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ENGINEERING MATHEMATICS - [17MAT41]

MODULE-4 PROBABILITY DISTRIBUTIONS

SEM-4

Part-I

INTRODUCTION: COURSE OUTCOME : [Develop probability distribn of discrete, continuous random variable, & joint distrbn prob in digital signal processing, inform theory & design eng]

Probability :- In real time statements, we make where we use probability unknowingly, here are some normal conversations we make ; where there is existence of probability.

Ex : 1) I have a good chance of being selected to the job.
2) It might rain today.
3) I might get 100/100 in mathematics.
4) Toss of a coin (Heads or tail).

Many Examples gives an insight of probability.

Mathematicaly;

Probability :- If the outcome of a trial consist n exhaustive, mutually exclusive, equally possible cases, of which m of them are favorable cases to an event E, then probability of happening of every event E, usually denoted by : P(E) / p is defined by ;

$$P(E) = p = \frac{\text{no/. of favorable cases}}{\text{no/. of possible cases}} = \frac{m}{n}.$$

Note :

* MODULE-1 (NUM METHODS) [CO: Solve 1st & 2nd order ODE, using Single & multistep Numerical methods]

1) If $P(E) = 1$, E is called a sure event.
If $P(E) = 0$, E is called an impossible event.

2) $p + q = 1$ and $P(E) + P(\bar{E}) = 1$, where; $q = \frac{n-m}{n}$

Examples :-

1) The probability of getting : (a) a number greater than 2.
(b) an ~~even~~ odd number
(c) an even number

when a die is thrown.

Soln : (a) Number of possible outcomes $= \boxed{6 = n}$ // die has 6 faces.
Number of favorable outcomes $= \boxed{4 = m}$

↳ Since ; no's greater than 2 are : $\underbrace{3, 4, 5, 6}$ = 4 outcomes

∴ probability of getting no greater than 2 $= \frac{m}{n} = \frac{\cancel{4}\,2}{\cancel{6}\,3} = \frac{2}{3}$ //

(b) Number of favorable outcomes (m) = 3, since ; odd no's are $\underbrace{1, 3, 5}$ = 3 outcomes

∴ probability of getting odd number $= \frac{m}{n} = \frac{\cancel{3}\,1}{\cancel{6}\,2} = \frac{1}{2}$ //

(c) Number of favorable outcomes (m) = 3 , since, even no's are $\underbrace{2, 4, 6}$ = 3 outcomes

∴ probability of getting even number $= \frac{m}{n} = \frac{\cancel{3}}{\cancel{6}\,2} = 1/2$ //

PROBABILITY DISTRIBUTIONS & JOINT PROBABILITY

DISTRIBUTIONS.

Define Random variable :-
In a random experiment, if a real variable is associated with every outcome then, it is called a Random variable / Stochastic variable.

Ex:- Consider an experiment of tossing 2 coins :-

$S = \{HH, HT, TH, TT\}$.

Define, X = no of heads, then.
$X = \{0, 1, 2\}$ is the random variable on S.

Random Experiment :- An Experiment whose outcome is unpredictable is known as Random Experiment.

Sample space :- The set of all possible outcomes of a random experiment is known as the sample space.

Eg:- Tossing a coin : $S = \{H, T\}$

Event : An event is a subset of the sample space.

Probability :- If E is an event of sample space S, then the probability of E is defined by;

$$P(E) = \frac{\text{Favorable no. of events}}{\text{Total no of outcomes}} = \frac{O(E)}{O(S)}$$ → order of E, order of S.

Discrete random variable :-
If a random variable takes finite or countably infinite number of values, then it is called discrete random variable.

Ex:- 1) Throwing a die & observing the numbers on the face.
2) Tossing a coin & observing the outcome.

Continuous random variable :- If a random variable takes non-countable infinite numbers of values, then it is non-discrete (or) Continuous random variable.

Ex :- 1) Observing a pointer on a Voltmeter
2) Conducting a survey on the life of electric bulbs.

Probability function :-
If for each value x_i of a discrete random variable X, we assign a real number $p(x_i)$ $\ni$: i) $p(x_i) \geq 0$
ii) $\sum_i p(x_i) = 1$.
then the function $p(x)$ is probability function.

Discrete probability function :-
If the probability that X takes the values x_i is p_i, then $P(X = x_i) = p_i$ (or) $p(x_i)$, the set of values $[x_i, p(x_i)]$ is called discrete probability fn.

* The function : $P(x)$ is called probability density function. (pdf)

(or) * The function : $P(x)$ is called probability mass function (pmf)

The mean and variance of the discrete distribution is :-

Mean : $\boxed{\mu = E[X] = \sum_i x_i P(x_i)}$

Variance : $V = E[X^2] - (E[X])^2$

$$\boxed{V = \sum_i (x_i - \mu)^2 . p(x_i)}$$

Standard deviation : $\boxed{\sigma = \sqrt{V}}$, where V is the variance.

Problems & Solutions :-

1) Show that the following distributions represent a discrete probability distribution (pdf), Find its mean & variance.

x	10	20	30	40
$p(x)$	1/8	3/8	3/8	1/8

probability ~~distn~~ discrete / density fn :-

A function $P(x)$ is said to be a probability density/discrete function ; if i) $P(x_i) \geq 0$.
ii) $\sum_i P(x_i) = 1$.

Soln :- Given

x	10 (x_1)	20 (x_2)	30 (x_3)	40 (x_4)
$p(x)$	1/8 $p(x_1)$	3/8 $p(x_2)$	3/8 $p(x_3)$	1/8 $p(x_4)$

Since, WKT ;

$P(x)$ is probability discrete fn ; if i) $P(x_i) \geq 0$
ii) $\sum_i P(x_i) = 1$

Consider ; $i=1$, $p(x_1) = 1/8 > 0$. $i=3$, $p(x_3) = 3/8 > 0$
$i=2$, $p(x_2) = 3/8 > 0$. $i=4$, $p(x_4) = 1/8 > 0$.

$\therefore$ clearly; $p(x) > 0$.

Now, to find : $\sum_i p(x_i)$

Consider; $\sum_{i=1}^{4}(p(x)) = p(x_1) + p(x_2) + p(x_3) + p(x_4)$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$

$\therefore \boxed{\sum(p(x)) = 1}$

$\therefore$ Since 2 Conditions are satisfied.

$\therefore p(x)$ is a probability density function.

Now, we find : Mean & Variance.

$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4$

$10 + 20 + 30 + 40$

Mean $= \mu = E[X] = \sum_i x_i p(x_i)$

$= [x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 p(x_3) + x_4 p(x_4)]$

$= \cancel{10}^{5} \cdot \frac{1}{\cancel{8}_4} + \cancel{20}^{5} \times \frac{3}{\cancel{8}_4} + \cancel{30}^{15} \times \frac{3}{\cancel{8}_4} + \cancel{40}^{5} \times \frac{1}{\cancel{8}}$

$\therefore \boxed{\mu = 25}$, Mean.

Now, Variance $V = E[X^2] - (E[X])^2$

Since; $\mu = E[X]$, therefore; $E[X^2] = \sum x^2 p(x)$

$E[X] = \mu = \sum x\, p(x)$

$\therefore E[X^2] = \sum x^2 P(x) = [x_1^2 p(x_1) + x_2^2 p(x_2) + x_3^2 p(x_3) + x_4^2 p(x_4)]$

$\therefore \boxed{E[X^2] = 700}$

$\therefore V = E[X^2] - (E[X])^2$

$= 700 - (25)^2$

$\therefore \boxed{V = 75}$

2) The probability function of a finite random variable x, is given by the table; (5)4

x	-2	-1	0	1	2	3
$p(x)$	0.1	K	0.2	$2K$	0.3	K.

Find the value of K, mean & Variance.

Soln:-

	x_1	x_2	x_3	x_4	x_5	x_6
x	-2	-1	0	1	2	3
$p(x)$	0.1	K	0.2	$2K$	0.3	K.
	$p(x_1)$	$p(x_2)$	$p(x_3)$	$p(x_4)$	$p(x_5)$	$p(x_6)$.

Since, Given that; $P(X)$ is ~~the~~ probability discrete/density function;

$\therefore$ i) $p(X) \geq 0$.
ii) $\sum p(X) = 1$. } satisfies.

$\therefore$ Consider; $\sum p(X) = 1$

$\therefore \; p(x_1) + p(x_2) + p(x_3) + p(x_4) + p(x_5) + p(x_6) = 1.$

$\Rightarrow [0.1 + K + 0.2 + 2K + 0.3 + K] = 1.$

$4K = 0.4$

$\boxed{K = 0.1}$

$\therefore \; P(X) \geq 0.$

Now, we Compute, Mean: $\mu = E[X] = \sum x \cdot P(x).$

$[x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3) + x_4 p(x_4) + x_5 p(x_5) + x_6 p(x_6)]$

$= [-2 \times 0.1 + (-1) \times (K) + (0)(0.2) + 1 \times (2K) + 2(0.3) + 3(K)]$, $K = 0.1$

$\therefore \boxed{\mu = 0.8}$, Mean..

$E[X^2] = \sum x^2 P(x)$, $E[X^2] = x_1^2 p(x_1) + x_2^2 p(x_2) + \cdots$, $\boxed{E[X^2] = 2.8}$

$\therefore$ Variance, $V = E[X^2] - (E[X])^2$

$= 2.80 - (0.8)^2$

$\therefore \boxed{V = 2.16.}$

3) Find the value of K such that the following distribution represents a finite probability distribution, Hence find its mean & standard deviation. Also find:

$p(x \leq 1)$, $p(x > 1)$, $P(-1 < x \leq 2)$.

x	-3	-2	-1	0	1	2	3
$p(x)$	K	$2K$	$3K$	$4K$	$3K$	$2K$	K.

Soln:- Given:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x	-3	-2	-1	0	1	2	3
$p(x)$	K	$2K$	$3K$	$4K$	$3K$	$2K$	K.
	$p(x_1)$	$p(x_2)$	$p(x_3)$	$p(x_4)$	$p(x_5)$	$p(x_6)$	$p(x_7)$

Since, Given that, the following distribution represents a finite probability, $\Rightarrow$ 1) $p(x) \geq 0$

2) $\Sigma p(x) = 1$.

Consider $\Rightarrow \sum_{i=1}^{7} p(x_i) = 1$

$\therefore [p(x_1) + p(x_2) + p(x_3) + p(x_4) + p(x_5) + p(x_6) + p(x_7)] = 1.$

$\Rightarrow [K + 2K + 4K + 3K + 3K + 2K + K] = 1.$

$16K = 1 \quad \Rightarrow \boxed{K = 0.0625}$

Now, we find mean, $\mu = E[x] = \sum_i x_i p(x_i)$.

$\mu = x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3) + x_4 p(x_4) + x_5 p(x_5) + x_6 p(x_6) + x_7 p(x_7)$

$\boxed{\mu = 0}$, Mean.

$E[x^2] = \Sigma x^2 p(x)$, $\boxed{E[x^2] = 40K = 2.5}$

$\therefore \quad V = E[x^2] - (E[x])^2$

$= 2.5 - 0$, $\boxed{V = 2.5}$, Variance.

$\therefore$ S.D, $\sigma = \sqrt{V} = \sqrt{2.5}$

$\therefore \boxed{\sigma = 1.5811}$, Standard deviation.

Now, we find ;

i) $P(x \leq 1)$, Since from the table, the values of x , which is ≤ 1 are:-

$x_1 = -3$, $x_2 = -2$, $x_3 = -1$, $x_4 = 0$, $x_5 = 1$ $\longrightarrow$ (less than or Equal to 1)

$$\therefore p(x \leq 1) = p(-3) + p(-2) + p(-1) + p(0) + p(1)$$
$$= K + 2K + 3K + 4K + 3K$$
$$= 13K$$
$$\therefore \boxed{p(x \leq 1) = 0.8125}$$

ii) $p(x > 1)$, Since, from table, values of $x > 1$ are:-

$x_6 = 2$, $x_7 = 3$

$$\therefore p(x > 1) = p(x_6) + p(x_7)$$
$$= p(2) + p(3)$$
$$= 2K + K = 3K$$
$$\therefore \boxed{p(x > 1) = 0.1875}$$

iii) $p(-1 < x \leq 2) = p(0) + p(1) + p(2)$
$$= 4K + 3K + 2K$$
$$= 9K$$
$$\therefore \boxed{p(-1 < x \leq 2) = 0.5625}$$

4) A random variable x has the following probability function for various values of x.

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find: i) K iii) $p(x \geq 6)$

ii) $p(x < 6)$ iv) $p(3 < x \leq 6)$.

Soln :- Since, following distribution represents discrete probability fn : 5)

By defn ; We have : i) $P(X) \geq 0$

ii) $\sum p(X) = 1$.

Consider : $\sum p(X) = 1$.

i) $\Rightarrow p(x_1) + p(x_2) + p(x_3) + p(x_4) + p(x_5) + p(x_6) + p(x_7) + p(x_8) = 1$.

$0 + K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$.

$9K + 10K^2 = 1$

$10K^2 + 9K - 1 = 0$.

$$\begin{array}{c} -10 \\ \wedge \\ +10 \; -1 \end{array}$$

$10K^2 + 10K - 1K - 1 = 0$.

