9) The prob that a news reader commits no mistake in reading the news is $1/e^3$. Find the prob that on a particular news broadcast he commits

i) only 2 mistakes

ii) more than 3 mistakes

iii) atmost 3 mistakes.

>> Taking a note of the prob given in the data as

$1/e^3 = e^{-3}$ we consider

$p(x) = \frac{m^x e^{-m}}{x!}$ where $x \to$ committe mistake

by data $p(x=0) = e^{-3}$ and hence

$e^{-m} = e^{-3} \Rightarrow m = 3$

$p(x) = e^{-3} \frac{3^x}{x!}$

i) prob of commits only 2 mistake is $p(2)$

$p(2) = e^{-3} \cdot \frac{3^2}{2!} = 0.2241$ $\quad e^{-3} = 0.04979 = \underline{0.0498}$

ii) $p(x > 3) = 1 - p(x \le 3)$

$= 1 - [p(0) + p(1) + p(2) + p(3)]$

$= 1 - 0.0498 \left[1 + \frac{3}{1} + \frac{3^2}{2!} + \frac{3^3}{3!}\right]$

$= 1 - 0.0498 [1 + 3 + 4.5 + 4.5]$

$= 1 - 0.6474 = \underline{0.3526}$

iii) $p(x \le 3) = p(0) + p(1) + p(2) + p(3)$

$= \underline{0.6474}$

Continuous Probability Distributions.

Def^n:- If for every x belonging to the range of a continuous random variable X, we assign a real number $f(x)$ satisfying the conditions

i) $f(x) \geq 0$ 2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

then $f(x)$ is called a continuous probability function (c.p.f) (or) probability density function (p.d.f)

Mean and variance

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$\text{Variance } (V) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$$

I) Find the foll^{ng} fu^{ns} is a Prob (p.d.f) or (c.d.f)

i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

cond^n for p.d.f are

$f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

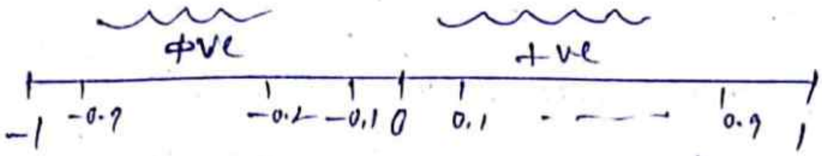
$$\left.\begin{aligned} 2x &= 2(0.1) = 0.2 \\ &= 2(0.2) = 0.4 \end{aligned}\right\} f(x) \geq 0$$

Clearly $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f_1(x)\,dx = \int_0^1 2x\,dx = \frac{2x^2}{2}\Big|_0^1 = x^2\Big|_0^1 = 1 - 0 = 1$$

$\therefore f_1(x)$ is a p.d.f / c.d.f

ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$



clearly ~~$f_2(x) \leq 0$~~ $f_2(x)$ can rewrite as

$$f_2(x) = \begin{cases} 2x & -1 < x < 0 \\ 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In $-1 < x < 0$, $f_2(x) = 2x < 0$ is less than zero.

further $\int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-1}^{1} 2x\,dx = [x^2]_{-1}^{1} = 0$

Both the conditions are not satisfied

$\therefore$ $f_2(x)$ is not a p.d.f / c.d.f.

3) Find the value of c such that $f(x) = \begin{cases} \frac{x}{6} + c, & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$ is a p.d.f. Also find $P(1 \leq x \leq 2)$

» $f(x) \geq 0$, if $c \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie $\int_0^3 \left(\frac{x}{6} + c\right) dx = 1$

$\frac{x^2}{12} + cx \Big|_0^3 = 1$

$\frac{3^2}{12} + 3c = 1$

$\frac{9}{12} + 3c = 1$

$\frac{3}{4} + 3c = 1$

$3c = 1 - \frac{3}{4} = \frac{4-3}{4} = 1/4$

$3c = 1/4$

$c = \frac{1}{4 \cdot 3} = \frac{1}{12}$

$\boxed{c = \frac{1}{12}}$

https://hemanthrajhemu.github.io

Now $P(1 \le x \le 2) = \int_1^2 f(x)\,dx$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2 = \left(\frac{4}{12} + \frac{2}{12}\right) - \left(\frac{1}{12} + \frac{1}{12}\right)$$

$$= \frac{6}{12} - \frac{2}{12} = \frac{4}{12} = \frac{1}{3}$$

Thus $\boxed{P(1 \le x \le 2) = 1/3}$

2) Find the constant k such that $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

is a p.d.f. Also find
i) $p(1 < x < 2)$
ii) $p(x \le 1)$
iii) $p(x > 1)$
iv) Mean and variance

>> $f(x) \ge 0$ if $k \ge 0$. Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie $\int_0^3 kx^2\,dx = 1$

$$k\frac{x^3}{3}\Big|_0^3 = 1 \Rightarrow \frac{k}{3}[3^3] = 1 \Rightarrow \frac{27k}{3} = 1 \Rightarrow$$

$$\Rightarrow k = \frac{3}{27} = \frac{1}{9} \Rightarrow \boxed{k = \frac{1}{9}}$$

i) $p(1 < x < 2) = \int_1^2 f(x)\,dx$

$$= \int_1^2 kx^2\,dx$$

$$= k\frac{x^3}{3}\Big|_1^2$$

$$= \frac{k}{3}[8 - 1]$$

$$\frac{k}{3}[7] = \frac{7}{3}\left(\frac{1}{9}\right)$$

$$= \frac{7}{27}.$$

Thus $\boxed{p(1 < x < 2) = \frac{7}{27}}$

ii) $P(x \le 1) = \int_0^1 f(x)\,dx = \int_0^1 kx^2\,dx = \int_0^1 \frac{x^2}{9}\,dx$

$= \frac{x^3}{27}\Big|_0^1 = \frac{1}{27}$

iii) $P(x > 1) = \int_1^3 f(x)\,dx = \int_1^3 \frac{x^2}{9}\,dx = \frac{x^3}{27}\Big|_1^3 = \frac{27}{27} - \frac{1}{27} = 1 - \frac{1}{27}$

$= \frac{26}{27}$

iv) Mean $E(\mu) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x\frac{x^2}{9}\,dx = \int_0^3 x\cdot\frac{x^2}{9}\,dx$

$\mu = \int_0^3 \frac{x^3}{9}\,dx = \frac{x^4}{4\times 9}\Big|_0^3 = \frac{1}{36}\left[x^4\right]_0^3 = \frac{1}{36}\left[3^4 - 0^4\right]$

$= \frac{1}{36}\left[81 - 0\right] = \frac{81}{36} = \frac{9}{4}$

Variance $(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$

(or)

$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= \int_0^3 x^2 \cdot \frac{x^2}{9}\,dx - \left(\frac{9}{4}\right)^2$

$= \int_0^3 \frac{x^4}{9}\,dx - \left(\frac{9}{4}\right)^2$

$= \frac{x^5}{5\times 9}\Big|_0^3 - \left(\frac{9}{4}\right)^2$

$= \frac{1}{45}\left[3^5\right] - \frac{81}{16}$

$= \frac{243}{45} - \frac{81}{16}$

$= \frac{81}{15} - \frac{81}{16} = \frac{81}{240} = \frac{27}{80}$

3) Find k such that $f(x) = \begin{cases} kxe^{-x}, & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

is a p.d.f. Find the mean.

>> $f(x) \geq 0$ if $k \geq 0$. Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie $\int_0^1 kxe^{-x}\,dx = 1 \Rightarrow k\int_0^1 xe^{-x}\,dx = 1$ Apply uv-Rule

$$k\left[x\frac{e^{-x}}{(-1)} - (1)\frac{e^{-x}}{(-1)^2}\right]_0^1 = 1$$

$$k\left[-xe^{-x} - e^{-x}\right]_0^1 = 1$$

$$k\left[-1e^{-1} - e^{-1} - (0 - e^{-0})\right] = 1$$

$$k\left[-\frac{1}{e} - \frac{1}{e} + 1\right] = 1$$

$$k\left(1 - \frac{2}{e}\right) = 1$$

$$k = \frac{1}{1-\frac{2}{e}} = \frac{1}{\frac{e-2}{e}} = \frac{e}{e-2}$$

$$\therefore \boxed{k = \frac{e}{e-2}}$$

$$\text{Mean }(M) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_0^1 x \cdot kxe^{-x}\,dx$$

$$= \int_0^1 x \cdot \frac{e}{e-2} xe^{-x}\,dx$$

$$= \frac{e}{e-2}\int_0^1 x^2 e^{-x}\,dx$$

$$= \frac{e}{e-2}\left[+x^2\frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^1$$

$$= \frac{e}{e-2}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^1$$

$$= \frac{e}{e-2}\left[-e^{-1} - 2e^{-1} - 2e^{-1} - (0 - 0 - 2e^{-0})\right]$$

$$= \frac{e}{e-2}\left[-\frac{1}{e} - \frac{2}{e} - \frac{2}{e} + 2\right]$$

$$= \frac{e}{e-2}\left[\frac{-5}{e} + 2\right]$$

$$= \frac{e}{e-2}\left[2 - \frac{5}{e}\right]$$

$$= \frac{e}{e-2}\left[\frac{2e-5}{e}\right]$$

$$= \frac{2e-5}{e-2}$$

4) Is the foll^{ng} fu^n a density fu^n?

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$ also find (1, 2) interval.