$10K(K+1) - 1(K+1) = 0$

$(10K-1)(K+1) = 0$, $10K = 1$ $\boxed{K = 1/10}$ $\boxed{K = -1}$

Now, we find ;

ii) $P(x<6)$

$= P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= 8K + K^2$

$= 0.8 + 0.01$

$\boxed{P(x<6) = 0.81}$

iii) $P(x \geq 6)$

$P(x \geq 6) = P(6) + P(7)$

$= 2K^2 + 7K^2 + K$

$\boxed{P(x \geq 6) = 0.19}$

iv) $P(3 < x \leq 6) = P(4) + P(5) + P(6)$

$= 3K + K^2 + 2K^2$

$= 0.3 + 0.03$

$\therefore \boxed{P(3 < x \leq 6) = 0.33}$.

5) A coin is tossed 3 times, let X denote the number of heads, showing up, find the distribution of X. Also find the mean and variance. (7)

Soln:- Let $S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$.

where; X = number of heads.

$\therefore X = \{0, 1, 2, 3\}$ // (where X contains no of possibilities of heads, when 3 coins are tossed)

no head, 1 head, 2 heads, 3 heads.

Now, we compute :-

$P(X=0) = \frac{\text{Fav no of Events}}{\text{Total no of Events}} = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$

$P(X=0) \rightarrow$ probability of getting 0 heads.

$P(X=1) \rightarrow$ probability of getting 1 heads

$P(X=2) \rightarrow$ probability of getting 2 heads

$\therefore$ The probability distribution table is given by;

X	0	1	2	3
P(x)	1/8	3/8	3/8	1/8.

$\therefore$ By defn; i) $P(X) \geq 0$
ii) $\sum P(X) = 1$.

Now, we compute : Mean; $\mu = E[X] = \sum x \cdot P(x)$.

$= x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + x_4 P(x_4)$

$\boxed{\mu = 1.5}$, Mean.

$$E[X^2] = \sum x^2 p(x)$$
$$\boxed{E[X^2] = 3}$$

Now, we Compute ; Variance; $V = E[X^2] - (E[X])^2$

$$= 3 - (1.5)^2$$
$$\boxed{V = 0.75}, \text{Variance.}$$

6) Find : $E[X]$, $E[X^2]$ and σ^2 for the probability function defined by the table.

x	1	2	3	-----	n
$p(x)$	K	$2K$	$3K$	-----	nK.

Soln :- By defn :- i) $P(X) \geq 0$

ii) $\sum p(x) = 1$.

Consider; $\sum_{i=1}^{n} p(x_i) = 1$.

$$[p(x_1) + p(x_2) + p(x_3) + \cdots + p(x_n)] = 1.$$

$$\Rightarrow K + 2K + 3K + \cdots + nK = 1.$$

$$K[\underbrace{1+2+3+\cdots+n}] = 1. \quad \text{// summation of n-series.}$$

$$K \frac{n(n+1)}{2} = 1$$

$$\therefore \boxed{K = \frac{2}{n(n+1)}}$$

Now, Mean, $\mu = E[X] = \sum x \cdot p(x)$

$$= x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3) + \cdots + x_n p(x_n).$$
$$= 1(K) + 2(2K) + 3(3K) + \cdots + n(nK).$$
$$= K + 4K + 9K + \cdots + n^2 K.$$
$$= K[1^2 + 2^2 + 3^2 + \cdots + n^2] \quad , \text{Since, } K = \frac{2}{n(n+1)}$$

$$= \frac{2}{n(n+1)}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{2(2n+1)}{6}$$

$$\therefore \boxed{\mu = \frac{2n+1}{3}}$$

$$E[x^2] = \sum x^2 p(x). = [k + 8k + 27k + \dots + n^3 k.]$$

$$= k[1 + 8 + 27 + \dots + n^3]$$

$$= k[\underbrace{1^3 + 2^3 + 3^3 + \dots + n^3}]$$

$$= \frac{2}{n(n+1)}\left[\left[\frac{n(n+1)}{2}\right]^2\right] = \frac{n^2(n+1)^2}{4} \cdot \frac{2}{n(n+1)}.$$

$$\boxed{E[x^2] = \frac{n(n+1)}{2}}$$

$$\therefore \sigma = \sqrt{V} \quad , \quad \sigma^2 = V$$

$$= E[x^2] - (E[x])^2$$

$$= \frac{n(n+1)}{2} - \left(\frac{2n+1}{3}\right)^2$$

$$= \frac{n(n+1)}{2} - \frac{4n^2 + 1 + 4n}{9}.$$

$$= \frac{9n^2 + 9n - 8n^2 - 2 - 8n}{18}$$

$$= \frac{n^2 + n - 2}{18}$$

$$\therefore \boxed{\sigma^2 = \frac{(n-1)(n+2)}{18}}$$

7) A random variable x has probability fn, $p(x) = 2^{-x}$, $x = 1, 2, 3, \dots$, ST: $p(x)$ is a probability fn,

Also find :- i) $P(x \text{ even})$ iii) $P(x \geq 5)$

ii) $P(x \div 3)$

Soln: Given; $P(x) = 2^{-x} = \frac{1}{2^x}$.

Clearly; $p(x) \geq 0$.

Consider; $\sum p(x) = \sum_{x=1}^{\infty} \frac{1}{2^x}$.

$\frac{1}{2} = \frac{1}{2}$

$\frac{1}{2^2} = \frac{1}{4}$ ....

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots$ } Geometric series.

$= \frac{a}{1-r}$, $\underline{a = 1/2}$, $\underline{r = 1/2}$

$= \frac{1/2}{1-1/2}$

$\therefore \boxed{\sum p(x) = 1}$ $\therefore$ $p(x)$ is a probability function.

i) $p(x \text{ even}) = p(2) + p(4) + p(6) + \cdots$

$= 1/2^2 + 1/2^4 + 1/2^6 + \cdots$ [Infinite GP].

$= \frac{a}{1-r}$, $\underline{a = 1/4}$, $\underline{r = 1/4}$

$= \frac{1/4}{1-1/4} = \frac{1}{4} \times 4/3$ $\therefore \boxed{p(x \text{ even}) = 1/3.}$

ii) $p(x \div \text{by } 3) = p(3) + p(6) + p(9) + \cdots$

$= 1/2^3 + 1/2^6 + 1/2^9 + \cdots$

$= \frac{a}{1-r} = \frac{1/8}{1-1/8} \Rightarrow \therefore \boxed{p(x \div 3) = 1/7.}$

iii) $p(x \geq 5) = p(5) + p(6) + p(7) + p(8) + \cdots$

$= 1/2^5 + 1/2^6 + 1/2^7 + 1/2^8 + \cdots$

$= \frac{1/32}{1-1/2} \Rightarrow \therefore \boxed{p(x \geq 5) = 1/16.}$

—— * ——

BINOMIAL DISTRIBUTION :-

If p is the probability of success and q is the probability of failure, the probability of x successes out of n trials is given by; $\boxed{P(x) = nC_x P^x . q^{n-x}}$

where, $q = 1-p$ is called Binomial distribution / Bernoulli's distribution.

(Binomial distribution) : formula.

$$\sum p(x) = q^n + nC_1 q^{n-1} p + nC_2 q^{n-2} p^2 + \ldots + p^n = (q+p)^n = 1^n = 1$$

That is : $\sum p(x) = (q+p)^n = 1^n = 1$

Mean of Binomial distribution :-

$$\text{Mean}, (\mu) = \sum_{x=0}^{n} x . p(x) \quad , \text{WKT} ; \ p(x) = nC_x P^x . q^{n-x}.$$

$$= \sum_{x=0}^{n} x . nC_x P^x . q^{n-x} \quad // \ nC_r = \frac{n!}{r!(n-r)!} , \ nC_x = \frac{n!}{x!(n-x)!}$$

$$\mu = \sum_{x=0}^{n} x \left[\frac{n!}{x!(n-x)!}\right] p^x . q^{n-x}. \quad // \ n! = n(n-1)! \quad x! \ x(x-1)!$$

$$= \sum_{x=0}^{n} \cancel{x} \left[\frac{n(n-1)!}{(n-x)! \cancel{x}(x-1)!}\right] p^x . q^{n-x}.$$

// if $\sum_{x=0}^{n} \frac{n(n-1)!}{(n-0)!(0-1)!} p^0 q^{n-0} = \frac{n(n-1)!}{n(n-1)!(-1)} q^n$

if $x=0$, $\sum_{x=0}^{n} = (-1)$ is not possible.

if $x=1$ $\sum_{x=1}^{n} \frac{n(n-1)!}{(n-1)!(1-1)!} p^1 q^{n-1} = \frac{npq^{n-1}}{0!} = \frac{npq^{n-1}}{1}$

$\therefore$ if $x=1$, $\sum_{x=1}^{n}$ = +ve value

$$\mu = \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^{x-1} . q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} . p^{x-1} . q^{(n-1)-(x-1)}.$$

// $p^x = p^{x-1} . p$

$q^{n-x} = q^{(n-1)-(x-1)}$

$(n-x)! = [(n-1)-(x-1)]!$

$$= np \sum_{x=1}^{n} {}^{(n-1)}C_{x-1} \ p^{x-1} . q^{(n-1)-(x-1)}$$

// $nC_r = \frac{n!}{r!(n-r)!}$

$$\mu = np \sum_{x=1}^{n} \underbrace{(n-1)C_{x-1}\, p^{x-1} \cdot q^{(n-1)-(x-1)}} \sim ①$$

WKT; By Binomial distribution formula; $\sum p(x) = (q+p)^n = 1^n = 1$. $\sim$ ⊛$_1$

Also; $p(x) = nC_x\, p^n . q^{n-x}$ // $\sum_{x=1}^{n} n-1\, C_{x-1} . p^{x-1} . q^{(n-1)-(x-1)} = \underline{\underline{\sum p(x-1)}}$

$\sum p(x) = \sum n C_x\, p^n q^{n-x}$

Eqn ⊛$_1$ $\Rightarrow$ $\sum (p(x-1)) = (q+p)^{n-1}$ — ⊛$_2$

$\Rightarrow$ By using ⊛$_2$ in Eqn ①; we get;

$\mu = np(q+p)^{n-1} = np\,\underline{\underline{(1)}}^{n-1}$.

$\therefore \boxed{\mu = np.}$, Mean.

<u>Variance</u>:- $V = \sum_{x=0}^{n} x^2 p(x) - \mu^2$ // $E[X^2] = \sum_{x=0}^{n} x^2 p(x)$.

Now; $\sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} [x(x-1)+x]\, p(x)$.

$= \sum_{x=0}^{n} x(x-1).\underline{\underline{p(x)}} + \underbrace{\sum_{x=0}^{n} x.p(x)}$. // $p(x)$

$= \sum_{x=0}^{n} x(x-1)\left[\underline{\underline{nC_x}} . p^x . q^{n-x}\right] + \mu$ // $\sum_{x=0}^{n} x p(x) = \underline{\underline{\mu = np}}$

$= \sum_{x=0}^{n} x(x-1) \left[\frac{n!}{x!\,(n-x)!}\right] p^x q^{n-x} + np$.

$= \sum_{x=0}^{n} \cancel{x(x-1)} \left[\frac{n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!\,(n-x)!}\right] p^{x-2} . p^2 . q^{n-2-x-2}$ // $p^x = p^{x-2} . p^2$
$q^{n-x} = q^{(n-2)-(x-2)}$

$+ np$. // $n! = n(n-1)(n-2)!$
$x! = x(x-1)(x-2)!$
$(n-x)! = (n-2-x-2)!$

$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!\; p^{x-2} q^{n-2-x-2}}{(x-2)!\,[(n-2)-(x-2)]!} + np$

$$= n(n-1)p^2 \sum_{x=2}^{n} \underbrace{\frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!}} p^{x-2} \cdot q^{(n-2)-(x-2)} + np$$

// if $\underline{x=0}$, $\sum_{x=0}^{n} \frac{(n-2)!}{(-2)!\,[n]!} p^{-2} q^{n-2} = \underline{(-2)}$
(-ve value).

$$= n(n-1)p^2 \underbrace{\sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} \cdot q^{n-2-x-2}} + np$$

// if $\underline{x=2}$, $\sum_{x=2}^{n} \frac{(n-2)!}{(2-2)!\,(n-2)!} p^0 q^{n-2}$
$= \underline{q^{n-2}}$ +ve
(+ve value)

$$= n(n-1)p^2 \cdot [(q+p)^{n-2}] + np.$$

// $(q+p)^n = 1^n$

$$= n(n-1)p^2\, [(1)^{n-2}] + np.$$

$$\boxed{\sum x^2 P(x) = n(n-1)p^2 + np.}$$

$\therefore$ Variance, $V = \sum x^2 P(x) - \mu^2$ // $\underline{\mu = np}$

$$= n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - \underbrace{np^2 + np}_{\text{Common}} - \cancel{n^2p^2}$$

$$= np - np^2$$

$$= np(1-p) \quad // \ \underline{q = 1-p}$$

$\therefore \boxed{V = npq.}$, Variance.

Standard deviation :- (σ)

$$\boxed{\sigma = \sqrt{V} = \sqrt{npq} = S.D.}$$

Thus, the proof of μ, V and σ for Binomial distribution.

Problems & Solutions :-

1) When a Coin is tossed 4 times, find the probability of getting
   i) Exactly one head.
   ii) Atmost 3 heads.
   iii) Atleast 2 heads.