>> we observe $f(x) \geq 0$. Also we must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

ie $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_0^{\infty} e^{-x}\,dx = \frac{e^{-x}}{-1}\Big|_0^{\infty} = -[e^{-\infty} - e^{-0}]$$

$$= -[0-1] = 1 \quad \because e^{-\infty} = 0$$

Hence $f(x)$ is a probability fu^n (p.d.f) / (c.d.f).

$$P(1 < x < 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 e^{-x}\,dx$$

$$= -[e^{-x}]_1^2$$

$$= -[e^{-2} - e^{-1}]$$

$$= -e^{-2} + e^{-1}$$

$$= \left[\frac{1}{e} - \frac{1}{e^2}\right] = 0.3679 - 0.1353$$

$$= \underline{\underline{0.2326}}$$

5) A random variable x has the foll^ng d.f

$$p(x) = \begin{cases} kx^2, & -3 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate k and find i) $P(1 \le x \le 2)$ ii) $P(x \le 2)$ iii) $P(x > 1)$

>> $p(x) \geq 0$ if $k \geq 0$ and also we must've $\int_{-\infty}^{\infty} p(x)\,dx = 1$

ie $\int_{-3}^{3} kx^2\,dx = 1 \Rightarrow k\frac{x^3}{3}\Big|_{-3}^{3} = 1$

$$\begin{array}{r} 27 \\ 27 \\ \hline 54 \end{array}$$

$$\Rightarrow \frac{k}{3}\left[27 - (-3)^3\right] = 1$$

$$\Rightarrow \frac{k}{3}\left[27 + 27\right] = 1$$

$$\frac{k}{\cancel{3}}\left[\cancel{54}^{18}\right] = 1$$

$$\boxed{k = \frac{1}{18}}$$

i) $P(1 \le x \le 2) = \int_1^2 kx^2\,dx = \int_1^2 \frac{x^2}{18}\,dx = \frac{x^3}{54}\Big|_1^2 = \frac{8}{54} - \frac{1}{54} = \frac{7}{54}$

ii) $P(x \le 2) = \int_{-3}^{2} \frac{x^2}{18}\,dx = \frac{x^3}{54}\Big|_{-3}^{2} = \frac{1}{54}\left[2^3 - (-3)^3\right]$

$$= \frac{1}{54}\left[8 + 27\right]$$

$$= \frac{1}{54}\left[35\right] = \frac{35}{54}$$

iii) $P(x > 1) = \int_1^3 \frac{x^2}{18}\,dx = \frac{x^3}{54}\Big|_1^3$

$$= \frac{1}{54}\left[27 - 1\right] = \frac{\cancel{26}^{13}}{\cancel{54}_{27}} = \frac{13}{27}$$

6) Find k so that the foll^{ng} fu^n can serve as a p.d.f. of a random variable.

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}$$

>> we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

i.e. $\int_0^{\infty} kxe^{-4x^2} dx = 1$

putting $4x^2 = t \Rightarrow 8x\,dx = dt$

$\Rightarrow dx = \frac{dt}{8x} \Rightarrow x\,dx = \frac{dt}{8}$

put $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\therefore \int_{t=0}^{\infty} k \cdot e^{-t} \frac{dt}{8} = 1$

$\frac{k}{8}\int_0^{\infty} e^{-t}dt = 1 \Rightarrow \frac{k}{8}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} = 1$

$\Rightarrow \frac{-k}{8}[0-1] = 1$

$\Rightarrow \frac{k}{8} = 1 \Rightarrow \boxed{k = 8}$

7) A random variable x has the density fu^n $f(x) = k/1+x^2$, $-\infty < x < \infty$. Determine k and hence evaluate i) $P(x \ge 0)$ ii) $P(0 < x < 1)$

>> we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

i.e. $\int_{-\infty}^{\infty} \frac{k}{1+x^2} dx = 1$

Since the integral is even

$2\int_0^{\infty} \frac{k}{1+x^2} dx = 1$

$2k[\tan^{-1}x]_0^{\infty} = 1$

$2k[\pi/2 - 0] = 1$

$2k\left[\frac{\pi}{2}\right] = 1$

$\boxed{k = 1/\pi}$

Now $p(x \geq 0) = \int_0^\infty f(x)\,dx = \frac{1}{\pi}\int_0^\infty \frac{1}{1+x^2}\,dx$

$$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^\infty = \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}$$

Thus $\boxed{p(x \geq 0) = \frac{1}{2}}$

Also $p(0 < x < 1) = \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx$

$$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^1 = \frac{1}{\pi}\left[\frac{\pi}{4} - 0\right] = \frac{1}{4}$$

Thus $\boxed{p(0 < x < 1) = \frac{1}{4}}$

8) The time t years required to complete a software Project has p.d.f of the form

$$f(t) = \begin{cases} kt(1-t), & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find k and also the prob that the project will be Completed in less than 4 months.

>> we must have $\int_{-\infty}^{\infty} f(t)\,dt = 1$

ie $\int_0^1 kt(1-t)\,dt = 1$

$\Rightarrow k\int_0^1 (t^2 - t)\,dt = 1$

$\Rightarrow k\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 = 1$

$\Rightarrow k\left[\frac{1}{3} - \frac{1}{2}\right] = 1$

$\Rightarrow k\left[\frac{2-3}{6}\right] = 1$

$-\frac{k}{3} = 1$

$\Rightarrow k\int_0^1 (t - t^2)\,dt = 1$

$\Rightarrow k\left(\frac{t^2}{2} - \frac{t^3}{3}\right)_0^1 = 1$

$\Rightarrow k\left[\frac{1}{6}\right] = 1$

$\therefore \boxed{k = 6}$

prob that the project will be completed in 4 months is equivalent to find $p(0 < t < 1/3)$

Since t is in years

$$\therefore p(0<t<1/3) = \int_0^{1/3} 6(t-t^2)\,dt$$

5 apple – 100 Rs
1 apple – ?
$\frac{1 \times 100}{5} = 20$

$$= 6\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^{1/3}$$

12 months = 1 year
4 months = ?
$\frac{4 \times 1}{12} = \frac{1}{3}$ yr

$$= 6\left[\frac{t^2}{2}\right]_0^{1/3} - 6\left[\frac{t^3}{3}\right]_0^{1/3}$$

$$= 3\left[t^2\right]_0^{1/3} - 2\left[t^3\right]_0^{1/3}$$

$$= \frac{7}{27}$$

Note:- If X is a continuous random variable with prob (p.d.f) $f(x)$ the fun $F(x)$ defined by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{——— (1)}$$

is called the commulative distribution function (c.d.f) of X

It is evident from (2) we've

$F(x) = P(X \le x) = P(-\infty < X \le x)$ and $\frac{d}{dx}[F(x)] = f(x)$ is a continuous random variable.

9) Find the c.d.f for the foll^ng p.d.f of a random variable x.

i) $f(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ ii) $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$

>> if $f(x)$ is the p.d.f then the c.d.f $F(x) = \int_{-\infty}^{x} f(x)dx$

i) $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$

$= 0 + \int_{0}^{x} (6x - 6x^2)dx = \underline{3x^2 - 2x^3}.$

$\therefore$ c.d.f $= 3x^2 - 2x^3$ if $0 \le x \le 1$

ii) $F(x) = \int_{0}^{x} \frac{x}{4} e^{-x/2} dx$

$= \frac{1}{4} \left[x \frac{e^{-x/2}}{-1/2} - \frac{e^{-x/2}}{1/4} \right]_0^x$

$= \frac{1}{4} \left[-2 (x e^{-x/2})_0^x - 4 (e^{-x/2})_0^x \right]$

$= \frac{1}{4} \left[-2 (x e^{-x/2} - 0) - 4 (e^{-x/2} - 1) \right]$

$= \frac{1}{4} \left[-2x e^{-x/2} - (4e^{-x/2} - 1) \right]$

$= \frac{1}{4}_2 \cancel{6x e^{-x/2}}$

$= \frac{1}{4} \left[-2x e^{-x/2} - 4e^{-x/2} + 1 \right]$

c.d.f $= 1 - e^{-x/2} - (x/2) e^{-x/2}$, if $0 < x < \infty$

10) A continuous random variable has the distribution fun

$$F(x) = \begin{cases} 0 & x \le 1 \\ c(x-1)^4 & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$ Find c and also the pdf

>> WKT the p.d.f $f(x) = \frac{d}{dx}[F(x)]$

$$\therefore f(x) = \begin{cases} 0 & x \le 1 \\ 4c(x-1)^3 & 1 \le x \le 3 \\ 0 & x > 3 \end{cases}$$

$f(x) \geq 0$ for $c \geq 0$ and we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

ie $\int_1^3 4c(x-1)^3 dx = 1$

$4c\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$

$c\left[(x-1)^4\right]_1^3 = 1$

$c\left[(3-1)^4 - (1-1)^4\right] = 1$

$c\left[2^4 - 0^4\right] = 1$

$c[16] = 1$

$\boxed{c = \frac{1}{16}}$

Thus p.d.f $f(x) = \frac{(x-1)^3}{4}$ where $1 \le x \le 3$

Exponential Distribution.

The continuous probability distribution having the proba density fun (p.d.f) $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \text{ where } \alpha > 0$$

is known as the exponential distribution.

Evidently $f(x) \geq 0$ and we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \alpha \int_{0}^{\infty} e^{-\alpha x}\,dx$$

$$= \cancel{\alpha}\left[\frac{e^{-\alpha x}}{-\cancel{\alpha}}\right]_0^{\infty} = -(e^{-\infty} - 1) = 1$$

Thus $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

$f(x)$ satisfy both the cond^ns required for a continuous prob fun c.p.f / prob. density fun (p.d.f)