Soln:- Since, a coin is tossed 4 times, $n=4$ (Total 4 outcomes)

W.K.T; The probaility of getting head is $= \boxed{p=\frac{1}{2}}$ // In one trial, probability or chance of Getting head is 1 time out of 2 possibilities

Since, $q = 1-p$

$q = 1-\frac{1}{2}$, $\boxed{q=\frac{1}{2}}$

$\therefore$ By Binomial distribution, we have;

$p(x) = {}^nC_x \cdot p^x q^{n-x}$ —① , $n=4$, $p=\frac{1}{2}$, $q=\frac{1}{2}$

$= {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

$= {}^4C_x \left(\frac{1}{\cancel{2}}\right)^{\cancel{x}} \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{\cancel{2}}\right)^{\cancel{-x}}$

$\therefore \boxed{p(x) = \frac{{}^4C_x}{16}}$

i) The probability of Getting exactly one head is ;

[That is : Out of 4 possibilities, all 3 outcomes should be tails and only one (1) outcome, should be head]

$\therefore x=1$ (1 head) // ${}^nC_r = \frac{n!}{(n-r)!\,r!}$

$\therefore p(x) = \frac{{}^4C_1}{16} = \frac{\cancel{4}^1}{\cancel{16}_4} = 0.25$ // ${}^nC_1 = n$

$\therefore \boxed{p(x) = 1/4 = 0.25}$

ii) probability of getting atmost 3 heads. (11)4

[That is: out of 4 outcomes; 3 possible outcomes should be heads, remaining one(1) possibility should be tail.]

Atmost 3 heads (max) 3 heads $\Rightarrow x \leq 3$, where, $x=0$, $x=1$, $x=2$, $x=3$, $x=4$

$x=0 \downarrow$ 1 Tail, $x=1 \downarrow$ Head, $x=2 \downarrow$ Head, $x=3 \downarrow$ Head, $x=4 \downarrow$ tail.

4 outcomes.

$$\therefore p(x \leq 3) = p(0) + p(1) + p(2) + p(3)$$

$$= \frac{4C_0}{16} + \frac{4C_1}{16} + \frac{4C_2}{16} + \frac{4C_3}{16}$$

// $nC_0 = 1$

$nC_r = \frac{n!}{(n-r)!\,r!}$

$$= \frac{1}{16} + \frac{1}{4} + \frac{6}{16} + \frac{4}{16}$$

$$\boxed{p(x \leq 3) = 0.9375}$$

iii) Probability of getting atleast 2 heads;

[That is: out of outcomes, 2 outcomes should be head.

$$\therefore p(x \leq 2) = p(0) + p(1) + p(2)$$

$$= \frac{4C_0}{16} + \frac{4C_1}{16} + \frac{4C_2}{16}$$

$$= \frac{1}{16} + \frac{1}{4} + \frac{6}{16}.$$

$$\therefore \boxed{p(x \leq 2) = \frac{11}{16} = 0.6875}$$

$$p(x \geq 2) = p(2) + p(3) + p(4)$$

$$= \frac{4C_2}{16} + \frac{4C_3}{16} + \frac{4C_4}{16}$$

$$= \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$$

$$\boxed{p(x \geq 2) = 11/16 = 0.6875}$$

2) A pair of dice is thrown twice, find the probability of scoring 7 points,

1) once.
2) Twice
3) ~~Thrice~~ Atleast once.

Soln :- Given; $n = 2$ [A dice is thrown twice].

$\therefore$ The probability of scoring 7 points is : Since, the dice is thrown twice.

* 1st time : 3 outcomes
* 2nd time : 3 outcomes

$\left.\right\}$ 3+3 = 6 outcomes

$\therefore$ The prob of scoring 7 pts is $= \frac{\cancel{6}^{1}}{\cancel{36}_{6}}$

$\boxed{p = 1/6}$

[That is : when pair of di...
is thrown once (1st time) : prob of scoring 7 pts is : $\{\boxed{6}\boxed{1}, \boxed{5}\boxed{2}, \boxed{4}\boxed{3}\} = 3$ o.

pair, pair, pair — pair of dice ....

when pair of dice is thrown 2nd time : prob of scoring 7 points is : $\{\boxed{6}\boxed{1}, \boxed{5}\boxed{2}, \boxed{4}\boxed{3}\}$ = 3 outcomes]

Total no of outcomes : 1st time : $\{\underbrace{6,1/5,2/4,3/3,4/1,6/5,2}_{6 \text{ outcomes}}\}$

2nd time : 6 outcomes.

$\Rightarrow 6 \times 6 = 36$ total outcomes.

WKT; $q = 1 - p$.

$q = 1 - 1/6$, $\boxed{q = 5/6}$

By Binomial distribution ; $p(x) = nC_x p^x q^{n-x}$

$p(x) = 2C_x (1/6)^x (5/6)^{2-x}$ — (1)

i) The prob of scoring 7 points once is ; $x=1$

$P(1) = 2C_1 (1/6)^1 (5/6)^{2-1} = \cancel{2}(1/\cancel{6}_3)(5/6)$

$\therefore \boxed{p(1) = \frac{5}{18} = 0.2778.}$

ii) prob. of scoring 7 point twice is ; $x=2$

$P(2) = 2C_2 (1/6)^2 (5/6)^{2-2} = 1.(1/36)(1)$

$\boxed{p(2) = 1/36 = 0.0278.}$

iii) Prob. of scoring 7 points atleast once ;

$p(x \geq 1) = p(1) + p(2)$

$= 0.2778 + 0.0278$

$\boxed{p(x \geq 1) = 0.3056.}$

3) The probability that a person aged 60 years will live upto 70 is 0.65 is : 0.65, what is the probability that out of 10 persons aged 60 atleast 7 of them will live up to 70.

Soln :- Let Given ; $\boxed{n=10}$ 10 persons are aged 60.

Given that ; The probability that a person aged 60 will live up to 70 is : 0.65 $\Rightarrow \boxed{p = 0.65}$

$\therefore q = 1-p$

$q = 1 - 0.65$, $\boxed{q = 0.35}$

Now, By binomial distribution :- $p(x) = nC_x p^x q^{n-x}$.

$p(x) = 10C_x (0.65)^x (0.35)^{10-x}$ — (1).

Now, to find: ($p(x \geq 7)$), ie : The probability, that out of 10 persons aged 60 atleast 7 of them will live up to 70

is : $p(x \geq 7) = p(7) + p(8) + p(9) + p(10)$ // Since ; only 10 persons are there.

$= 10C_7 (0.65)^7 (0.35)^{10-7} + 10C_8 (0.65)^8 (0.35)^{10-8} + 10C_9 (0.65)^9 (0.35)^{10-9}$

$+ 10C_{10} (0.65)^{10} (0.35)^{10-10}$.

But $\Rightarrow$ $10C_7 = \frac{10(10-1)(10-2)}{(10-7)!}$ // $nC_r = \frac{n!}{(n-r)!r!} = 10C_7 = \frac{10!}{(10-7)!\,7!}$

$\Rightarrow 10C_7 = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$. // $= \frac{10(10-1)(10-2)(10-3)\cancel{\cdots}\ldots}{7!\,3!} = \frac{10(10-1)(10-2)\cancel{7!}}{\cancel{7!}\,3!}$

$\Rightarrow 10C_8 = \frac{10 \cdot 9}{1 \cdot 2} = 45$ (or) $10C_2$, $10C_9 = 10$ (or) $10C_1$, $10C_{10} = 1$

$\therefore p(x \geq 7) = 120(0.65)^7 (0.35)^3 + 45(0.65)^8 (0.35)^2 + 10(0.65)^9 (0.35)^1 + 1(0.65)^{10} (1)$

$\therefore \boxed{p(x \geq 7) = 0.5138}$.

4) The number of telephone lines busy at an instant of time is a binomial variable, with probability 0.2, If at a instant, 10 lines are chosen at random, what is the probability that ; i) 5 lines are busy ii) 2 lines are busy. iii) All lines are busy.

Soln:- $n=10$ (No of telephone lines choosed).

Given ; $\boxed{p=0.2.}$ $q=1-p$, $q=1-0.2$, $\boxed{q=0.8}$

By binomial distribution ; $p(x) = nC_x\, p^x \cdot q^{n-x}$.

$\therefore p(x) = 10C_x \cdot (0.2)^x (0.8)^{10-x}$ — (1).

(i) Prob. of that, 5 lines are busy ; $x=5$.

$p(5) = 10C_5\,(0.2)^5 (0.8)^{10-5}$ = $\dfrac{10(10-1)(10-2)(10-3)(10-4)\cancel{(10-5)!}}{5!\,(10-5)!} = 10C_5$

$10C_5 = \dfrac{\overset{2}{\cancel{10}}\,\overset{3}{\cancel{(9)}}\,\overset{2}{\cancel{(8)}}(7)(6)}{\cancel{5\times4\times3\times2\times1}} = 36\times7 = 252$

$\therefore p(5) = 252(0.2)^5(0.8)^5$

$\boxed{p(5) = 0.0264}$

(ii) Prob that 2 lines are busy; $x=2$; $p(2) = 10C_2\,(0.2)^2(0.8)^{10-2}$.

$P(2) = 45(0.2)^2(0.8)^8$

$\boxed{P(2) = 0.3020}$

// $10C_2 = \dfrac{10(10-1)(10-2)\cdots(10-8)!}{2!\,(10-2)!}$

// $10C_2 = \dfrac{\overset{5}{\cancel{10}}(9)\cancel{(8)(7)(6)(5)(4)(3)}}{8!\;2}$

$10C_2 = 45$

(iii) prob that All lines are busy ;

$p(10) = 10C_{10}\,(0.2)^{10}(0.8)^0$

$p(10) = 1.024\times10^{-7}$

of
at any

5) 2 persons A & B play a game in which their chances of winning are in the ratio 3:2, Find A's chance of winning atleast 3 games out of 6 games played.

$3:2 = 3+2 = 5$

Soln :- $n=6$.

The probability that A wins the game :- $\boxed{p = \frac{3}{5}}$ → A wins game 3 times / A and B wins game

$q = 1-p = 1-3/5$

$\boxed{q = 2/5}$

By Binomial distribution ;

$p(x) = nC_x \cdot p^x \cdot q^{n-x}$.

$p(x) = 6C_x (3/5)^x (2/5)^{6-x}$ — ①.

Now, To find prob of A's chance of winning atleast 3 games out of 6 games played is ; $p(x \geq 3)$

$p(x \geq 3) = p(3)+p(4)+p(5)+p(6)$

$\therefore p(x \geq 3) = p(3)+p(4)+p(5)+p(6) + 6C_4 (3/5)^4 (2/5)^2 + 6C_5 (3/5)^5 (2/5) + 6C_6 (3/5)^6 (q)$

$= 6C_3 (3/5)^3 (2/5)^3 + 6C_4 (3/5)^4 (2/5)^2 + 6C_5 (3/5)^5 (2/5) + 6C_6 (3/5)^6 (q)$

$6C_3 = \frac{6(6-1)(6-2)\cancel{(6-3)!}}{3!(6-3)!}$

$= \frac{6(5)(4)}{3\times2\times1} = 20$.

$\boxed{p(x \geq 3) = 0.8208}$

6) In a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2, Out of 1000 such samples, how many should be expected to contain atleast 3 defective parts.

Soln :- $n = 20$ (Mean no of defective samples).

Since, Mean no of samples defectives = 2

$\boxed{Mean = \mu = 2}$, $n = 20$

WKT $\Rightarrow \mu = np$.

Also; $p = \frac{\mu}{n} = \frac{2}{20} = \boxed{0.1 = p}$

Also; $q = 1 - p$.

$q = 1 - 0.1 \Rightarrow \boxed{q = 0.9}$

By Binomial distribution; $p(x) = nC_x \, p^x . q^{n-x}$.

$p(x) = 20C_x (0.1)^x (0.9)^{20-x}$ — (1).

Now, Probability of atleast 3 defective parts is; // (3 or more defects) atleast

$$P(x \geq 3) = 1 - (P(x < 3))$$

$$= 1 - \{P(0) + P(1) + P(2)\}$$

$$= 1 - \{20C_0 (0.1)^0 (0.9)^{20} + 20C_1 (0.1)^1 (0.9)^{19} + 20C_2 (0.1)^2 (0.9)^{18}\}.$$

$\boxed{P(x \geq 3) = 0.3231}$

| $nC_0 = 1$
| $nC_n = 1$
| $nC_1 = n$.

$\therefore$ Out of 1000 samples, the expected no of samples that contain atleast 3 defective parts $= 0.3231 \times 1000$.

$= 323.1$

$\simeq 323$

(14) 4

7) If the mean & SD of no of correctly answered questions given to 4096 students are 2.5 & $\sqrt{1.875}$, Find an estimate of no of student answering correctly, (a) 8 or more questions (atleast)
(b) 2 or less question
(c) 5 questions.