Mean and Standard Deviation of the Exponential Distribution.. Mean $(\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$= \int_{-\infty}^{0} x \cdot \cancel{f(x)}\,dx + \int_{0}^{\infty} x \cdot f(x)\,dx = \int_{0}^{\infty} x \cdot \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_{0}^{\infty} x e^{-\alpha x}\,dx = \alpha \left[x \cdot \frac{e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{\alpha^2}\right]_0^{\infty}$$

$$= \alpha \left[-\frac{1}{\alpha}\left[x e^{-\alpha x}\right]_0^{\infty} - \frac{1}{\alpha^2}\left[e^{-\alpha x}\right]_0^{\infty}\right]$$

$$= \alpha \left[-\frac{1}{\alpha}\left[\infty e^{-\infty} - 0\right] - \frac{1}{\alpha^2}\left[e^{-\infty} - 1\right]\right]$$

$$= \alpha \left[0 + \frac{1}{\alpha^2}\right] = \frac{1}{\alpha} \qquad \because e^{-\infty} = 0$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$$

$$\sigma^2 = \cancel{\int_{-\infty}^{0} (x-\mu)^2 f(x)\,dx} + \int_{0}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$$

$$\sigma^2 = \int_{0}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$$

$$= \alpha \int_{0}^{\infty} (x-\mu)^2 \cdot e^{-\alpha x}\,dx.$$

$$= \alpha \left[(x-\mu)^2 \frac{e^{-\alpha x}}{-\alpha} - 2(x-\mu)\frac{e^{-\alpha x}}{\alpha^2} + 2(1-0)\frac{e^{-\alpha x}}{-\alpha^3} \right]_0^{\infty}$$

$$= \alpha \left[\frac{-1}{\alpha}\left\{(x-\mu)^2 e^{-\alpha x}\right\}_0^{\infty} - \frac{2}{\alpha^2}\left\{(x-\mu)\, e^{-\alpha x}\right\}_0^{\infty} - \frac{2}{\alpha^3}\left\{e^{-\alpha x}\right\}_0^{\infty} \right]$$

$$= \alpha \left[-\frac{1}{\alpha}\left\{(\infty-\mu)^2 e^{-\infty} - (0-\mu)^2 \cdot 1\right\} - \frac{2}{\alpha^2}\left\{(\infty-\mu)e^{-\infty} - (0-\mu)\right\} - \frac{2}{\alpha^3}\left\{e^{-\infty} - 1\right\} \right]$$

$$= \alpha \left[+\frac{1}{\alpha}\left\{0 + \mu^2\right\} - \frac{2}{\alpha^2}\left\{0 + \mu\right\} - \frac{2}{\alpha^3}\left\{0 - 1\right\} \right]$$

$$\because e^{-\infty} = 0$$

$$= \alpha \left[+ \frac{\mu^2}{\alpha} - \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3} \right] \quad \text{but } \mu = \frac{1}{\alpha}$$

$$= \alpha \left[\frac{1}{\alpha^3} - \cancel{\frac{2}{\alpha^3}} + \cancel{\frac{2}{\alpha^3}} \right]$$

$$\sigma^2 = \alpha \cdot \left[\frac{1}{\alpha^3} \right] = \frac{1}{\alpha^2}$$

Hence $\sigma = \frac{1}{\alpha}$

Thus <u>ED</u>,

$$\text{Mean}(\mu) = \frac{1}{\alpha}\ ; \quad S.D(\sigma) = \frac{1}{\alpha}$$

$$\therefore \underline{\mu = \sigma} \quad \underline{\text{mean} = S.D}$$

1) The kilometre run (in thousands of k.ms) without any sort of problem in respect of a certain vehicle is a random variable having p.d.f

$$f(x) = \begin{cases} \frac{1}{40} e^{-x/40}, & x \geq 0 \\ 0, & x \leq 0 \end{cases}$$

Find the prob that the vehicle is trouble free

i) atleast for 25000 k.ms

ii) atmost for 25000 k.ms

iii) between 16000 to 32000 k.ms

>> Here x is the random variable representing kilometre in multiples of 1000 regarding trouble free run by the vehicle.

i) To find $P(x \geq 25)$

$$P(x \geq 25) = \cancel{1 - P(x < 25)}$$

$$= \cancel{1 -} \int_{25}^{\infty} \frac{1}{40} e^{-x/40} dx$$

$$= \cancel{1 -} \frac{1}{40} \cdot \frac{e^{-x/40}}{-1/40} \Big|_{25}^{\infty}$$

$$= \cancel{1 +} e^{-x/40} \Big|_{25}^{\infty}$$

$$= \cancel{1 +} \left[e^{-\infty} - e^{-25/40}\right]$$

$$= \cancel{1 +} \left[0 - e^{-5/8}\right]$$

$$= \cancel{1 -} e^{-5/8}$$

$$= \int_{25}^{\infty} \frac{1}{40} e^{-x/40} dx$$

$$= \frac{1}{40} \cdot \frac{e^{-x/40}}{-1/40} \Big|_{25}^{\infty}$$

$$= -\left(e^{-\infty} - e^{-25/40}\right)$$

$$= -\left(0 - e^{-5/8}\right)$$

$$= e^{-5/8}$$

$$= 0.5353$$

Thus $P(x \geq 25) = 0.5353$

ii) To find $P(x \le 25)$

$$P(x \le 25) = \int_0^{25} \frac{1}{40} e^{-x/40} dx$$

$$= \frac{1}{\cancel{40}} \cdot \frac{e^{-x/40}}{-1/\cancel{40}} \Big|_0^{25} = -\left(e^{-x/40}\right)_0^{25}$$

$$= -\left(e^{-25/40} - 1\right) = -e^{-5/8} + 1$$

$$= 1 - e^{-5/8}$$

$$= 1 - 0.5353$$

$$\therefore \boxed{P(x \le 25) = 0.4647}$$

$\frac{\cancel{16}^{\cancel{8}\,4}}{\frac{\cancel{32}}{\cancel{40}\,\cancel{20}\,\cancel{10}\,5}}$

iii) To find $P(16 \le x \le 32)$

$$P(16 \le x \le 32) = \int_{16}^{32} \frac{1}{40} e^{-x/40} dx$$

$\frac{\cancel{16}^{\cancel{4}\,2}}{\cancel{40}\,\cancel{10}\,5}$

$$= -\left(e^{-x/40}\right)_{16}^{32}$$

$$= -\left(e^{-32/40} - e^{-16/40}\right)$$

$$= -\left(e^{-4/5} - e^{-2/5}\right)$$

$$= e^{-2/5} - e^{-4/5}$$

$$= 0.6703 - 0.4493$$

$$= \underline{\underline{0.221}}$$

2) If x is an exponential variate with mean 3 find i) $p(x>1)$ ii) $p(x<3)$ (38)

>> The p.d.f of the exponential distribution is given by $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$

The mean of this distribution is given by $\frac{1}{\alpha}$

by data $\frac{1}{\alpha} = 3 \Rightarrow \alpha = \frac{1}{3}$

Hence $f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & 0<x<\infty \\ 0, & \text{otherwise.} \end{cases}$

i) $p(x>1) = \int_1^\infty \frac{1}{3} e^{-x/3} dx$

$= \frac{1}{\cancel{3}} \frac{e^{-x/3}}{-1/\cancel{3}} \Big|_1^\infty$

$= -(e^{-x/3})_1^\infty$

$= -(e^{-\infty} - e^{-1/3})$

$= -(0 - e^{-1/3})$

$= e^{-1/3}$

$p(x>1) = e^{-1/3} = \underline{0.7165}$

ii) $p(x<3) = \int_0^3 \frac{1}{3} e^{-x/3} dx$

$= \frac{1}{\cancel{3}} \frac{e^{-x/3}}{-1/\cancel{3}} \Big|_0^3$

$= -(e^{-x/3}) \Big|_0^3$

$= -(e^{-3/3} - e^{-0})$

$= -(e^{-1} - 1)$

$= 1 - \frac{1}{e}$

$= 1 - 0.3679$

$= \underline{0.6321}$

3) if x is an exponential variate with mean 5. evaluate i) $p(0<x<1)$ ii) $p(-\infty<x<10)$

iii) $p(x \le 0 \text{ or } x \ge 1)$

>> $f(x) = \alpha e^{-\alpha x}, \quad 0<x<\infty$ Mean $= \frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$

Thus $f(x) = \begin{cases} \frac{1}{5} e^{-x/5}, & 0<x<\infty \\ 0 & \text{others} \end{cases}$

i) $p(0<x<1) = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{5} e^{-x/5}\,dx = \frac{1}{\cancel{5}}\left[\frac{e^{-x/5}}{-1/\cancel{5}}\right]_0^1$

$= -\left(e^{-x/5}\right)_0^1 = -\left(e^{-1/5} - 1\right) = 1 - e^{-1/5}$

$= 1 - e^{-1/5} = 1 - 0.8187 = 0.1813$

$\therefore$ $\underline{p(0<x<1) = 0.1813}$

ii) $p(-\infty<x<10) = \int_{-\infty}^{10} f(x)\,dx = \int_{-\infty}^{0} \cancel{f(x)}^{\,0}\,dx + \int_0^{10} f(x)\,dx$

$= \int_0^{10} \frac{1}{5} e^{-x/5}\,dx = \frac{1}{\cancel{5}} \left(\frac{e^{-x/5}}{-1/\cancel{5}}\right)\Big|_0^{10} = -\left(e^{-x/5}\right)\Big|_0^{10}$

$= -\left(e^{-10/5} - 1\right) = -\left(e^{-2} - 1\right) = 1 - \frac{1}{e^2}$

$= 1 - 0.1353 = \underline{\underline{0.8647}}$

iii) $p(x \le 0 \text{ or } x \ge 1) = p(x \le 0) + p(x \ge 1)$

$= \int^0 f(x)\,dx + \int_1^{\infty} f(x)\,dx = 0 + \int_1^{\infty} \frac{1}{5} e^{-x/5}\,dx$

$= \frac{1}{\cancel{5}} \frac{e^{-x/5}}{-1/\cancel{5}}\Big|_1^{\infty} = -\left(e^{-x/5}\right)_1^{\infty} = -\left(e^{-\infty} - e^{-1/5}\right)$

$= -\left(0 - e^{-1/5}\right) = e^{-1/5}$

$= \underline{\underline{0.8187}}$

4) The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the prob that a random call made from this booth

i) ends less than 5 minutes.

ii) between 5 and 10 minutes.

we have ED $f(x) = \alpha e^{-\alpha x}$, $x > 0$

Mean $= 5$ by data $\frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$

Hence $f(x) = \alpha e^{-\alpha x} = \frac{1}{5} e^{-x/5}$ is the p.d.f

i) $P(x<5) = \int_0^5 f(x)\,dx = \int_0^5 \frac{1}{5} e^{-x/5}\,dx$

$= \cancel{\frac{1}{5}} \frac{e^{-x/5}}{\cancel{-1/5}}\Big|_0^5 = -\left(e^{-x/5}\right)_0^5 = -\left(e^{-1} - 1\right)$

$= 1 - \frac{1}{e}$

$= 1 - 0.3679$

$= \underline{\underline{0.6321}}$

ii) $P(5 < x < 10) = \int_5^{10} \frac{1}{5} e^{-x/5}\,dx$

$= \frac{1}{5} \frac{e^{-x/5}}{-1/5}\Big|_5^{10} = -\left(e^{-x/5}\right)_5^{10}$

$= -\left(e^{-10/5} - e^{-5/5}\right) = -\left(e^{-2} - e^{-1}\right)$

$= \frac{1}{e} - \frac{1}{e^2}$

$= 0.3679 - 0.1353$

$= \underline{\underline{0.2326}}$

5) In a Certain town the duration of a shower is exponentially distributed with mean 5 minutes. what is the prob that a shower will last for.

i) 10 minutes or more

ii) less than 10 minutes

iii) b/w 10 and 12 minutes.

>> The p.d.f of the exponential distribution is given by

$f(x) = \alpha e^{-\alpha x}$, $\alpha > 0$ and mean $= \frac{1}{\alpha} = 5 \Rightarrow 5 = \frac{1}{\alpha}$

$\Rightarrow \alpha = 1/5$

$\therefore f(x) = \frac{1}{5} e^{-x/5}$

i) $P(x \geq 10) = \int_{10}^{\infty} \frac{1}{5} e^{-x/5} = \frac{1}{5} \frac{e^{-x/5}}{-1/5} = -\left(e^{-x/5}\right)\Big|_{10}^{\infty}$

$= -\left(e^{-\infty/5} - e^{-10/5}\right) = -\left(0 - e^{-2}\right) = e^{-2} = \frac{1}{e^2}$

$= \underline{\underline{0.1353}}$

ii) $P(x < 10) = \int_0^{10} \frac{1}{5} e^{-x/5} dx = -\left(e^{-x/5}\right)_0^{10} = -\left(e^{-2} - 1\right)$

$= 1 - \frac{1}{e^2} = \underline{\underline{0.8647}}$

iii) $P(10 < x < 12) = -\left(e^{-x/5}\right)_{10}^{12}$

$= -\left(e^{-12/5} - e^{-2}\right)$

$= e^{-2} - e^{-12/5}$

$= \frac{1}{e^2} - \frac{1}{e^{12/5}}$

$= \underline{\underline{0.0446}}$

Normal Distribution

The continuous Prob distribution having the Prob density fu^n $f(x)$ given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

put $t^2 = (x-\mu)^2/2\sigma^2$

$$\Rightarrow t = (x-\mu)/\sqrt{2}\sigma$$

$$x = \mu + \sqrt{2}\sigma t$$

$$dx = \sqrt{2}\sigma\,dt$$

$$t \to -\infty \text{ to } \infty$$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \cancel{\sqrt{2}\sigma}\,dt$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt$$

$$= \frac{1}{\sqrt{\pi}}\, 2\int_{0}^{\infty} e^{-t^2}\,dt$$

But $\int_0^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$ by gamma function.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

Thus $f(x)$ represents a Prob density fu^n

Mean and Standard Deviation of the Normal D.

$$\text{Mean } (M) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-(x-m)^2/2\sigma^2}\,dx$$

put $t^2 = \frac{(x-m)^2}{2\sigma^2} \Rightarrow t = \frac{(x-m)}{\sqrt{2}\sigma}$

$\Rightarrow x = m + \sqrt{2}\sigma t$

$dx = \sqrt{2}\sigma\,dt \qquad t \to -\infty \text{ to } \infty$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (m + \sqrt{2}\sigma t)\, e^{-t^2}\, \sqrt{2}\cancel{\sigma}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (m + \sqrt{2}\sigma t)\, e^{-t^2}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} m e^{-t^2}\,dt + \sqrt{2}\sigma t e^{-t^2}\,dt$$

$$= \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2}\,dt$$

$$= \frac{m}{\sqrt{\pi}}\, 2\int_{0}^{\infty} e^{-t^2}\,dt + 0$$

Here 2nd integral is Zero by a standard property since te^{-t^2} is an odd fun

$$= \frac{2m}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2}$$

Gamma fun

$$= m$$

Thus $\boxed{\text{mean} = m}$

Hence we can say that the mean of the normal distribution is equal to the mean of the given distribution.

$\therefore$ Variance $(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot e^{-(x-\mu)^2/2\sigma^2}\,dx$$

put $t^2 = \frac{(x-\mu)^2}{2\sigma^2} \Rightarrow t = \frac{(x-\mu)}{\sqrt{2}\sigma}$

$x = \mu + \sqrt{2}\sigma t \qquad t \to -\infty$ to ∞

$$\therefore = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} \cancel{\sqrt{2}\sigma}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2}\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt$$

Now $$= \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^2 e^{-t^2}\,dt$$

Now $\int_0^{\infty} t^2 e^{-t^2}\,dt = \left[t^2 e^{-t^2}\,dt \right]$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\, 2\int_0^{\infty} t^2 e^{-t^2}\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t\,(2t\,e^{-t^2})\,dt$$

wkt $\int uv\,dt = u\int v\,dt - \int\int v\,dt \cdot u'\,dt$

taking $u = t$, $v = 2t e^{-t^2}$ and nothing that $\int v\,dt = -e^{-t^2}$ we now

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left\{ \left[\cancel{t(-e^{-t^2})} \right]_0^{\infty} - \int_0^{\infty} -(e^{-t^2})\,1.dt \right.$$

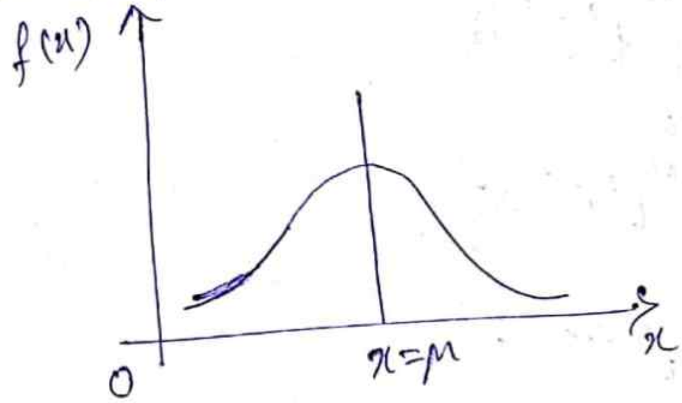
$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \sigma^2$$

Thus $\boxed{\text{Variance} = \sigma^2}$

Hence we can say that the variance / S.D of the ND is equal to the variance / S.D of the given distribution

Note :- The graph of the prob. fun $f(x)$ is a bell shaped curve symmetrical about the line $x=\mu$ and is called the normal prob. curve.

The shape of the curve is as follows.



The line $x=\mu$ divides the total area under the curve which is equal to 1 into two equal parts. The area to the right as well as to the left of the line $x=\mu$ is 0.5

Standard Normal Distribution.

* $Z = \frac{x-\mu}{\sigma}$ is called the standard normal variate (S.N.V)

* $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2}\, dz$

area under the standard normal cure from $z = 0$ to z

Note :- Standard Results.

1) $P(-\infty \le z \le \infty) = 1$

2) $P(-\infty \le z \le 0) = 1/2$

3) $P(0 \le z \le \infty)$ (or) $P(z \ge 0) = \frac{1}{2}$

4) $P(-\infty < z < z_1) = P(-\infty < z \le 0) + P(0 \le z \le z_1)$

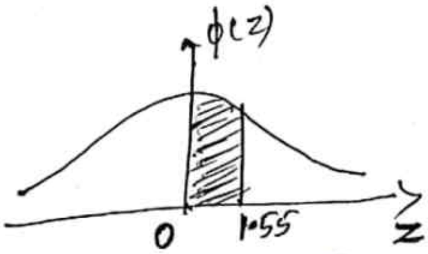
$\Rightarrow P(z < z_1) = 0.5 + \phi(z_1)$

5) $P(z > z_2) = P(z \ge 0) - P(0 \le z < z_2)$

$P(z > z_2) = 0.5 - \phi(z_2)$

Illustration -1

To find the area under the standard normal curve b/w $z = 0$ and 1.55



Theoritically the are

$\frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz = \phi(z)$

$\frac{1}{\sqrt{2\pi}} \int_0^{1.55} e^{-z^2/2} dz = \phi(1.55)$

$\Rightarrow \phi(1.55) = 0.4394$

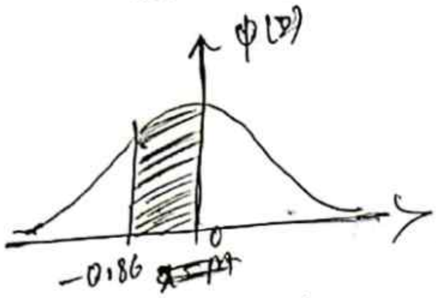
$1.55 \Rightarrow \begin{array}{r} 1.5 \\ 0.05 \\ \hline 1.55 \end{array}$

Equivalently we have

$P(0 \le z \le 1.55) = \underline{0.4394}$

Ex – ②

Area of the standard normal curve b/w $z = -0.86$ and $z = 0$



$$Area = \frac{1}{\sqrt{2\pi}} \int_{-0.86}^{0} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{0.86} e^{-z^2/2} dz \quad \text{by Symmetry}$$

$$= \phi(z)$$

$0.86 = 0.8 + 0.06 = 0.86$

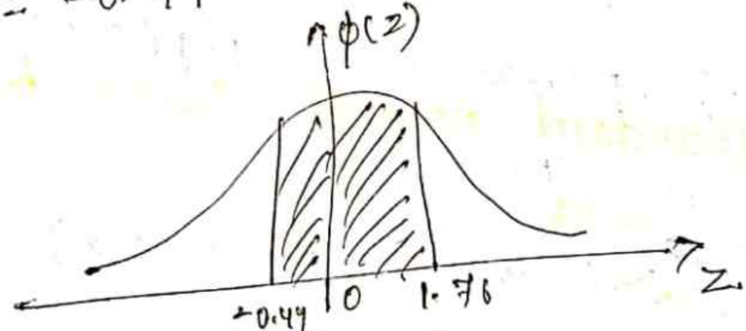
$\therefore$ area $= \phi(0.86) = 0.3051$

Equivalently

$P(-0.86 \le z \le 0) = 0.3051$

Ex – ③

Area of the standard normal curve b/w $z = -0.44$ and $z = 1.76$



$$Area = \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{1.76} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{0} e^{-z^2/2} dz + \int_{0}^{1.76} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{0.44} e^{-z^2/2} dz + \int_{0}^{1.76} e^{-z^2/2} dz \quad \text{by Symmetry}$$