Soln :- Given; $\mu = 2.5$, Mean.
$S.D = \sigma = \sqrt{1.875}$

WKT; $\mu = np$, $\sigma = \sqrt{npq}$.
$\sigma = \sqrt{\mu q} \Rightarrow q = \frac{\sigma^2}{\mu} = \frac{1.875}{2.5}$, $\boxed{q = 0.75}$

WKT; $p = 1-q$.
$\boxed{p = 0.25}$

By Binomial distribution; $P(x) = nC_x p^x q^{n-x}$.
$P(x) = 10C_x (0.25)^x (0.75)^{10-x}$ — (1)

(i) Prob. of no of stdnt answering Correctly 8 or more questions;
$P(x \geq 8) = P(8) + P(9) + P(10) = 10C_8 (0.25)^8 (0.75)^2 + 10C_9 (0.25)^9 (0.75)^1 + 10C_{10} (0.25)^{10} (0.75)^0$.
$\therefore \boxed{P(x \geq 8) = 0.0004}$ $\therefore$ Out of 4096 stds, expected value no of stds who answered 8 or more questions correctly $= 0.0004 \times 4096$
$= 1.6384 \simeq \boxed{2 = p(x \geq 8)}$

ii) $p(x \leq 2) = p(0) + p(1) + p(2)$.
$= 10C_0 (0.25)^0 (0.75)^{10} + 10C_1 (0.25)^1 (0.75)^9 + 10C_2 (0.25)^2 (0.75)^8$
$\therefore p(x \leq 2) = 0.5256$. $\therefore$ Out of 4096 stds, expected no of stds who answer 2 or less questions Correctly $= 0.5256 \times 4096$
$\boxed{p(x \leq 2) = 2153}$

iii) $P(5)$
$= 10C_5 (0.25)^5 (0.75)^5 = \boxed{0.0584 = p(5)}$
$\therefore$ out of 4096 stds, the expected no of stds who answer 5 questions correctly $= 0.0584 \times 4096 \simeq \boxed{239 = p(5)}$

— * —

POISSON' DISTRIBUTION :-

The probability function defined by; $p(x) = \frac{e^{-\mu}\mu^{x}}{x!}$

where; $\mu = np$ is called poisson's Distribution.

Poisson's Distribution :-

Poisson's distribution is regarded as the limiting form of the binomial distribution, when n is very large $(n \to \infty)$ and probability of success p is very small $(p \to 0)$, so that np tends to a fixed finite constant μ.

Consider, the Binomial distribution; $p(x) = nC_x\, p^x . q^{n-x}$ — (1)

$$p(x) = \frac{n!}{(n-x)!\,x!} p^x . q^{n-x} \qquad // \; nC_x = \frac{n!}{(n-r)!\,r!}$$

(n common)

$$p(x) = \frac{n(n-1)(n-2)\cdots\cdots[n-(x-1)]\cancel{(n-x)!}\cdots(1)}{\cancel{(n-x)!}\;\; x!\;\; .q^{x}} . p^x . q^{n}$$

$$p(x) = \frac{n . n[1-1/n][1-2/n]n \cdots\cdots n[1-(\frac{x-1}{n})]}{x!\,q^x} p^x . q^n$$

$$// \; n . n^2 \cdots\cdots n^x = n^x$$

$$p(x) = n^x \frac{[1-1/n][1-2/n]\cdots\cdots[1-(\frac{x-1}{n})]}{x!\,q^x} . p^x . q^n$$

$$= \frac{(np)^x [1-1/n][1-2/n]\cdots\cdots[1-(\frac{x-1}{n})] . q^n}{x!\,q^x} \qquad // \; \mu = np$$

$$p(x) = \frac{(\mu)^x (1-1/n)(1-2/n)\cdots\cdots(1-(\frac{x-1}{n})) . q^n}{x!\,q^x} \sim (2)$$

Consider; $\lim_{n\to\infty} q^n = \lim_{n\to\infty}(1-p)^n$ // $\mu = np$ // $p = \frac{\mu}{n}$

$\lim_{n\to\infty} q^n = \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n$

$$\boxed{\lim_{n\to\infty} q^n = e^{-\mu}}$$

// $\lim_{n\to\infty}(1+1/n)^n = e$

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$\lim_{n\to\infty}\left(1+\left(-\frac{\mu}{n}\right)\right)^n = e^{-\mu}$

$\lim_{p\to 0} q^x = \lim_{p\to 0}(1-p)^x$.

$$\boxed{\lim_{p\to 0} q^x = 1}$$

Also, the factors; $(1-1/n)(1-2/n)\cdots\left(1-\left(\frac{x-1}{n}\right)\right)$ tends to $\longrightarrow 1$ as $n\to\infty$.

$\therefore$ The Eqn ② reduces to :- $\boxed{P(x) = \frac{e^{-\mu}\mu^x}{x!}}$

MEAN :- $\mu = \sum_{x=0}^{\infty} x \cdot p(x) = \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!} x.$

$= \sum_{x=1}^{\infty} \frac{\cancel{x}\cdot e^{-\mu}\mu^x}{\cancel{x}(x-1)!} = \sum_{x=1}^{\infty} \frac{e^{-\mu}\mu^{x-1}\mu^1}{(x-1)!} = e^{-\mu}\cdot\mu \sum_{x=1}^{\infty}\frac{\mu^{x-1}}{(x-1)!}$

$= e^{-\mu}\cdot\mu\left\{1+\frac{\mu}{1!}+\frac{\mu^2}{2!}+\cdots\right\}$ // $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$

$= e^{-\mu}\cdot\mu\left[e^{\mu}\right]$

$= e^{-\mu+\mu}\cdot\mu = e^0\mu$

$$\boxed{\text{Mean} = \mu}$$

Variance:- $V = E[X^2] - (E[X])^2$

$V = E[X^2] - \mu^2$ — (3)

WKT;

$$E[X^2] = \sum_{x=0}^{\infty} x^2 \cdot p(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x]\, p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)\, p(x) + \underbrace{\sum_{x=0}^{\infty} x \cdot p(x)} \quad // \ \mu = \sum x \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)\, p(x) + \mu$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\mu}\mu^x}{x!} + \mu$$

$$= \sum_{x=2}^{\infty} \cancel{x(x-1)} \frac{e^{-\mu}\mu^x}{\cancel{x(x-1)}(x-2)!} + \mu$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\mu} \cdot \mu^{(x-2)+2}}{(x-2)!} + \mu$$

$$= e^{-\mu} \cdot \mu^2 \left[\sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!} \right] + \mu$$

$$= e^{-\mu} \mu^2 \left\{ 1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \cdots \right\} + \mu$$

$E[X^2] = e^{-\mu}\mu^2 e^{\mu} + \mu$ — use in (3), we get.

$\therefore V = e^{-\mu} \cdot \mu^2 e^{\mu} + \mu - \mu^2$

$\therefore \boxed{V = \mu}$

Hence, the proof of Mean & Variance of Poisson's Distribution.

* **NOTE:-**

1. $P(x \geq n) = 1 - P(x < n)$ *
2. Avg. rate = Mean = μ
3. $P(x > n) = 1 - P(x \leq n)$
4. .

(17) 4

POISSON'S DISTRIBUTION:-

1) Alpha particles are emitted by a radioactive source at an average rate of 5 in 20 minutes interval, using Poisson's distribution, find probability that there will be:
i) 2 Emissions ii) Atleast 2 emissions in 20 minutes interval.

Soln :- Given; Avg. rate = $\mu = 5$, Mean.

WKT; By Poisson's distribution; $p(x) = \frac{e^{-\mu} \cdot \mu^{x}}{x!}$

$$p(x) = \frac{e^{-5} \cdot 5^{x}}{x!} \quad \text{—} (1).$$

i) Probability that, there will be 2 emissions is;

$x = 2$, $p(2) = \frac{e^{-5} 5^{2}}{2!} = \frac{e^{-5}(25)}{2}$

$$\therefore \boxed{p(2) = 0.0842}$$

ii) Probability that there will be atleast 2 emissions;

$$p(x \geq 2) = 1 - p(x<2)$$
$$= 1 - \{p(0) + p(1)\}$$
$$= 1 - \{e^{-5} + e^{-5} \cdot 5\}$$
$$\boxed{p(x \geq 2) = 0.9596.}$$

2) A car hire firm has 2 cars, which it hires out day by day, the demand of a car on each day is distributed as a poisson distribution with mean 1.5.
Calculate the probability that on a certain day
i) Neither car is used.
ii) Some demands are refused.

*) Probability : WKT ; Given ; $\mu = 1.5$

WKT ; Poison's distribution is : $p(x) = \frac{e^{-\mu} \mu^{x}}{x!}$

i) Probability that neither car is used. (no car is used).

$\therefore x = 0, \ p(0) = \frac{e^{-1.5} \cdot (-1.5)^{0}}{0!}$ // $0! = 1$

$\boxed{p(0) = 0.2231.}$

ii) Probability that some demands are refused.

= prob that there will be more than 2 demands.

$p(x > 2) = 1 - (p(x \leq 2))$

$= 1 - \{ p(0) + p(1) + p(2) \}$

$= 1 - \{ 0.223 + 0.3347 + 0.2510 \}$

$\boxed{p(x > 2) = 0.1912.}$

3.) The probability that, an individual suffers a bad reaction from certain injection is : 0.002, Using Poisson's distribution, determine probability that out of 1000 individuals : (i) Exactly 2
(ii) More than 2. , will suffer from bad reaction.

Soln :- Given; $p = 0.002$
$n = 1000$.

$\therefore \mu = np$.
$\mu = 1000 \times 0.002$. , $\boxed{\mu = 2}$

WKT; By poisson's distribution; $p(x) = \dfrac{e^{-\mu}\mu^{x}}{x!}$

$$p(x) = \frac{e^{-2}(2)^{x}}{x!} \quad \text{— (1)}.$$

(i) Prob. that out of 1000 individuals, Exactly 2 suffer from bad reaction is ; $x=2$

$$P(2) = \frac{e^{-2}\,2^{2}}{2!}$$

$$\boxed{P(2) = 0.2707.}$$

(ii) Prob that out of 1000 individuals, more than 2 suffer from bad reaction is ; $(x>2)$

$$\Rightarrow p(x>2) = 1 - P(x \leq 2)$$
$$= 1 - \{p(0) + p(1) + p(2)\}$$
$$= 1 - \left\{ \frac{e^{-2}2^{0}}{0!} + \frac{e^{-2}2^{1}}{1!} + \frac{e^{-2}2^{2}}{2!} \right\}$$

$$\therefore \boxed{p(x>2) = 0.3233.}$$

4) Given that, 2% of the fuses manufactured by a firm are defective, Find by poisson's distribution, the probability that a box containing 200 fuses has:

i) Atleast 1 defective fuse.

ii) 3 or more defective fuses.

Soln:- Given; $p = 2\% = \frac{2}{100} = \boxed{0.02 = p}$

$\boxed{n = 200}$

WKT; $\mu = np$

$\mu = 200 \times 0.02$

$\boxed{\mu = 4}$

WKT; By Poisson's distribution;

$\Rightarrow P(x) = \frac{e^{-4} 4^x}{x!} \sim (1)$.

(i) The prob. that box containing 200 fuses, having atleast 1 defective fuse is;

$p(x \geq 1) = 1 - P(x < 1)$

$1 - p(0)$

$\Rightarrow 1 = \frac{e^{-4} 4^0}{0!}$

$\therefore \boxed{p(x \geq 1) = 0.9817}$

(ii) The prob., that box containing 200 fuses, have 3 or more defective fuses is;

$p(x \geq 3) = 1 - P(x < 3)$.

$= 1 - \{p(0) + p(1) + p(2)\} \cancel{+ p(}$

$= 1 - \left\{ e^{-4} + \frac{e^{-4} 4}{1} + \frac{e^{-4} 4^2}{2} \right\}$

$\boxed{\therefore p(x \geq 3) = 0.7619}$

5) The probability that a news reader commits no mistakes, is : $\frac{1}{e^3}$, Find the probability that ; on a particular news broadcast, he commits , i) Only 2 mistakes.
ii) More than 3 mistakes.
iii) Atmost 3 mistakes.

Soln:- Given ; $p(0) = \frac{1}{e^3}$ // probability that, newsreader commits no mistakes.

WKT; By Poisson's distribution ; $p(x) = \frac{e^{-\mu}\mu^x}{x!}$ ~ ①

put, $x=0$; $p(0) = \frac{e^{-\mu}\mu^0}{0!} = e^{-\mu}$ // $p(0) = \frac{1}{e^3}$.

$p(0) = e^{-\mu}$.

$\Rightarrow \frac{1}{e^3} = e^{-\mu} \Rightarrow \frac{1}{e^3} = \frac{1}{e^{\mu}}$ (Bases are same ; powers are equal).

$\Rightarrow \boxed{\mu = 3.}$, Mean.

$\therefore$ ① $\Rightarrow p(x) = \frac{e^{-3}(3)^x}{x!}$ - ②.

(i) prob. that ; on particular news, he commits only 2 mistakes is ; $p(2) = \frac{e^{-3}3^2}{2!} \Rightarrow \boxed{p(x=2) = 0.2240.}$

(ii) prob. that on particular news, he comits more than 3 mistakes is ; $p(x>3) = 1 - p(x\leq 3)$
$= 1 - \{p(0)+p(1)+p(2)+p(3)\}$
$\therefore \boxed{p(x>3) = 0.3528.}$

(iii) prob. that on particular news, he comits Atmost 3 mistake is ; $p(x\leq 3) = p(0)+p(1)+p(2)+p(3)$
$\therefore \boxed{p(x\leq 3) = 0.6472}$.

6) The no of accident in a year by taxi drivers in a city follows a poisson distribution, with mean 3, Out of 1000 taxi drivers, find approximately the no of drivers with; i) No accident
ii) More than 3 accidents in a year.

Soln: - Given; $\mu = 3$

$n = 1000$

WKT; By poisson's distribution; $p(x) = \frac{e^{-\mu}\mu^x}{x!}$

$$p(x) = \frac{e^{-3}3^x}{x!} \quad \text{--- (1)}$$

(i) Prob that; no of drivers; with no accident in a year,

$\Rightarrow p(0) = \frac{e^{-3}3^0}{0!}$, $\boxed{p(0) = 0.0498}$ (for 1 taxi driver).

$\therefore$ The no of drivers with no accident in year $= 0.0498 \times 1000$

$= 49.8 \simeq 50$

$\therefore \boxed{p(0) = 50}$

(ii) $p(x>3) = 1 - P(x \leq 3)$

$= 1 - \{p(0) + p(1) + p(2) + p(3)\}$

$= 1 - \{0.0498 + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2} + \frac{e^{-3}3^3}{6}\}$

$= 0.3528$

$\therefore$ The no of taxi drivers with 3 accidents in a year

is ; 0.3528×1000

$\boxed{p(x>3) = 353.}$

4) In a certain factory, turning out razor blades, there is a chance of 0.002 for any blade to be defective, The blades are supplied in packets of 10, using Poisson distribution, find the approximate no of packets Containing : i) No defective blade ii) 1 def. blade iii) 2 def. blade in a consignment of 10,000 packets. (20)

Soln : Given : $p = 0.002$.

$n = 10$.

WKT; $\mu = np$

$= 10 \times 0.002$, $\boxed{\mu = 0.02}$, Mean.

WKT; Poisson's distribution ; $p(x) = \frac{e^{-\mu}\mu^{x}}{x!}$

$$p(x) = \frac{e^{-0.02}(0.02)^{x}}{x!} \quad -(1).$$

(i) Probability that there is no defective blade;

$\Rightarrow p(0) = e^{-0.02} = \boxed{0.9802 = p(0)}$

$\therefore$ No of packets containing no defective blades $= 10{,}000 \times 0.9802$

$\boxed{p(0) = 9802}$

(ii) $p(2) = \frac{e^{-0.02}(0.02)^{2}}{2!} = 0.0002$

$\therefore$ No of packets containing 2 defective blades $= 10{,}000 \times 0.0002$

$\therefore \boxed{p(2) = 2}$

— * —

CONTINUOUS PROBABILITY DISTRIBUTION :-

The probability distributions, where the random variable varies continuously over an interval, is called Continuous-probability distribution.

A function $P(x)$ is said to be a probability density function [probability mass function (PMF)], if :

i) $P(x) \geq 0$

ii) $\int_{-\infty}^{\infty} P(x).dx = 1.$

For any specified variable t, the function $F(t)$ is defined by ; $F(t) = P(x \leq t) = P(x < t)$ is called Cumulative distribution function (CDF).

$\therefore$ Mean, $\boxed{\mu = E[X] = \int_{-\infty}^{\infty} x.p(x).dx.}$

$\therefore$ Variance, $V = E[X^2] - (E[X])^2$ (or)

$\therefore \boxed{V = \int_{-\infty}^{\infty} (x-\mu)^2 p(x).dx}$

$\therefore$ Standard deviation; $\boxed{\sigma = \sqrt{V}.}$

— * —

$$= \int_0^{2.5} p(x).dx$$

$$= \int_0^{2.5} e^{-x}.dx = -e^{-x}\Big|_0^{2.5}$$

$$= -[e^{-0.25} - e^0]$$

$$\therefore \boxed{F(2.5) = 0.9179}$$

2) A Continuous random variable x has the probability density

$$p(x) = \begin{cases} 2e^{-2x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Evaluate ; i) $E[X]$ ii) $E[X^2]$ iii) Variance iv) S.D.

Soln :-

WKT; Mean, $\mu = E[X] = \int_{-\infty}^{\infty} x.p(x).dx$

$$= \int_0^{\infty} x(2e^{-2x}).dx$$

$$= 2\int_0^{\infty} \underset{\mathrm{I}}{x} . \underset{\mathrm{II}}{e^{-2x}}.dx. = 2\left[x.\frac{e^{-2x}}{-2} - \left(-\frac{e^{-2x}}{4}\right)(1)\right]_0^{\infty}.$$

(prod. rule)

$$= \cancel{2}\left[\frac{e^0}{\cancel{4}_2}\right]$$

$$\therefore \boxed{\mu, E[X] = \frac{1}{2}}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p(x).dx$$

$$= 2\int_{0}^{\infty} x^2 e^{-2x}.dx.$$

$$= 2\left[x^2\left(\frac{e^{-2x}}{-2}\right) - \left(\frac{e^{-2x}}{4}\right)2x + \frac{e^{-2x}}{8}2\right]_0^{\infty}$$

$$= 2.\frac{1}{4}$$

$$\boxed{E[X^2] = 1/2}$$

$$\therefore V = E[X^2] - (E[X])^2 \Rightarrow \boxed{V = 1/4}$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$S.D = \sigma = \sqrt{V}, \quad \boxed{\sigma = 0.5}$$

$$\sigma = \sqrt{1/4}$$

3) A random variable x, has the density function;

$$P(x) = \begin{cases} Kx^2 & -3 \leq x \leq 3. \\ 0 & \text{elsewhere.} \end{cases}$$

Evaluate :- i) $P(1 \leq x \leq 2)$

ii) $P(x \leq 2)$

iii) $P(x > 1)$

Soln:- By defn :- $\int_{-\infty}^{\infty} p(x).dx = 1.$

$$= \int_{-\infty}^{\infty} Kx^2.dx = 1$$

$$= K\int_{-\infty}^{\infty} x^2.dx = 1.$$

$$K\left[\frac{x^3}{3}\right]_{-\infty}^{\infty} = K\left[\frac{3^3}{3} - \frac{(-3)^3}{-3}\right] = 1.$$

$$= K\left[\frac{x^3}{3}\right]_{-3}^{+3} = 1$$

$$= K\left[\frac{3^3}{3} - \frac{(-3)^3}{3}\right] = 1 \quad , \quad K[9+9] = 1$$

$$\therefore \boxed{K = 1/18 = 0.0556.}$$

(i) $p(1 \leq x \leq 2) = \int_1^2 p(x).dx$

$$= \int_1^2 Kx^2.dx = \frac{1}{18}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{(8-1)}{54} = \frac{7}{54} = \underline{\underline{0.1296}}$$

(ii) $p(x \leq 2) = \int_{-3}^{2} p(x).dx$

$$= \int_{-3}^{2} Kx^2.dx = \frac{1}{18 \times 3} x^3 = \frac{1}{54}(8+27)$$

$$= \frac{35}{34} = \underline{\underline{0.6481}}$$

(iii) $p(x > 1) = \int_1^3 p(x).dx = \int_1^3 Kx^2 dx = \frac{1}{54} x^3 \Big|_1^3$

$$= \frac{1}{54}[27-1] = \frac{13}{27} = 0.4815 //$$

3) The prob...
is :

3) The probability density of continuous random variable, (23) 4

if : $p(x) = \begin{cases} Kx(1-x)e^{x} & 0 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$

find K & Evaluate mean & S.D of distribution.

Soln :- By defn : $\int_{-\infty}^{+\infty} p(x)\, dx = 1$

$\Rightarrow \int_0^1 K(x)(1-x)e^{x}\, dx = 1$

$= K\int_0^1 \underset{I}{(x-x^2)}\, \underset{II}{e^{x}}\, dx = 1.$

$= K\left[(x-x^2)e^{x} - (1-2x)e^{x} + (-2)e^{x}\right]_0^1 = 1$

$= K\left[+e - 2e - (1-2)\right] = 1$

$= K[-e+3] = 1$

$\boxed{K = \frac{1}{3-e}}$

WKT;

$\therefore \mu = \int_{-\infty}^{\infty} x.\, p(x)\, dx$

$= \int_0^1 x \cdot Kx(1-x)e^{x}\, dx = K\int_0^1 (x^2-x^3)e^{x}\, dx.$

$\mu = K\left\{(x^2-x^3)e^{x} - (2x-3x^2)e^{x} + (2-6x)e^{x} - (-6)e^{x}\right\}_0^1$

$= K\{e - 4e + 6e - (2+6)\}$

$= K\{3e-8\} \Rightarrow \mu = \frac{3e-8}{3-e}$, $\boxed{\mu = 0.5496.}$

$\therefore S.D = \sigma = \sqrt{V}$

Now, we find ; $E[X^2] = \int_{-\infty}^{+\infty} x^2 p(x)\, dx$

$= \int_0^1 (x^2\, Kx(1-x)e^{x})\, dx$

$= K\int_0^1 (x^3-x^4).e^{x}.dx.$

$= K\left\{(x^3-x^4)e^{x} - (3x^2-4x^3)e^{x} + (6x-12x^2)e^{x} - (6-24x)e^{x} + (-24)e^{x}\right\}_0^1$

$= K\{-11e+30\}$, $E[X^2] = \frac{30-11e}{3-e}$

$$\boxed{E[X^2] = 0.3511}$$

Variance, $V = E[X^2] - (E[X])^2$

$= 0.3511 - (0.5496)^2$

$$\boxed{V = 0.0490}$$

$\therefore$ SD, $\sigma = \sqrt{V}$

$$\boxed{\sigma = 0.2214.}$$

4) The probability density fn of a continous random variable x is given by ; $p(x) = Ke^{-|x|}$, $-\infty < x < \infty$.

Show that ; $K = \frac{1}{2}$, find mean, variance & SD of distribution

Soln :- By defn :- $\int_{-\infty}^{+\infty} p(x)\,dx = 1$

$\int_{-\infty}^{\infty} K e^{-|x|}.dx = 1.$ // $e^{-|x|} = e^{-x}$

$= K.2\int_{0}^{\infty} e^{-x}.dx = 1.$

$= 2K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1.$

$= -2k(0-1) = 1$

$$\boxed{K = 1/2.}$$

$\therefore$ Mean, $\mu = E[X] = \int_{-\infty}^{\infty} x.p(x).dx$

$= \int_{-\infty}^{\infty} x.K.e^{-|x|}.dx$

$= K\int_{-\infty}^{\infty} x.e^{-|x|}.dx$ // [Since, $e^{-|x|}$ is odd function. Integration of odd fn is zero]

$$\boxed{\mu = 0}$$

Now; $E[X^2] = \int_{-\infty}^{\infty} x^2 p(x).dx$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 K.e^{-|x|}.dx$$

$$= K.2\int_0^{\infty} \underset{\mathrm{I}}{x^2}.\underset{\mathrm{II}}{e^{-x}}.dx.$$

$$= 2K\left\{x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(e^{-x}\right) + 2\left(\frac{+e^{-x}}{-1}\right)\right\}_0^{\infty}$$

$$= 2K(-(-2))$$

$$= \boxed{E[X^2] = 2}$$

$$\therefore V = E[X^2] - (E[X])^2$$

$$= 2 - 0.$$

$$\boxed{V = 2}$$

$$\therefore SD,\ \sigma = \sqrt{V}$$

$$\boxed{\sigma = 1.4142}$$

Repeat: Find the constant K so that; $p(x) = \begin{cases} Kxe^{-x} & 0 < x < 1 \\ 0 & , \text{otherwise.} \end{cases}$ if probability density fn, find, μ, V, σ?

$K = \dfrac{e}{e-2}$ $\mu = \dfrac{2e-5}{e-2}$ $E[X^2] = \dfrac{6e-16}{e-2}$

$$V = \frac{2e^2 - 8e + 7}{e^2 - 4e + 4}$$

The random variable x has the density fn; $p(x) = \dfrac{K}{1+x^2}$ $-\infty < x < \infty$, Determine K & evaluate;

i) $P(x \geq 0)$

ii) $P(0 < x < 1)$

Soln :- By defn : $\int_{-\infty}^{+\infty} \phi(x).dx = 1$

$$= \int_{-\infty}^{+\infty} \frac{K}{1+x^2}.dx = 1$$

$$= K \int_{-\infty}^{+\infty} \underbrace{\frac{1}{1+x^2}}.dx = 1.$$

$$= K \left(\tan^{-1}x\right)_{-\infty}^{\infty} = 1$$

$$K\left[\pi/2 + \pi/2\right] = 1$$

$$\therefore \boxed{K = 0.3183}$$

(i) $P(x \geq 0)$

$$\therefore = \int_0^{\infty} P(x).dx = K\int_0^{\infty} \frac{1}{1+x^2}.dx.$$

$$= K \tan^{-1}x \Big|_0^{\infty} = K.\pi/2 = 1/2$$

$$\therefore \boxed{P(x \geq 0) = 1/2}$$

(ii) $P(0 < x < 1)$

$$= \int_0^1 P(x).dx$$

$$= K\int_0^1 \frac{1}{1+x^2}.dx$$

$$= K \tan^{-1}(x)\Big|_0^1 = K.\pi/4$$

$$\therefore \boxed{P(0 < x < 1) = 1/4}$$

NORMAL DISTRIBUTION :-

(25)4

The continuous probability distribution having the probability density function (pdf), $f(x)$ is given by;

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

where; $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently; the following 2 conditions are satisfied;

1) $f(x) \geq 0$.

2) $\int_{-\infty}^{+\infty} f(x)dx = 1$.

MEAN & STANDARD DEVIATION OF NORMAL DISTRIBUTION :-

1) Mean, $\boxed{\mu = \text{Mean}}$, The mean of normal distribution is equal to the mean of the given distribution.

2) Variance; $\boxed{V = \sigma^2}$

3) Standard deviation; $\boxed{\sigma = SD}$

Hence, Variance & S.D. of normal distribution is equal to V & SD of given distribution.

* EXPONENTIAL DISTRIBUTION :-

The continuous probability distribution having the probability density function $f(x)$ given by;

$$f(x) = \begin{cases} \alpha \cdot e^{-\alpha x} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

is known as Exponential distribution.

* The 2 necessary conditions to be satisfied are :-

1) $f(x) > 0$.
2) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$.