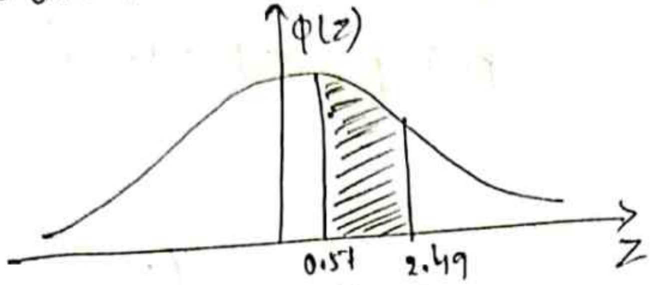
$= \phi(0.44) + \phi(1.76)$

$= 0.1700 + 0.4608 = 0.6308$

Equivalently $P(-0.44 \le z \le 1.76) = \underline{0.6308}$

Ex-4

Area of the standard normal curve b/w $z = 0.57$ to $z = 2.49$



Required area = (Area b/w $z = 0$ to 2.49) − (Area b/w $z = 0$ to 0.57)

$= \phi(2.49) - \phi(0.57)$

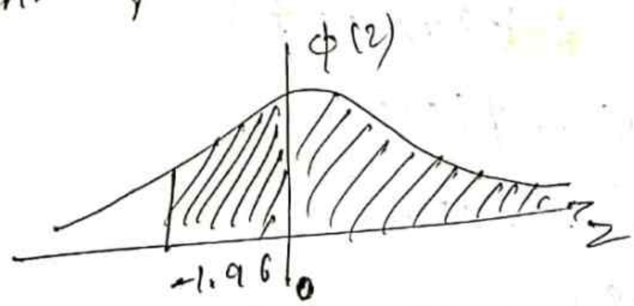
$= 0.4936 - 0.2157$

$= 0.2779$

Equivalently $P(0.57 \leq z \leq 2.49) = \underline{\underline{0.2779}}$

Ex-5

Area of the standard normal curve to the right of $z = -1.96$



Required area = (Area b/w $z = -1.96$ to 0) + (Area to the right of $z = 0$)

$= ($ Area b/w $z = 0$ to $1.96) +$ " by symmetry

$= \phi(1.96) + 0.5 = 0.4750 + 0.5 = 0.9780$

equivalently $P(z \geq -1.96) = \underline{\underline{0.975}}$

Evaluate the foll^ng prob with the help of normal prob tables.

i) $P(Z \geq 0.85)$

ii) $P(-1.64 \leq Z \leq -0.88)$

iii) $P(Z \leq -2.43)$

iv) $P(|Z| \leq 1.94)$

proof i) $P(Z \geq 0.85)$

WKT $P(Z > Z_2) = P(Z \geq 0) - P(0 \leq Z \leq Z_2)$

$\Rightarrow P(Z \geq 0.85) = P(Z \geq 0) - P(0 \leq Z \leq 0.85)$

$= 0.5 - \phi(0.85)$

$= 0.5 - 0.3023$

$= 0.1977$

$\therefore P(Z \geq 0.85) = 0.1977$.

ii) $P(-1.64 \leq Z \leq -0.88)$

By Symmetry $P(-1.64 \leq Z \leq -0.88) = P(0.88 \leq Z \leq 1.64)$

" $= P(0 \leq Z \leq 1.64) - P(0 \leq Z \leq 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.3106$

$= 0.1389$

$\therefore P(-1.64 \leq Z \leq -0.88) = 0.1389$.

iii) $P(Z \le -2.43)$

~~wkt $P(Z <$~~

$P(Z \le -2.43) = P(Z \ge 2.43)$

wkT $P(Z \ge Z_2) = P(Z \ge 0) - P(0 \le Z < Z_2)$

$= P(Z \ge 0) - P(0 \le Z < 2.43)$

$= 0.5 - \phi(2.43)$

$= 0.5 - 0.4925$

$= 0.0075$

$\therefore P(Z \le -2.043) = \underline{\underline{0.0075}}$

iv) $P(|Z| \le 1.94) = P(-1.94 \le Z \le 1.94)$

$= 2P(0 \le Z \le 1.94)$

$\because \int_{-\infty}^{\infty} = 2\int_{0}^{\infty}$

$= 2\,\phi(1.94)$

$= 2(0.4738)$

$= \underline{\underline{0.9476}}$

$\therefore P(|Z| \le 1.94) = \underline{\underline{0.9476}}$.

2) if x is a normal variate with mean 30 and S.D 5 find the prob that

i) $26 \le x \le 40$ ii) $x \ge 45$

>> we have standard normal variate

(S.N.V) $\Rightarrow Z = \frac{x-\mu}{\sigma} = \frac{x-30}{5}$

$$Z = \frac{x-30}{5}$$

$$\Rightarrow x - 30 = 5Z$$

$$\Rightarrow \boxed{x = 30 + 5Z} \longrightarrow (1)$$

i) To find $P(26 \le x \le 40)$

if $x = 26$, $Z = -0.8$

if $x = 40$, $Z = 2$

$(1) \Rightarrow 26 = 30 + 5Z$
$26 - 30 = 5Z$
$-4 = 5Z$
$Z = -4/5 = \underline{\underline{-0.8}}$

Hence we need to find

$P(-0.8 \le Z \le 2)$

Thus $P(-0.8 \le Z \le 2) = P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$

$= P(0 \le Z \le 0.8) + P(0 \le Z \le 2)$

$= \phi(0.8) + \phi(2)$

$= 0.2881 + 0.4772$

$= 0.7653$

$\therefore P(26 \le x \le 40) = \underline{0.7653}$

iii) To find $P(x \geq 45)$

if $x = 45$, $z = 3$ and hence we have to find

$$P(z \geq 3) = P(z \geq 0) - P(0 \leq z \leq 3)$$
$$= 0.5 - \phi(3)$$
$$= 0.5 - 0.4987$$
$$= 0.0013$$

$\therefore P(x \geq 45) = 0.0013$

3) if x is normally distributed with mean 12 and S.D 4. find the foll^{ng}.

i) $P(x \geq 20)$ ii) $P(x \leq 20)$

>> we've S.N.V $z = \frac{x-\mu}{\sigma} = \frac{x-12}{4}$

$$\Rightarrow \boxed{x = 12 + 4z}$$

if $x = 20$, $z = 2$

we've to find $P(z \geq 2)$ and $P(z \leq 2)$

Now $P(z \geq 2) = P(z \geq 0) - P(0 \leq z \leq 2)$

$$= 0.5 - \phi(2) = 0.5 - 0.4772 = 0.0228$$

Also $P(z \leq 2) = P(-\infty \leq z \leq 0) + P(0 \leq z \leq 2)$

WKT Result $= 0.5 + \phi(2)$

$$= 0.5 + 0.4772$$
$$= 0.9772$$

Thus $P(x \geq 20) = 0.0228$ and $P(x \leq 20) = 0.9772$.

4) The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard D 5. Find the no of students whose marks will be

i) less than 65 ii) more than 75 iii) b/w 65 and 75

>> Let x represent the marks of students.

By data $\mu = 70$, $\sigma = 5$

Hence S.N.V $z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$

$$\boxed{x = 70 + 5z}$$

i) $p(x < 65)$

if $x = 65$, $z = -1$ and hence we have to find

$p(z < -1)$

$\Rightarrow p(z<-1) = p(z>1)$

$= p(z \geq 0) - p(0 \leq z \leq 1)$

$= 0.5 - \phi(1)$

$= 0.5 - 0.3413 = 0.1587$

$\therefore$ no of students scoring less than 65 marks

$= 1000 \times 0.1587 = 158.7 \simeq 159$

ii) if $x = 75$, $z = 1$ we need to find $p(z > 1)$

$p(z>1) = p(z \geq 0) - p(0 < z < 1)$

$= 0.5 - \phi(1) = 0.1587$

$\therefore$ no of students scoring more than 75 marks

$= 1000 \times 0.1587 = 158.7 \simeq 159$

iii) we've to find

$$P(-1<x<1) = 2P(0<z<1)$$
$$= 2\phi(1)$$
$$= 2(0.3413)$$
$$= 0.6826$$

$\therefore$ no of students scoring marks b/w 65 and 75

$$= 1000 \times 0.6826 = 682.6 \simeq 683$$

5) ~~In a normal distribution 31% of the~~

In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours.

if a firm purchases 2500 bulbs find the no of bulbs that are likely to last for

i) more than 2100 hours.

ii) less than 1960 hours

iii) b/w 1900 to 2100 hours.

>> By data $\mu = 2000$, $\sigma = 60$

S.N.V $\quad z = \dfrac{x-\mu}{\sigma} = \dfrac{x-2000}{60}$

$$\Rightarrow z = \frac{x-2000}{60}$$

$$\boxed{x = 2000 + 60z}$$

i) To find $p(x>2100)$

$x = 2000 + 60z$
$2100 - 200 = 60z$
$100 = 60z$
$z = \frac{100}{60} = 1.67$

if $x = 2100$, $z = 1.67$

$p(x>2100) = p(z>1.67)$
$= p(z \geq 0) - p(0<z<1.67)$
$= 0.5 - \phi(1.67)$
$= 0.5 - 0.4525$
$= 0.0475$

$\therefore$ no of bulbs that are likely to last for more than 2100 hour is

$2500 \times 0.0475 = 118.75 \simeq 119$

ii) To find $p(x<1950)$

if $x = 1950$, $z = -0.83$

$p(x<-0.83) = p(z>0.83)$
$= p(z \geq 0) - p(0<z<0.83)$
$= 0.5 - \phi(0.83)$
$= 0.5 - 0.2967$
$= 0.2033$

$\therefore$ no of bulbs that are likely to last for less than 1950 hour is

$2600 \times 0.2033 = 508.25 \simeq \underline{508}$

iii) To find $P(1900 < x < 2100)$

ie. if $x = 1900$, $z = -1.67$

if $x = 2100$, $z = 1.67$

$P(-1.67 < z < 1.67) = 2P(0 < z < 1.67)$

$= 2\phi(1.67)$

$= 2(0.4525)$

$= 0.905$

$\therefore$ no of bulbs that are likely to last b/w 1900 and 2100 hours

$= 2500 \times 0.905 = 2262.5 \simeq 2263$

Q) In a normal distribution 31% of the items are under 45 and 8% of the items are over 64. Find the mean and S.D of the distribution.