* Mean & Standard deviation of Exponential distribution :-

Mean, $\boxed{\mu = \frac{1}{\alpha}}$

Variance, $\boxed{V = \frac{1}{\alpha^2}}$

Standard deviation ; $\boxed{\sigma = \frac{1}{\alpha}}$

· Problems & Solutions :-

1) Find which of the following functions is a probability density function.

Soln i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

iv) $f_4(x) = \begin{cases} 2x, & 0 < x \leq 1. \\ 4-4x, & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$

ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

iii) $f_3(x) = \begin{cases} |x|, & |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

Soln:- WKT: Conditions satisfied for a "probability density function" are :-

1) $f(x) \geq 0$.

2) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$.

Clearly ; $\underline{\underline{2x \geq 0}}$. // $x \geq 0$ 1st condition is satisfied.

(2) $\Rightarrow \int_{-\infty}^{\infty} f_1(x).dx = \int_0^1 2x.dx. = \left[2.\frac{x^2}{2}\right]_0^1$

$= [1-0]$

$\therefore \boxed{\int_{-\infty}^{+\infty} f_1(x).dx = 1.}$

Hence, 2 Conditions are satisfied.

$\Rightarrow f_1(x)$ is a "probability density function". (pdf)

ii) $f_2(x) = \begin{cases} 2x & , -1<x<1 \\ 0 & , \text{otherwise.} \end{cases}$

Soln:-

$f(x) = 2x \quad , -1<x<1$

$f(x) = -ve < 0$

$\therefore f(x) \geq 0$ Condition is not satisfied.

Consider (2) $\Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx$

$= \int_{-1}^{+1} 2x \cdot dx = \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}}\right]_{-1}^{+1}$

$= [\cancel{1-1}]^{0} = 0 \neq 1.$

$\therefore \int_{-1}^{+1} f(x)\,dx \neq 1$

$\therefore$ 2 Conditions are not satisfied.

$\therefore$ It is not pdf

iii) $f_3(x) = \begin{cases} |x| & , |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

Soln:- Since ; $f_3(x) = |x| \geq 0$ (1) Condn. is satisfied

(2) $\Rightarrow \int_{-\infty}^{+\infty} f_3(x)\,dx = \int_{-1}^{+1} |x|\,dx$

Here; $|x| = \begin{cases} -x & , \text{if } -1<x<0 \\ +x & , \text{if } 0<x<1. \end{cases}$

$\therefore \int_{-1}^{+1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx = -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$

$\therefore$ It is a pdf

iv) $f_4(x) = \begin{cases} 2x & , 0 < x \leq 1 \\ 4-4x & , 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$

Soln:- Clearly; $f_4(x) = 2x > 0$, $f_4(x) \geq 0$, ① Condn is satisfied in $0 < x \leq 1$.

Clearly; $f_4(x) = 4-4x$ is negative in : $1 < x < 2$.

∴ The 1st condition is not satisfied.

∴ $f_4(x)$ is not a pdf

2) Find the value of c such that;

$f(x) = \begin{cases} x/6 + c, & 0 \leq x \leq 3. \\ 0 & \text{elsewhere} \end{cases}$ is a probability density function.

Also find : $P(1 \leq x \leq 2)$.

Soln :- given that; $f(x)$ is a pdf

then, it satisfies :- 1) $f(x) \geq 0$.

2) $\int_{-\infty}^{+\infty} f(x).dx = 1$.

Consider; ② $\Rightarrow \int_{-\infty}^{+\infty} f(x)\, dx = 1$.

$$\int_0^3 \left(x/6 + c\right) dx = 1.$$

$$= \left[\frac{x^2}{12} + cx\right]_0^3 = 1.$$

$$= \frac{3}{4} + 3c = 1 \quad , \boxed{c = 1/12}$$

Now, to find : $P(1 \leq x \leq 2) = \int_1^2 f(x).dx.$

$$= \int_1^2 \left(x/6 + 1/12\right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}\left[(4+2) - (1+1)\right]$$

$$\therefore \boxed{p(1 \leq x \leq 12) = 1/3}$$

3) Find the constant K such that;

$$f(x) = \begin{cases} Kx^2 & 0<x<3 \\ 0 & \text{otherwise.} \end{cases}$$ is a p.d.f, Also Compute :

i) $p(1<x<2)$ ii) $P(x \leq 1)$ iii) $P(x>1)$

iv) Mean v) Variance.

Soln:- Since, Given that; $f(x)$ is pdf

$\Rightarrow$ 1) $f(x) \geq 0$

2) $\int_{-\infty}^{+\infty} f(x) dx = 1.$

Consider; ② $\Rightarrow \int_0^3 Kx^2.dx = 1.$

$$\Rightarrow \left[\frac{Kx^3}{3}\right]_0^3 = \left[\frac{K}{3}(3^3 - 0^3)\right] = \frac{K}{\cancel{3}_1} \cancel{27}^{9} = 1$$

$$\therefore \boxed{K = 1/9.}$$

Now, to find:

i) $P(1<x<2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x^2}{9}\,dx$

$= \left[\frac{x^3}{27}\right]_1^2 = \boxed{\frac{7}{27} = P(1<x<2)}$

ii) $P(x \leq 1) = \int_0^1 f(x)\,dx$

$= \int_0^1 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_0^1$

$\therefore \boxed{P(x \leq 1) = 1/27}$

iii) $P(x>1) = \int_1^3 f(x)\,dx$

$= \int_1^3 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_1^3$

$\therefore \boxed{P(x>1) = \frac{26}{27}}$

iv) Mean $= \mu = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx$

$= \int_0^3 x \cdot \frac{x^2}{9}\,dx = \left[\frac{x^4}{36}\right]_0^3 = \frac{81}{36} = \boxed{\frac{9}{4} = \mu}$

v) Variance $= V = \int_{-\infty}^{+\infty} x^2 f(x)\,dx - \mu^2$

$V = \int_0^3 x^2 \cdot \frac{x^2}{9} - \left(\frac{9}{4}\right)^2$

$V = \left[\frac{x^5}{45}\right]_0^3 - \frac{81}{16} = \frac{81}{240} = \boxed{\frac{27}{80} = V}$

—x—

EXPONENTIAL PROBABILITY :-

1) In a certain town, the duration of shower is exponential distributed with mean 5 minutes., what is the probability that a shower will last for :-

i) Less than 10 minutes. ii) 10 minutes (or) more.

Soln :- Given : Mean, $\boxed{\mu = 5}$

WKT; $\mu = 1/\alpha$ in Exp distribution.

$\frac{1}{\alpha} = 5$, $\boxed{\alpha = 1/5}$

WKT; $p(x) = \alpha e^{-\alpha x}$ in Exp distribution.

$p(x) = \frac{1}{5} e^{-\frac{x}{5}}$ ~ (1).

i) The probability that shower will last for : Less than 10 mins :-

$P(x<10) = \int_0^{10} p(x)\,dx. = \int_0^{10} \frac{1}{5}. e^{-x/5}.dx = \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5}\right]_0^{10}$

$p(x<10) = -(e^{-2}-1) \Rightarrow \boxed{P(x<10) = 0.8647}$

ii) Probability that shower will last for : 10/more mins. :-

$P(x \geq 10) = \int_{10}^{\infty} p(x).dx = \frac{1}{5}\int_{10}^{\infty} e^{-x/5} dx. = \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5}\right]_{10}^{\infty}$

$= \{0 - e^{-2}\} = e^{-2}$

$\therefore \boxed{P(x \geq 10) = 0.1353.}$

29 4

2) The length of telephone conversation has an exponential distribution with a mean of 3 minutes, find the probability that the call : i) Ends ~~at~~ in 3 minutes.
ii) Takes b/w 3 mins & 5 mins

Soln :- given : $\mu = \frac{1}{\alpha} = 3.$

$$\Rightarrow \boxed{\alpha = 1/3}$$

WKT; from Exponential distribution ; $p(x) = \alpha . e^{-\alpha x}$, $x \geq 0$.

$$\therefore p(x) = \frac{1}{3} . e^{-x/3} \quad -①.$$

i) The probability that ; the Call ends in 3 minutes ;

$$p(x \leq 3) = \int_0^3 p(x).dx. = \int_0^3 \frac{1}{3} . e^{-x/3} . dx.$$

$$= \frac{1}{\cancel{3}} . \left[\frac{e^{-x/3}}{-1/\cancel{3}} \right]_0^3 = -\left[e^{-3/3} - e^0 \right] = -\left[e^{-1} - 1 \right]$$

$$\therefore \boxed{p(x \leq 3) = 0.6321}$$

ii) The probability that ; call takes b/w 3 mins & 5 mins ;

$$p(3 < x < 5) = \int_3^5 p(x).dx = \int_3^5 1/3 . e^{-x/3} . dx.$$

$$\frac{1}{3}\int_3^5 e^{-x/3} . dx = \frac{1}{\cancel{3}} \left[\frac{e^{-x/3}}{-1/\cancel{3}} \right]_3^5$$

$$= -\left[e^{-5/3} - e^{-3/3} \right] = -\left[e^{-5/3} - e^{-1} \right]$$

$$\therefore \boxed{p(3 < x < 5) = 0.1790.}$$

3) The life of a Compressor manufactured by a Company is known to be 200 months on an average following the exponential distribution,

Find the probability that the life of Compressor is;

i) less than 200 months.

ii) b/w 100 months & 25 years.

Soln:- Given; $\mu = 1/\alpha = 200$, $\boxed{\alpha = \frac{1}{200}}$

WKT; $P(x) = \alpha . e^{-\alpha x}$.

$\therefore p(x) = \frac{1}{200} . e^{-x/200}$ ~ (1).

i) The probability that life of Compressor is : Less than 200 months is;

$$p(x<200) = \int_0^{200} p(x).dx = \int_0^{200} \frac{1}{200} . e^{-x/200} dx.$$

$$= \frac{1}{\cancel{200}}\left[\frac{e^{-x/200}}{-1/\cancel{200}}\right]_0^{200} = -\left[e^{-200/200} - e^0\right] = -\left[e^{-1} - 1\right]$$

$$\therefore \boxed{p(x<200) = 0.6321.}$$

ii) The probability that life of Compressor is b/w 100 months & 25 yrs;

$\Rightarrow \boxed{100 \text{ months}}$

$\Rightarrow 25$ yrs

$\Rightarrow 25 \times 12 = \boxed{300 \text{ years}}$ months

$$p(100<x<300) = \int_{100}^{300} p(x).dx$$

$$= \int_{100}^{300} \frac{1}{200} . e^{-x/200} dx.$$

$$= \frac{1}{\cancel{200}}\left[\frac{e^{-x/200}}{-1/\cancel{200}}\right]_{100}^{300} = -\left[e^{-300/200} - e^{-100/200}\right]$$

$$\therefore \boxed{p(100<x<300) = 0.3834}$$

4) The life of an Invertors (Generators) manufactured by a Company is known to be 100 months on an average following the exp distribution, Find probability that the life of Inverter is;

1) Less than 100 months.

2) b/w 100 & 15 years. $(15 \times 12 m) = 180$ months.

The ... di...

(30) 4

i) The sales per day in a shop is exponentially distributed with average sale amounting to ₹100 and net profit 8%, find the probability that; profit exceeds ₹30, on 2 Consequtive days.

Soln:- Given that; $\mu = \frac{1}{\alpha} = 100$, $\boxed{\alpha = \frac{1}{100}}$

WKT; $\phi(x) = \alpha \cdot e^{-\alpha x}$.

$$\phi(x) = \frac{1}{100} \cdot e^{-x/100} \quad \text{—(1)}.$$

Let "A" be the amount for which the profit is 8%;

$\Rightarrow A \cdot 8\% = 30$ rupees. // for prod, A if 8% profit given

$A \cdot \frac{8}{100} = 30$, $A = \frac{3000}{8}$

$\therefore \boxed{A = 375}$ // Actual price of product, A.

the probability of the profit exceeding ₹30 is;

1) Now, to find the probability of the profit exceeding ₹30 is;

$\Rightarrow P(\text{profit} \geq 30)$

$\Rightarrow P(\text{sales} \geq 375)$. [if profit should exceed 30, then sales should exceed 375].

WKT; $\phi(x) = \int f(x)\,dx$.

$$p(s \geq 375) \Rightarrow P(p \geq 30) = \int_{375}^{\infty} \frac{1}{100} \cdot e^{-x/100} \cdot dx = \frac{1}{100} \left[\frac{e^{-x/100}}{-1/100}\right]_{375}^{\infty}$$

$$= -\left\{e^{-\infty} - e^{-3.75}\right\}$$

$\therefore p(s \geq 375) = p(p \geq 30) = 0.0235$

$\therefore$ The prob., that profit exceeds ₹30 on 2 Consequtive days is:-

$= 0.0235 \times 0.0235$

$= 0.0006$

6) The daily turnover in medical shop is exp distributed with ₹ 6000 avg, with net profit of 8%. Find prob that profit exceeds ₹ 500, on randomly choosen day

Soln :- Given ; $\mu = 1/\alpha = 6000$, $\boxed{\alpha = 1/6000}$

WKT; $P(x) = \alpha e^{-\alpha x}$, $p(x) = \frac{1}{6000} \cdot e^{-x/6000}$ — ①.

let A be the amount for which profit is 8% ;

$\Rightarrow A \cdot 8\% = 500$, $A \cdot \frac{8}{100} = 500$.