>> Let μ and σ be the mean and S.D of the normal distribution

By data $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$

we have S.N.V $z = \frac{x - \mu}{\sigma}$

when $x = 45$, $z = \frac{45 - \mu}{\sigma} = z_1$ (say)

$x = 64$, $z = \frac{64 - \mu}{\sigma} = z_2$ (say)

Thus we have

$P(Z < z_1) = 0.31$ and $P(Z > z_2) = 0.08$

ie $0.5 + \phi(z_1) = 0.31$ and $0.5 - \phi(z_2) = 0.08$

$\Rightarrow \phi(z_1) = -0.19$ and $\phi(z_2) = 0.42$

Referring to the normal probability tables we have

$0.1915 (\simeq 0.19) = \phi(0.5)$ and $0.4192 (\simeq 0.42) = \phi(1.4)$

$\therefore \phi(z_1) = -\phi(0.5)$ and $\phi(z_2) = \phi(1.4)$

$\Rightarrow z_1 = -0.5$ and $z_2 = 1.4$

ie $\dfrac{45 - \mu}{\sigma} = -0.5$ and $\dfrac{64 - \mu}{\sigma} = 1.4$

$45 - \mu = -0.5\sigma$ | $64 - \mu = 1.4\sigma$

$-\mu + 0.5\sigma = -45$ | $-\mu + 1.4\sigma = -64$

$\Rightarrow -\mu + 0.5\sigma = -45$
$-\mu - 1.4\sigma = -64$

two equations 2 unknown is given by

$\boxed{\mu = 50}$ $\boxed{\sigma = 10}$ on solving we get

Thus mean = 50 and S.D = 10

$0.19 = p(0.5)$
$0.42 = \phi(1.4)$

Q) In a Examination 7% of students score less than 35% marks and 89% of students score less than 60% marks. Find the mean and SD of the marks are normally distributed, It is given that if $p(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz$ then $P(1.2263) = 0.39$ and $P(1.4757) = 0.43$

>> Let μ and σ be the mean and S.D of N.D

By data we have $P(x < 35) = 0.07$, $P(x < 60) = 0.89$

S.N.V $\Rightarrow$ $z = \frac{x - \mu}{\sigma}$

when $x = 35$, $z = \frac{35 - \mu}{\sigma} = z_1$ (say)

$x = 60$, $z = \frac{60 - \mu}{\sigma} = z_2$ (say)

Hence we have

$P(z < z_1) = 0.07$ and $P(z < z_2) = 0.89$

$0.5 + \phi(z_1) = 0.07$ and $0.5 + \phi(z_2) = 0.89$

$\phi(z_2) = 0.39$

$\phi(z_1) = -0.43$

using the given data in the RHS of these we have

$\phi(z_1) = -\phi(1.4757)$ and $\phi(z_2) = \phi(1.2263)$

$\Rightarrow z_1 = -1.4757$ and $z_2 = 1.2263$

ie. $\frac{35-\mu}{\sigma} = -1.4757$ and $\frac{60-\mu}{\sigma} = 1.2263$

$35-\mu = -1.4757\sigma$ | $60-\mu = 1.2263\sigma$

$\Rightarrow -\mu + 1.475\sigma = -35$ | $-\mu - 1.2263\sigma = -60$

$\therefore \boxed{\mu = 48.65}$ and $\boxed{\sigma = 9.25}$ on solving

Thus $\boxed{mean = 48.65}$ and $\boxed{S.D = 9.25}$

Joint Probability Distributions

Joint Probability and Joint distribution.

If X and Y are two discrete random variables, we define the joint probability function of X and Y by

$$P(X=x, Y=y) = f(x, y)$$

where $f(x, y)$ satisfy the conditions

$$f(x, y) \geq 0 \quad \text{and} \quad \sum_x \sum_y f(x, y) = 1$$

The 2nd condⁿ means that the sum over all the values of x and y is equal to one.

f is also referred to as joint probability density function of X and Y in the respective order.

The set of values of this funⁿ $f(x_i, y_j) = J_{ij}$ for $i = 1, 2, \ldots m$ $j = 1, 2, \ldots n$ is called the joint Probability distribution of X and Y.) These values are presented in the form of a two way table called the joint probability table.

X \ Y	y_1	y_2	——	y_n	Sum
x_1	J_{11}	J_{12}	--	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	--	J_{2n}	$f(x_2)$
--	--	--	--	--	—
x_m	J_{m1}	J_{m2}	--	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	——	$g(y_n)$	1

Marginal Probability distributions

In the joint prob table, $f(x_1), f(x_2), --- f(x_m)$ represents the sum of all the entries in the first row, second row, --- m^{th} row, $g(y_1), g(y_2) ——— g(y_n)$ represents the sum of all the entries in the first column, second column, ——— n^{th} column.

ie

$f(x_1) = J_{11} + J_{12} + ——— + J_{1n}$	$g(y_1) = J_{11} + J_{21} + ——— + J_{m1}$
$f(x_2) = J_{21} + J_{22} + ——— + J_{2n}$	$g(y_2) = J_{12} + J_{22} + ——— + J_{m2}$
———	———
$f(x_m) = J_{m1} + J_{m2} + ——— + J_{mn}$	$g(y_n) = J_{1n} + J_{2n} + ——— + J_{mn}$

$\{f(x_1), f(x_2) --- f(x_m)\}$ and $\{g(y_1), g(y_2) -- g(y_m)\}$ are called marginal probability distributions of X and Y respectively.

It should be noted that,

$f(x_1) + f(x_2) + ---- + f(x_m) = 1$ and

$g(y_1) + g(y_2) + ——— + g(y_n) = 1$

This is equivalent to writing

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) = \sum_{1}^{m} \sum_{1}^{n} J_{ij} = 1$$

It means that the total of all the entries in the joint probability table is equal to 1.

Expectation, Variance, Covariance and Correlation

* If X is a discrete random variable taking values $x_1, x_2, \ldots x_n$ having prob. fun $f(x)$ then the Expectation of X denoted by $E(X)$ or μ_x is defined by the relation

$X : x_1, x_2, \ldots x_n$
$x : x_1, x_2, \ldots, x_n$

$$\mu_x = E(X) = \sum_i x_i f(x_i)$$

$$\mu_y = E(Y) = \sum_j y_j g(y_j)$$

Further $E(XY) = \sum_{i,j} x_i y_j J_{ij}$

* The covariance of X and Y denoted by $\text{cov}(X,Y)$ is defined by the relation.

$$\text{cov}(X,Y) = E(XY) - \mu_x \mu_y$$

* further the correlation of X and Y denoted by $\rho(X,Y)$ is defined by the relation

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$$

* $\sigma_x^2 = E(X^2) - \mu_x^2$

Now $E(X^2) = \sum x_i^2 f(x_i)$

* $\sigma_y^2 = E(Y^2) - \mu_y^2$

Now $E(Y^2) = \sum y_j^2 g(y_j)$

$\sigma_x^2 = E(X^2) - [E(X)]^2$
Now $E(X^2) = \sum x^2 f(x)$
$E(X) = \sum x f(x)$

$\sigma_y^2 = E(Y^2) - [E(Y)]^2$
$E(Y^2) = \sum y^2 g(y)$
$E(Y) = \sum y g(y)$

1) The joint distribution of two random variables X and Y is as follows.

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Compute the following.

a) $E(X)$ and $E(Y)$ b) $E(XY)$ c) σ_X and σ_Y

d) $\operatorname{cov}(X, Y)$ e) $\rho(X, Y)$

→ The distribution (marginal distribution) of X and Y is as follows.

This distribution is obtained by adding the all the respective row entries and also the respective column entries.

Distribution of X:

x_i	1	5
$f(x_i)$	1/2	1/2

Distribution of Y:

y_j	-4	2	7
$g(y_j)$	3/8	3/8	1/4

a) $E(X) = \sum x_i f(x_i)$

$= 1 \times \frac{1}{2} + 5 \times \frac{1}{2}$

$= \frac{1}{2} + \frac{5}{2}$

$= \frac{6}{2}$

$= 3$

Thus $\boxed{\mu_X = E(X) = 3}$

$E(Y) = \sum y_j g(y_j)$

$= -4 \times \frac{3}{8} + 2 \times \frac{3}{8} + 7 \times \frac{1}{4}$

$= -\frac{1 \times 3}{2} + 1 \times \frac{3}{4} + \frac{7}{4}$

$= \frac{-3}{2} + \frac{3}{4} + \frac{7}{4}$

$= \frac{-6+3+7}{4} = \frac{4}{4} = 1$

$\boxed{\mu_Y = E(Y) = 1}$

b) $E(XY) = \sum x_i y_j J_{ij}$

$= 1\times(-4)\times\frac{1}{8} + 1\times2\times\frac{1}{4} + 1\times7\times\frac{1}{8}$

$+ 5\times(-4)\times\frac{1}{4} + 5\times2\times\frac{1}{8} + 5\times7\times\frac{1}{8}$

$= -\frac{4}{8} + \frac{2}{4} + \frac{7}{8} - 5 + \frac{10}{8} + \frac{35}{8}$

$= -\frac{1}{2} + \frac{1}{2} + \frac{7}{8} - 5 + \frac{10}{8} + \frac{35}{8}$ $= \frac{7}{8} + \frac{10}{8} + \frac{35}{8} - 5$