$A = \frac{50000}{8}$, $\boxed{A = 6250}$

$\therefore$ The probability of profit exceeding ₹ 500 is ;

$\Rightarrow P(\text{profit} \geq 500)$

$\Rightarrow P(\text{sales} \geq 6250)$

$$\Rightarrow \int_{6250}^{\infty} \frac{1}{6000} e^{-x/6000} dx = \frac{1}{\cancel{6000}} \left[\frac{e^{-x/6000}}{-1/\cancel{6000}} \right]_{6250}^{\infty}$$

$$= -\left[e^{\infty} - e^{-6250/6000} \right]$$

$$= e^{-25/24}$$

$$= 0.3529$$

$\therefore$ probability that the profit exceeds ₹ 500, on a randomly choosen day is : 0.3529

NORMAL DISTRIBUTION :-

The probability fn ; $\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is called Normal distribution.

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot \phi(x) \, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx. \quad = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \quad \text{--- (1)}$$

put ; $z = \frac{x-\mu}{\sqrt{2}\sigma} \Rightarrow \left.\begin{array}{l} x = \mu + \sqrt{2}\sigma z \\ dx = \sqrt{2}\sigma dz \end{array}\right\}$ use in Eqn (1) ; we get.

$$\therefore \text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma z) \, e^{-z^2} \sqrt{2}\sigma \cdot dz$$

odd function $\Rightarrow \int \text{odd} = 0$

$$= \frac{1}{\sqrt{\pi}} \left\{ \mu \underbrace{\int_{-\infty}^{\infty} e^{-z^2} dz} + \sqrt{2}\sigma \cancel{\int_{-\infty}^{\infty} z \, e^{-z^2} \cdot dz} \right\}.$$

$$= \frac{1}{\sqrt{\pi}} \left\{ \mu \sqrt{\pi} \right\}.$$

$$\therefore \boxed{\text{Mean} = \mu}$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 \, \phi(x) \cdot dx$$

$$V = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx \quad \text{--- (2)}.$$

put ; $z = \frac{x-\mu}{\sqrt{2}\sigma}$, $\left.\begin{array}{l} x = \mu + \sqrt{2}\sigma z \\ dx = \sqrt{2} \cdot \sigma dz \end{array}\right\}$ use in Eqn (2), we get

$$\therefore V = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma z - \mu)^2 e^{-z^2} (\sqrt{2}\sigma dz).$$

$$V = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma z)^2 e^{-z^2} \sqrt{2}\sigma . dz.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 . e^{-z^2} . dz.$$

$$V = \frac{\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z (2z e^{-z^2}) dz \quad -(3).$$

Let; $I = \int 2z . e^{-z^2} . dz$, put : $t = -z^2$
$dt = -2z . dt$.

$$\therefore I = -\int e^t dt.$$

$$\boxed{I = -e^{-z^2}}$$

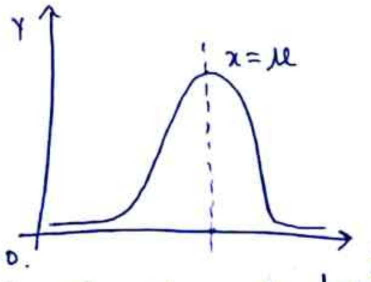
$$\therefore V = \frac{\sigma^2}{\sqrt{\pi}} \left\{ z(-e^{-z^2}) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} (-e^{-z^2}) . dz \right\}.$$

$$= \frac{\sigma^2}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}} \qquad \therefore \boxed{V = \sigma^2}$$

$$\therefore S.D = \sqrt{V}$$

$$\therefore \boxed{S.D = \sigma}$$

Note :- 1) The Graph of prob function, $p(x)$ is bell-shaped curve symmetrical about line, $x = \mu$ & is called Normal distribution Curve.

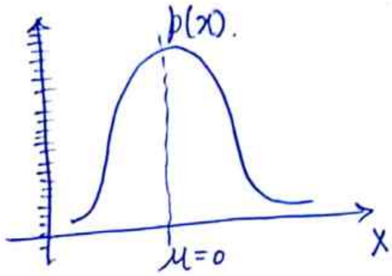


The line $x = \mu$ divides total area which is equal to 1, divided in to 2 Equal parts.

2) In Normal distribution, the limit values may be whole no's or decimal values.

STANDARD NORMAL DISTRIBUTION:-

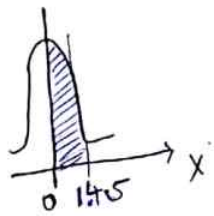
$$\phi(z) = \frac{1}{\sqrt{2\pi}}\left[\int_0^z e^{-\frac{z^2}{2}}\right] dz = A.$$



* problems & sdz:-

1) Evaluate the following:-

i) $P(0 \leq z \leq 1.45)$.



$\frac{0.3495}{2.50662} = 0.1394$

$P(0 \leq z \leq 1.45)$

$= A(1.45) = \phi(1.45)$ $\Rightarrow \boxed{z = 1.45}$

// WKT; theoretical; Area, $A = \frac{1}{\sqrt{2\pi}}\int_0^{1.45} e^{-z^2/2} . dz = \phi(z)$ ↳ (Do in calculator)

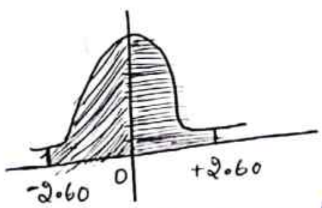
$\therefore P(0 \leq z \leq 1.45) = A = \phi$

$= A = \frac{1}{\sqrt{2\pi}}[1.069003]$.

$\Rightarrow 0.4264$

$A = 0.4264$

ii) $P(-2.60 \leq z \leq 0)$.



(Integral value can't be -ve)

$P(-2.60 \leq z \leq 0)$

$\Rightarrow P(0 \leq z \leq 2.60)$.

$\Rightarrow P(0 \leq z \leq 2.60)$

$\Rightarrow A(2.60)$

// $A = \frac{1}{\sqrt{2\pi}}\int_0^z e^{-z^2/2} . dz = \phi(z)$

$\boxed{z = 2.60}$

$A = \frac{1}{\sqrt{2\pi}}\int_0^{2.60} e^{-z^2/2} . dz = \phi(3.20)$

$A = 0.~~03531~~ 0.4953$

$\Rightarrow$ ~~0.03531~~

$\Rightarrow 0.4953$

3) $P(0 \leq z \leq 1.55)$.

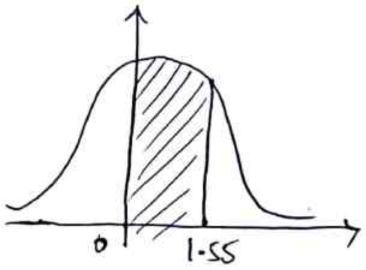
$\Rightarrow P(0 \leq z \leq \underline{1.55})$

$\Rightarrow A(1.55) \rightarrow \phi(z) \Rightarrow \underline{z = 1.55}$

WKT; $A = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} \, dz$. [should not substitute value of z, since no z value will be ther to integrate].

$A = \frac{1}{\sqrt{2\pi}} \int_0^{1.55} e^{-\frac{(1.55)^2}{2}} \, dz$

$A = \underline{\underline{0.4394}}$



4) $P(-3.4 \leq z \leq 2.65)$

$\Rightarrow P(-3.4 \leq z \leq 2.65)$

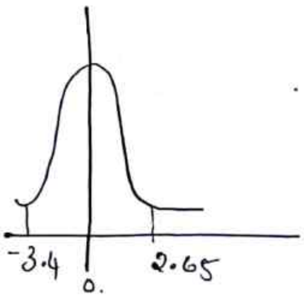
$\Rightarrow P(-3.4 \leq z \leq 0) + P(0 \leq z \leq 2.65)$

$\Rightarrow P(0 \leq z \leq 3.4) + P(0 \leq z \leq 2.65)$

$\Rightarrow A(3.4) + A(2.65)$

$= 0.49966 + 0.4960$

$= \underline{\underline{0.9957}}$



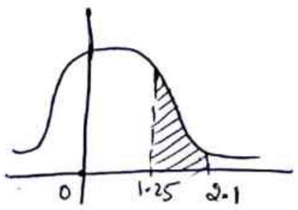
5) $P(1.25 \leq z \leq 2.1)$

$P(0 \leq z \leq 2.1) - P(0 \leq z \leq 1.25)$

$= A(2.1) - A(1.25)$

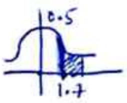
$= 0.4821 - 0.3944$

$= \underline{\underline{0.0877}}$



Asmt

6) $P(z \geq 1.7) \Rightarrow 0.5 - A(1.7)$
$= \underline{0.0446}$



7) $P(0 \leq z \leq 1.25)$.

8) $P(0 \leq z \leq 1.65)$.

Prob & Soln :-

1) An Analog signal received as detector may be modelled as normal random variable with mean 200 and variance 256 at fixed point of time, what is the probability that signal will exceed 240 μV?

Soln :- Given ; $\mu = 200$.

$$V = \sigma^2 = 256.$$

$$\Rightarrow \boxed{\sigma = 16}$$

∴ The standard normal variable is ; $z = \frac{x-\mu}{\sigma}$, $z = \frac{x-200}{16}$ — (1)

when ; $x = 240$

(1) ⇒ $z = \frac{240-200}{16}$, $\boxed{z = 2.5}$

∴ The probability that signal exceeds 240 μV is ;

$P(x > 240)$

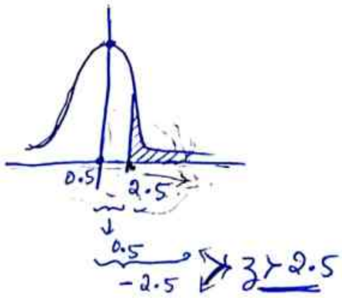
$P(z > 2.5) \Rightarrow 0.5 - A(2.5)$

$= 0.5 - 0.4938$

∴ $\boxed{p(x>240) = 0.0062}$

$$A = \frac{1}{\sqrt{2\pi}} \int_0^{2.5} e^{-z^2/2} \cdot dz.$$

$A = 0.4938$



2) A Certain machine makes electric resistors having a mean of 40 Ω & standard deviation of 2 Ω, Assuming that the resistance follows a normal distribution, what percentage of resistors will have resistance that exceeds 43 Ω?

Soln :- Given ; $\boxed{\mu = 40}$

$\boxed{\sigma = 2}$

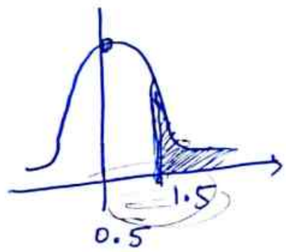
The standard normal variable is ;

$$z = \frac{x-\mu}{\sigma} = \frac{x-40}{2} \sim ①$$

when ; $x = 43\,\Omega$

$$① \Rightarrow z = \frac{43-40}{2} \quad , \boxed{z = 1.5}$$

∴ The probability that the signal exceeds $43\,\Omega$ is ;



$P(x > 43)$.

ie, $\Rightarrow P(z > 1.5)$.

$= 0.5 - A(1.5)$

$= 0.5 - 0.4332$

$= 0.0668$ (or) $6.68\,\%$

(34) 4

2) If the amount of cosmic radiation to which a random variable having normal distribution, if person is exposed while flying by jet, with mean, 4.35 & S.D, 0.59, find probability that amount of cosmic radiation to which person will be exposed on such flight is;

1) b/w 4 & 5   2) Atleast 5.5.

Soln :- Given ; $\mu = 4.35$, $\sigma = 0.59$

The standard normal variable is ; $z = \frac{x-\mu}{\sigma} = \frac{x-4.35}{0.59}$ — (1).

when ; $x=4$ (1) $\Rightarrow$ $z = \frac{4-4.35}{0.59}$, $\boxed{z = -0.5932}$

when ; $x=5$ (1) $\Rightarrow$ $z = \frac{5-4.35}{0.59}$, $\boxed{z = 1.1017}$.

when ; $x=5.5$ (1) $\Rightarrow$ $z = \frac{5.5-4.35}{0.59}$, $\boxed{z = 1.9492.}$

— 0 → 0 → +

(i) $p(4<x<5) \Rightarrow p(-0.5932 \leq z \leq 1.1017)$

$\Rightarrow P(-0.5932 \leq z \leq 0) + P(0 \leq z \leq 1.1017)$.

$\Rightarrow P(0 \leq z \leq 0.5932) + P(0 \leq z \leq 1.1017)$.

$\Rightarrow A(0.5932) + A(1.1017)$. //

$\Rightarrow 0.2224 + 0.3643$.

$\therefore \boxed{p(4<x<5) = 0.5867} //$.

(ii) $P(x \geq 5.5)$.

$\Rightarrow p(z \geq 1.9492)$

$\Rightarrow 0.5 - A(1.9492) \approx A(1.95)$

$\Rightarrow 0.5 - 0.4744$

$\therefore \boxed{P(x \geq 5.5) = 0.0256.}$

https://hemanthrajhemu.github.io

4) The mean weight of 500 studs at a certain school is 50kg & S.D is 6 kg, Assuming that weights are normally distributed, find no of students weighing :- i) b/w 40 & 50kg ii) more than 60 k

Soln:- Given; $\boxed{\mu = 50}$, $\boxed{\sigma = 6}$, $z = \frac{x-50}{6}$.

when, $x = 40$, $z = -1.6667$, $x = 60$, $z = 1.6667$

$x = 50$, $z = 0$.

(i) $P(40 < x < 50) \Rightarrow 0.4525$ (ii) $P(x > 60) = 0.0475$

$\therefore$ The no of stdts $= 0.0475 \times 500$
$= 23.75 \simeq 24$

5) The life of a certain type of electric lamps is normally distributed with mean 2040 hours, and S.D of 60 hrs. In a consignment of 2000 lamps, find how many would be expected to burn for :-
i) More than 2150 hrs.
ii) less than 1950 hrs
iii) b/w 1920 hrs & 2160 hrs.