$= \frac{-4+4+7-80+10+35}{8}$ $= \frac{17}{8} + \frac{35}{8} - 5$

$= \frac{\cancel{52}}{\cancel{8}} - 5$

$= \frac{\cancel{5280}}{8} = \frac{3}{2}$ $= \frac{13}{2} - 5$

$= \frac{13-10}{2}$

$= \frac{3}{2}$

Thus $\boxed{E(XY) = \frac{3}{2}}$

c) $\sigma_x^2 = E(X^2) - \mu_x^2$

Now $E(X^2) = \sum x_i^2 f(x_i)$

$E(X^2) = 1^2\left(\frac{1}{2}\right) + 5^2\left(\frac{1}{2}\right)$

$= 1\times\frac{1}{2} + 25\times\frac{1}{2}$

$= \frac{1}{2} + \frac{25}{2}$

$= \frac{26}{2} = 13$

$\therefore \sigma_x^2 = 13 - 3^2$

$= 13 - 9$

$\sigma_x^2 = 4$

$\Rightarrow \boxed{\sigma_x = 2}$

$\sigma_y^2 = E(Y^2) - \mu_y^2$

$E(Y^2) = \sum y_j^2 g(y_j)$

$= (-4)^2\cdot\frac{3}{8} + 2^2\left(\frac{3}{8}\right) + 7^2\cdot\frac{1}{4}$

$= 16\times\frac{3}{8} + 4\times\frac{3}{8} + 49\times\frac{1}{4}$

$= 6 + \frac{3}{2} + \frac{49}{4}$

$= \frac{24+6+49}{4}$

$= \frac{79}{4}$

$\therefore \sigma_y^2 = \frac{79}{4} - 1$

$= \frac{79-4}{4}$

$= \frac{75}{4}$

$\Rightarrow \sigma_y = \sqrt{75/4} = 4.33$

d) $cov(X, Y) = E(XY) - \mu_x \mu_y$

$= 3/2 - 3 \times 1$

$= \frac{3}{2} - 3 = \frac{3-6}{2} = -3/2$

$\therefore \boxed{cov(X, Y) = -3/2}$

e) $\rho(X, Y) = \frac{cov(X, Y)}{\sigma_x \sigma_y}$

$= \frac{-3/2}{2 \times 4.33} = -0.1732$

Thus $\boxed{\rho(X, Y) = -0.1732}$

2) The joint prob distribution table for two random variables X and Y is as follows.

X \ Y	−2	−1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal prob distributions of X and Y.

Also compute

a) Expectations of X, Y and XY

b) S.D of X, Y

c) Covariance of X and Y

d) Correlation of X and Y.

further verify that X and Y are dependent random variables. Also find $P(X+Y > 0)$

Marginal distributions of X and Y are got by adding all the respective row entries and the respective column entries.

x_i	1	2
$f(x_i)$	0.6	0.4

y_j	-2	-1	4	5
$g(y_j)$	0.3	0.3	0.1	0.3

a) $\mu_X = E(X) = \sum_i x_i f(x_i)$

$= 1\times 0.6 + 2\times 0.4$

$= 0.6 + 0.8$

$\boxed{\mu_X = 1.4}$

$\mu_Y = E(Y) = \sum y_j g(y_j)$

$= -2\times 0.3 + (-1)\times 0.3 + 4\times 0.1 + 5\times 0.3$

$\boxed{\mu_Y = 1}$

$E(XY) = \sum x_i y_j J_{ij}$

$= 1\times -2\times 0.1 + 1\times -1\times 0.2 + 1\times 4\times 0 + 1\times 5\times 0.3$
$+ 2\times -2\times 0.2 + 2\times -1\times 0.1 + 2\times 4\times 0.1 + 2\times 5\times 0$

$= -0.2 - 0.2 + 1.5 - 0.8 - 0.2 + 0.8 = 0.9$

$\boxed{E(XY) = 0.9}$

b) $\sigma_X^2 = E(X^2) - \mu_X^2$

$E(X^2) = \sum x_i^2 f(x_i)$

$= 1\times 0.6 + 4\times 0.4$

$= 2.2$

Now $\sigma_X^2 = 2.2 - (1.4)^2$

$= 2.2 - 1.96$

$\boxed{\sigma_X^2 = 0.24}$

$\Rightarrow \boxed{\sigma_X = 0.49}$

$\sigma_Y^2 = E(Y^2) - \mu_Y^2$

$E(Y^2) = \sum y_j^2 g(y_j)$

$= 4\times 0.3 + 1\times 0.3 + 16\times 0.1 + 25\times 0.3$

$= 10.6$

Now $\sigma_Y^2 = 10.6 - 1$

$\boxed{\sigma_Y^2 = 9.6}$

$\Rightarrow \boxed{\sigma_Y = 3.1}$

c) $cov(X, Y) = E(XY) - E(X)E(Y)$

$= 0.9 - (1.4) \times 1$

$= -0.5$

$\therefore \boxed{cov(X, Y) = -0.5}$

d) Correlation of X and $Y = \rho(X, Y) = \dfrac{cov(X, Y)}{\sigma_X \sigma_Y}$

ie $\rho(X, Y) = \dfrac{-0.5}{0.49 \times 3.1} = -0.3$

$\therefore \boxed{\rho(X, Y) = -0.3}$

if X and Y are independent random variables we must have $f(x_i)\, g(y_j) = J_{ij}$

It can be seen that $f(x_1)\, g(y_1) = 0.6\,(0.3) = 0.18$ and

$J_{11} = 0.1$ ie $f(x_1)\, g(y_1) \neq J_{11}$

Similarly for others also the cond^n is not satisfied.

Thus X & Y are dependent random variables.

To find $P(X+Y>0)$:

we have $X = \{x_i\} = \{x_1, x_2\} = \{1, 2\}$ respectively.

$Y = \{y_j\} = \{y_1, y_2, y_3, y_4\} = \{-2, -1, 4, 5\}$ resp.

$X+Y>0$ is possible when (X, Y) takes the values

$(x_1, y_3) = (1, 4)$; $(x_1, y_4) = (1, 5)$; $(x_2, y_2) = (2, -1)$;

$(x_2, y_3) = (2, 4)$; $(x_2, y_4) = (2, 5)$.

Hence $P(X+Y>0) = J_{13} + J_{14} + J_{22} + J_{23} + J_{24}$

$= 0 + 0.3 + 0.1 + 0.1 + 0$

$= 0.5$.

3) Suppose X and Y are independent random variables with the following respective distribution. find the joint distribution of X and Y Also verify that Cov (X, Y) = 0 (5)

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	−2	5	8
$g(y_j)$	0.3	0.5	0.2

→ Since X and Y are independent, the joint distribution $J(x, y)$ is obtained by using the definition $f(x_i)\, g(y_j) = J_{ij}$

J_{ij} are obtained on multiplication of the marginal entries.

The given data and the required J_{ij} is exhibited in the following table.

X \ Y	$y_1 = -2$	$y_2 = 5$	$y_3 = 8$	$f(x_i)$
$x_1 = 1$	J_{11}	J_{12}	J_{13}	0.7
$x_2 = 2$	J_{21}	J_{22}	J_{23}	0.3
$g(y_j)$	0.3	0.5	0.2	1

$J_{11} = 0.7 \times 0.3 = 0.21$

$J_{12} = 0.7 \times 0.5 = 0.35$

$J_{13} = 0.7 \times 0.2 = 0.14$

$J_{21} = 0.3 \times 0.3 = 0.09$

$J_{22} = 0.3 \times 0.5 = 0.15$

$J_{23} = 0.3 \times 0.2 = 0.06$

The joint distribution table is as follows.

X \ Y	−2	5	8	$f(x_i)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$	0.3	0.5	0.2	1

we have $cov(X, Y) = E(XY) - \mu_x \mu_y$

$\mu_x = E(X) = \sum x_i f(x_i)$

$= 1 \times 0.7 + 2 \times 0.3$

$\boxed{\mu_x = 1.3}$

$\mu_y = E(Y) = \sum y_j g(y_j)$

$= -2 \times 0.3 + 5 \times 0.5 + 8 \times 0.2$

$= 3.5$

$E(XY) = \sum x_i y_j J_{ij} = 4.55$

$\therefore \ cov(X, Y) = 4.55 - 1.3 \times 3.5 = 0$

Thus the result $\underline{cov(X, Y) = 0}$ for independent random variables X and Y is verified.

4) X and Y are independent random variables, X take values 2, 5, 7 with prob 1/2, 1/4, 1/4 respectively, Y take values 3, 4, 5 with prob 1/3, 1/3, 1/3

a) Find the joint prob distribution of X and Y.

b) S.T the covariance of X and Y is equal to zero.

c) Find the prob distribution of $Z = X + Y$.

6

>> Given data is as follows.

x_i	2	5	7
$f(x_i)$	1/2	1/4	1/4

y_j	3	4	5
$g(y_j)$	1/3	1/3	1/3

a) we have $J_{ij} = f(x_i).g(y_j)$ for $i,j = 1,2,3$

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$J_{11} = \frac{1}{6}$ $J_{12} = 1/6$, $J_{13} = 1/6$

$J_{21} = 1/12$ $J_{22} = 1/12$, $J_{23} = 1/12$

$J_{31} = 1/12$ $J_{32} = 1/12$, $J_{33} = 1/12$

The joint distribution table is as follows.

X \ Y	3	4	5	$f(x_i)$
2	1/6	1/6	1/6	1/2
5	1/12	1/12	1/12	1/4
7	1/12	1/12	1/12	1/4
$g(y_j)$	1/3	1/3	1/3	1

b) $\text{cov}(X;Y) = E(XY) - \mu_x \mu_y$

$\mu_x = E(X) = \sum x_i f(x_i)$

$= 2 \times \frac{1}{2} + 5 \times \frac{1}{4} + 7 \times \frac{1}{4} = 4$

$\mu_y = E(Y) = \sum_j y_j g(y_j) = 4$

$E(XY) = \sum x_i y_j J_{ij} = 16$

$\therefore \text{cov}(X,Y) = 16 - 4(4) = 0$

$\therefore \underline{\text{cov}(X,Y) = 0}$

c) $Z = X + Y$

Let $z_i = x_i + y_j$ and hence

$\{z_i\} = \{5, 6, 7, 8, 9, 10, 10, 11, 12\}$

$\Rightarrow \{z_i\} = \{5, 6, 7, 8, 9, 10, 11, 12\}$

The corresponding probabilities are,

$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}+\frac{1}{12}, \frac{1}{12}, \frac{1}{12}$

$\Rightarrow \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}$

The probability distribution of $Z = X+Y$ is as follows.

Z	5	6	7	8	9	10	11	12
p(z)	1/6	1/6	1/6	1/12	1/12	1/6	1/12	1/12

we note that $\sum p(z) = 1$

5) Given the following joint distribution of the random variables X and Y, find the corresponding marginal distribution. Also compute the covariance and the correlation of the random variables X and Y

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

$\therefore$ Marginal distributions of X and Y as follows. (7)

x_i	2	4	6
$f(x_i)$	1/4	1/2	1/4

y_j	1	3	9
$g(y_j)$	1/2	1/3	1/6

$$\text{cov}(X, Y) = E(XY) - \mu_x \mu_y \quad \text{——— (1)}$$

$$E(X) = \sum x_i f(x_i) = 2\left(\tfrac{1}{4}\right) + 4\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{4}\right) = 4$$

$$E(Y) = \sum y_j g(y_j) = 1\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{3}\right) + 9\left(\tfrac{1}{6}\right) = 3$$

$$E(XY) = \sum x_i y_j J_{ij} = 12$$

$$(1) \Rightarrow \text{Cov}(X, Y) = 12 - 4(3) = 0$$

Thus $\text{cov}(X, Y) = 0$

Correlation of X and Y $= \rho(X, Y) = \dfrac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$

Thus $\rho(X, Y) = 0$

6) The joint prob distribution of two discrete random variables X and Y is given by $f(x, y) = k(2x + y)$ where x and y are integers such that $0 \le x \le 2$, $0 \le y \le 3$

a) Find the value of the constant k.

b) Find the marginal prob distributions of X and Y.

c) Show that the random variables X and Y are dependent.

>> $X = \{x_i\} = \{0, 1, 2\}$ and

$Y = \{y_j\} = \{0, 1, 2, 3\}$

$f(x, y) = k(2x+y)$ and the joint prob distribution table is formed as follows.

X \ Y	0	1	2	3	Sum
0	0	k	2k	3k	6k
1	2k	3k	4k	5k	14k
2	4k	5k	6k	7k	22k
Sum	6k	9k	12k	15k	42k

a) we must have $42k = 1$

$\therefore \boxed{k = 1/42}$

b) Marginal prob. distribution is as follows.

x_i	0	1	2
$f(x_i)$	$6/42$	$14/42$	$22/42$
	$= 1/7$	$= 1/3$	$= 11/21$

y_j	0	1	2	3
$g(y_j)$	$6/42$	$9/42$	$12/42$	$15/42$
	$1/7$	$3/14$	$2/7$	$5/14$

c) It can be easily seen that

$$f(x_i)\, g(y_j) \neq P_{ij}$$

Hence the random variables are dependent.

7) Consider the joint prob distribution as in Example-6 and compute the following. ⑧

a) $E(X), E(Y), E(XY)$ b) $E(X^2), E(Y^2)$ c) σ_x, σ_y

>> a) $E(X) = \sum x_i f(x_i)$

$$= 0 \times \frac{6}{42} + 1 \times \frac{14}{42} + 2 \times \frac{22}{42} = \frac{58}{42} = \frac{29}{21}$$

$E(Y) = \sum y_j g(y_j)$

$$= 0 \times \frac{6}{42} + 1 \times \frac{9}{42} + 2 \times \frac{12}{42} + 3 \times \frac{15}{42} = \frac{13}{7}$$

$E(XY) = \sum x_i y_j T_{ij}$

$$= 0 + \left(0 + \frac{3}{42} + \frac{8}{42} + \frac{15}{42}\right) + \left(0 + \frac{10}{42} + \frac{24}{42} + \frac{42}{42}\right)$$

$$= \frac{102}{42} = \frac{17}{7}$$

b) $E(X^2) = \sum_i x_i^2 f(x_i)$

$$= 0 + 1 \times \frac{14}{42} + 4 \times \frac{22}{42} = \frac{102}{42} = \frac{17}{7}$$

$$E(Y^2) = \sum y_j^2 g(y_j) = \frac{192}{42} = \frac{32}{7}$$

c) $\sigma_x^2 = E(X^2) - \mu_x^2 = \frac{230}{441}$

$$\Rightarrow \sigma_x = \frac{\sqrt{230}}{21}$$

$$\sigma_y^2 = E(Y^2) - \mu_y^2 = \frac{495}{441}$$

$$\Rightarrow \sigma_y = \frac{\sqrt{495}}{21}$$

Thus $\sigma_x = 0.72$ and $\sigma_y = 1.06$

8) Consider the joint prob distribution as in Ex - 6. and find

a) $P(X=1, Y=2)$ b) $P(X=2, Y=1)$

c) $P(X\geq 1, Y\leq 2)$ d) $P(X+Y>2)$

Let $X=\{x_i\} = \{x_1, x_2, x_3\} = \{0, 1, 2\}$

$Y=\{y_j\} = \{y_1, y_2, y_3, y_4\} = \{0, 1, 2, 3\}$

also $J_{11} = 0$, $J_{12} = 1/42$, $J_{13} = 2/42$ $J_{14} = 3/42$

$J_{21} = 2/42$ $J_{22} = 3/42$ $J_{23} = 4/42$, $J_{24} = 5/42$

$J_{31} = 4/42$ $J_{32} = 5/42$ $J_{33} = 6/42$ $J_{34} = 7/42$

a) $P(X=1, Y=2) = f(x_2, y_3) = J_{23} = \frac{4}{42} = \frac{2}{21}$

b) $P(X=2, Y=1) = f(x_3, y_2) = J_{32} = 5/42$

c) $P(X\geq 1, Y\leq 2)$

(X, Y) take the values

$(1,0)$ $(1,1)$ $(1,2)$, $(2,0)$ $(2,1)$ $(2,2)$

They are respectively

(x_2, y_1) (x_2, y_2) (x_2, y_3) (x_3, y_1) (x_3, y_2) (x_3, y_3)

$\therefore P(X\geq 1, Y\leq 2) = J_{21} + J_{22} + J_{23} + J_{31} + J_{32} + J_{33}$

$= \frac{2}{42} + \frac{3}{42} + \frac{4}{42} + \frac{5}{42} + \frac{4}{42} + \frac{6}{42}$

$= \frac{24}{42} = \frac{4}{7}$

Thus $P(X\geq 1, Y\leq 2) = \frac{4}{7}$

d) To find $P(X+Y>2)$, (X,Y) take the values ⑨

$(0,3)\ (1,2)\ (1,3)\ (2,1)\ (2,2)\ (2,3)$

They are respectively

$(x_1,y_4)\ (x_2,y_3)\ (x_2,y_4)\ (x_3,y_2)\ (x_3,y_3)\ (x_3,y_4)$

$\therefore P(X+Y>2) = J_{14}+J_{23}+J_{24}+J_{32}+J_{33}+J_{34}$

$$= \frac{3+4+5+5+6+7}{42}$$

$$= \frac{30}{42} = \frac{5}{7}$$

Thus $P(X+Y>2) = \frac{5}{7}$

9) A fair coin is tossed thrice. The random variables X and Y are defined as follows.

X = 0 or 1 according as head or tail occurs on the first toss.

Y = Number of heads.

a) Determine the distributions of X and Y

b) determine the joint distribution of X and Y

c) obtain the expectations of X, Y and XY. Also find S.Ds of X and Y

d) Compute covariance and correlation of X and Y

>> The sample space S and the association of random variables X and Y is given by the foll^ng table.

S	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	0	0	0	0	1	1	1	1
Y	3	2	2	1	2	1	1	0

a) The prob distribution of X and Y is found as

$X = \{x_i\} = \{0, 1\}$ and $Y = \{y_j\} = \{0, 1, 2, 3\}$

$p(X = 0)$ is $\frac{4}{8} = \frac{1}{2}$

$p(X = 1)$ is $\frac{4}{8} = \frac{1}{2}$

$p(Y = 0)$ is $1/8$

$p(Y = 1)$ is $3/8$

$p(Y = 2)$ is $3/8$

$p(Y = 3)$ is $1/8$

Thus we have the following probability distribution of X and Y

x_i	0	1
$f(x_i)$	$1/2$	$1/2$

y_j	0	1	2	3
$g(y_j)$	$1/8$	$3/8$	$3/8$	$1/8$

b) The joint distribution of X and Y is found by Computing.,

$J_{ij} = P(X = x_i, Y = y_j)$ where we have

$x_1 = 0$, $x_2 = 1$ and $y_1 = 0$, $y_2 = 1$, $y_3 = 2$, $y_4 = 3$

$J_{11} = P(X = 0, Y = 0) = 0$

($X = 0$ implies that there is a head turn out and Y the total number heads 0 is impossible)

$J_{12} = P(X = 0, Y = 1) = \frac{1}{8}$ corresponding to the outcome HTT

$J_{13} = P(X = 0, Y = 2) = \frac{2}{8} = \frac{1}{4}$

out comes are HHT and HTH

$J_{14} = P(X = 0, Y = 3) = \frac{1}{8}$ out comes is HHH

$J_{21} = P(X = 1, Y = 0) = \frac{1}{8}$ out come is TTT

$J_{22} = P(X = 1, Y = 1) = \frac{2}{8} = \frac{1}{4}$

out comes are THT, TTH

$J_{23} = P(X = 1, Y = 2) = \frac{1}{8}$ out come is THH

$J_{24} = P(X = 1, Y = 3) = 0$ since the outcome is impossible.

The required joint prob distribution of x and y is as follows.

X \ Y	0	1	2	3	Sum
0	0	1/8	1/4	1/8	1/2
1	1/8	1/4	1/8	0	1/2
Sum	1/8	3/8	3/8	1/8	1

c) $\mu_x = E(x) = \Sigma x_i f(x_i)$

$= 0 \cdot \left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$

$\mu_y = E(y) = \Sigma y_j g(y_j)$

$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{12}{8} = \frac{3}{2}$

$E(xy) = \Sigma x_i y_j p_{ij} \quad = \frac{1}{2}$

$\sigma_x^2 = E(x^2) - \mu_x^2$

$\sigma_x^2 = \frac{1}{4}$

$\boxed{\sigma_x = 1/2}$

$\sigma_y^2 = E(y^2) - \mu_y^2$

$= 3/4$

$\boxed{\sigma_y = \sqrt{3}/2}$

d) $\text{cov}(x, y) = E(xy) - \mu_x \mu_y$

$= \frac{1}{2} - 3/4 = -1/4$

$\rho(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$

$= -\frac{1}{\sqrt{3}}$

Thus $\text{cov}(x, y) = -1/4$ and $\rho(x, y) = -1/\sqrt{3}$