Soln:- Given; $\boxed{\mu = 2040}$

$\boxed{\sigma = 60}$

The standard normal variable is ; $z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$ ~ (1).

when; $x = 2150$, (1) $\Rightarrow z = \frac{2150-2040}{60}$, $\boxed{z = 1.8333}$

when; $x = 1950$, (1) $\Rightarrow \boxed{z = -1.5}$

when; $x = 1920$, (1) $\Rightarrow \boxed{z = -2}$

when; $x = 2160$, (1) $\Rightarrow \boxed{z = 2}$

(i) $P(x > 2150)$
$= P(z > 1.8333)$
$= 0.5 - A(1.83) = 0.5 - 0.4664$
$= 0.0336$

$\therefore$ No of lamps $= 2000 \times 0.0336$
$= 67$

ii) $P(x < 1950)$

$= P(z < -1.5) = P(z > 1.5)$

$= 0.5 - A(1.5) = 0.5 - 0.4332$

$= \underline{\underline{0.0668}}$ $\therefore$ No of lamps $= 0.0668 \times 2000$

$= \underline{\underline{134}}$

iii) $P(1920 < x < 2160)$

$P(-2 < z < 2) = P(-2 < z < 0) + P(0 < z < 2)$

$\Rightarrow P(0 < z < 2) + P(0 < z < 2) = A(2) + A(2)$

$= 0.4772 + 0.4772$

$= \underline{\underline{0.9544}}$

$\therefore$ No of lamps $= 0.9544 \times 2000$

$= \underline{\underline{1909}}$

6) A sample of 100 dry battery cells produced by a certain company were tested for lengths of life & test yielded the data; mean $\rightarrow$ 12hrs & $\sigma = 3$hrs, Using Normal distribution, how many cells are expected to have their life :- (i) Greater than 15hrs

(ii) b/w 10 & 14 hrs

iii) less than 6 hrs.

Soln:- $\boxed{\mu = 12}$ $\boxed{\sigma = 3}$

S.N. Var $\Rightarrow z = \frac{x-\mu}{\sigma}$

when; $x = 15$, $\boxed{z = 1}$

when, $x = 10$, $\boxed{z = -0.6667}$

when $x = 14$, $\boxed{z = 0.6667}$

when $x = 6$, $\boxed{z = -2}$

(i) $P(x \geq 15) = 0.5 - A(1)$

$= \underline{\underline{0.1587}}$

$\therefore$ No of cells $= 1000 \times 0.1587$

$= \underline{\underline{16}}$

(ii) $P(10 < x < 14)$

$= P(0 < z < 0.6667) + P(0 < z < 0.6667)$

$= \underline{\underline{0.4974}}$ $\therefore$ No of cells $= \underline{\underline{50}}$

(iii) $P(x \leq 6) = 0.5 - A(2) = 0.0228$ $\therefore$ No of cells $= \underline{\underline{2}}$

7) In a normal distribution, 31% of items are under 45, and 8% of items are over 64, find mean & S.D of distribution. 8) Hint

$A(z_1) \rightarrow$ ___ , $z_1 =$

Soln:- Given;

The Standard Normal variable is; $z = \frac{x-\mu}{\sigma}$.

when; $x = 45$; $z = \frac{45-\mu}{\sigma} = z_1$

when; $x = 64$; $z = \frac{64-\mu}{\sigma} = z_2$.

$\therefore P(x < 45) = \frac{31}{100} = 31\% \Rightarrow P(x < 45) = 0.31$

$\Rightarrow P(z < z_1) = 0.31$.

$\Rightarrow 0.5 + A(z_1) = 0.31$.

$A(z_1) = 0.31 - 0.5$, $A(z_1) = -0.19$

$z_1 = \cancel{-0.31 - 0.19}$

$\boxed{z_1 = -0.5}$ $\rightarrow$ how? z_1

$\therefore P(x > 64) = 0.08 \Rightarrow P(x > 64) = 8\% = \frac{8}{100}$

$\Rightarrow P(z > z_2) = 0.08$

$\Rightarrow 0.5 - A(z_2) = 0.08$

$\Rightarrow A(z_2) = 0.42$

$= z_2 = 1.41$ $\rightarrow$ how?

we have; $z_1 = \frac{45-\mu}{\sigma} \Rightarrow \frac{45-\mu}{\sigma} = -0.5$

$\Rightarrow 45 - \mu = -0.5 \times \sigma$

$\Rightarrow \mu + 0.5\sigma = 45 \sim (1)$

we have; $z_2 = \frac{64-\mu}{\sigma} \Rightarrow \frac{64-\mu}{\sigma} = 1.41$

$\Rightarrow \mu + 1.41\sigma = 64 \sim (2)$.

Solving (1) & (2) $\rightarrow$ $\boxed{\mu = 49.9738}$

$\boxed{\sigma = 9.9476}$

$\mu + 0.5\sigma = 45$
$\mu + 1.41\sigma = 64$
$(-)\ (-)\ (-)$

$-0.91\sigma = -19$

$\boxed{\sigma = 9.947}$

$\boxed{\mu = 49.9738.}$

8) In a normal distribution 7% of items are under 35 & 89% of items are under 65, find the mean and variance given that ; $A(1.23) = 0.39$, $A(1.48) = 0.42$.

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Soln :- Given ; $P(x<35) = 0.07$, $P(x<65) = 0.89$, $z = \frac{x-\mu}{\sigma}$

when ; $x = 35$, $z = \frac{35-\mu}{\sigma} = z_1$

$x = 65$, $z = \frac{65-\mu}{\sigma} = z_2$

$P(x<35) = 0.07$
$P(z<z_1) = 0.07$
$0.5 + A(z_1) = 0.07$
$A(z_1) = -0.43$, $z_1 = -1.48$

$P(x<65) = 0.89$
$P(z<z_2) = 0.89$
$0.5 + A(z_2) = 0.89$
$A(z_2) = 0.39$, $z_2 = 1.23$

we have ; $\mu - 1.48\sigma = 35$ — ①
$\mu + 1.23\sigma = 65$ — ②

$\mu = 51.3868$
$\sigma = 11.0701$
$V = 122.5471$

9) Steel rods are manufactured to be 3cms in diameter, but they are acceptable if they are inside the limits. 2.99cms & 3.01 cms. It is observed that 5% are rejected as oversized and 5% are rejected as undersized. assuming that diameters are normally distributed, find mean & std. deviation of distribution, $A(1.65) = 0.45$.

Soln :- Given ; $P(x < 2.99) = 0.05$
$P(x > 3.01) = 0.05$

Standard normal variable is ; $z = \frac{x-\mu}{\sigma}$

when ; $x = 2.99$, $z = \frac{2.99-\mu}{\sigma} = z_1$ ~ ①

when ; $x = 3.01$, $z = \frac{3.01-\mu}{\sigma} = z_2$ — ②

we have ; $P(x < 2.99) = 0.05$

$\Rightarrow P(z < z_1) = 0.05$

$\Rightarrow 0.5 + A(z_1) = 0.05$

$A(z_1) = -0.45$ // By reffering to normal probability table; [Refer last page in KSC]

$z_1 = -1.65$

Also; $P(x > 3.01) = 0.05$

$P(z > z_2) = 0.05 \Rightarrow 0.5 - A(z_2) = 0.05$

$A(z_2) = 0.45$

$\boxed{z_2 = 1.65}$

$\therefore$ ① $\Rightarrow \dfrac{2.99 - \mu}{\sigma} = -1.65$ ② $\Rightarrow \dfrac{3.01 - \mu}{\sigma} = 1.65$

$\mu - 1.65\sigma = 2.99$ ④$_1$ $\mu + 1.65\sigma = 3.01$ ④$_2$

$\Rightarrow$ Solving Eqn; $\boxed{\mu = 3.}$, $\boxed{\sigma = 0.0061}$

10) Assign

A manufacturer doesnot know, the mean & S.D of diameters of ball bearings, he is producing, However, a seceiving system rejects all bearings larger than 2.4 cm & those under 1.8 cm in diameter, out of 1000 ball bearings, 8% are rejected as tooo small & 5.5% as too big, what is mean & σ of ball bearings produced.

[$p(x > 2.4) = 0.055$, $P(x < 1.8) = 0.08$]

— * —

JOINT PROBABILITY DISTRIBUTION :-

Joint probability function :- If X & Y are 2 discrete random variables, we define the joint probability function of X & Y by;

$P(X=x, Y=y) = f(x,y)$.

where; $f(x,y)$ satisfies the conditions ① $f(x,y) \geq 0$.

② $\sum_x \sum_y f(x,y) = 1$.

Joint probability distribution :-

The set of values of the function : $f(x_i, y_j) = J_{ij}$ for $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$ is called Joint probability distribution of X & Y.

Joint probability density function; f is reffered to as joint probability density function of X & Y.

where; $X \times Y = \{(x_1, y_1), (x_2, y_2) \ldots (x_m, y_n)\}$.

* Joint probability distribution table is Given below;

X \ Y	y_1	y_2	......	y_n	sum
x_1	J_{11}	J_{12}	...	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	...	J_{2n}	$f(x_2)$
⋮	⋮	⋮	...	J_{mn}	$f(x_m)$
x_m	J_{m1}	J_{m2}			
sum.	$g(y_1)$	$g(y_2)$	....	$g(y_n)$	1

Expectation :- If X is a discrete random variable, taking values $x_1, x_2, \ldots x_n$ having probability fn $f(x)$, then the expectation of X denoted by $E(X)$ or μ_X is defined by;

$$\mu_X = E[X] = \sum_{i=1}^{n} x_i f(x_i) \quad (\text{or}) \quad \sum x \cdot f(x)$$

Note :- If X & Y are 2 discrete random variables having joint probability function $f(x,y)$, then Expectations of X & Y are defined as follows;

$$\mu_X = E[X] = \sum_x \sum_y x \cdot f(x,y) = \sum_i x_i f(x_i)$$

$$\mu_Y = E[Y] = \sum_x \sum_y \cdot y f(x,y) = \sum_j y_j g(y_j).$$

Further; $E[XY] = \sum_{i,j} x_i y_j J_{ij}$

Variance :- The variance of X is denoted by : $V(X)$.

$$V(X) = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = E[(X-\mu)^2].$$

$\mu \to$ mean.

Standard deviation :- The S.D is denoted by : σ_X

$$\boxed{\sigma_X = \sqrt{V(X)}}$$

Covariance :- If X & Y are random variables having mean μ_X & μ_Y resp, then the Covariance of X & Y denoted by : $COV(X,Y)$ defined by;

$$COV(X,Y) = \sum_i \sum_j (x_i - \mu_X)(y_j - \mu_Y) J_{ij}$$

$$= E[(X-\mu_X)(Y-\mu_Y)].$$

$$\boxed{COV(X,Y) = E(XY) - \mu_X \mu_Y.}$$

Correlation of X & Y :-

The correlation of X & Y denoted by ; $\rho(X,Y)$ is defined by the relation ; $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$.

Note :-

1) If X & Y are independent random variables, then ;

i) $E(XY) = E(X).E(Y)$

ii) $Cov(X,Y) = 0$, and hence $\rho(X,Y) = 0$.

2) $V(X) = E(X^2) - [E(X)]^2$

* Problems & Soln's :-

1) The joint distribution of 2 random variables X & Y is as follows

X \ Y	y_1 = -4	y_2 = 2	y_3 = 7
x_1 = 1	J_{11} 1/8	J_{12} 1/4	J_{13} 1/8
x_2 5	J_{21} 1/4	J_{22} 1/8	J_{23} 1/8

Compute the following :-

a) $E[X]$ & $E[Y]$ b) $E[XY]$

c) σ_X & σ_Y d) $Cov(X,Y)$ e) $\rho(X,Y)$.

The distribution of X & Y is as follows :-

This distribution is obtained by adding all respective row entries & also respective columns ;

Distribution of X :-

x_i	x_1 1	x_2 5
$f(x_i)$	$f(x_1)=1/2$	$f(x_2)=1/2$.

Distribution of Y :-

y_j	y_1 −4	y_2 2	y_3 7
$g(y_j)$	3/8	3/8	1/4
	$g(y_1)$	$g(y_2)$	$g(y_3)$

$g(y_1) = 1/8 + 1/4$

$\boxed{g(y_1) = 3/8}$

$g(y_2) = 1/4 + 1/8$

$\boxed{g(y_2) = 3/8}$

$g(y_3) = 1/8 + 1/8$

$\boxed{g(y_3) = 1/4}$

$f(x_1) \Rightarrow 1/8 + 1/4 + 1/8$

$\boxed{f(x_1) = 1/2}$ $\quad \frac{1+2+1}{8} = \frac{4}{8} = \frac{1}{2}$

$f(x_2) = 1/4 + 1/8 + 1/8$.

$\boxed{f(x_2) = 1/2}$

⇒ Now, to Compute (a)

(a) $E[X]$ & $E[Y]$

WKT; $E[X] = \sum x_i f(x_i)$

$= x_1 f(x_1) + x_2 f(x_2)$

$= 1(1/2) + 5(1/2)$

$\boxed{E[X] = 3}$

$E[Y] = \sum y_j g(y_j)$

$= y_1 g(y_1) + y_2 g(y_2) + y_3 g(y_3)$

$= -4(3/8) + 2(3/8) + 7(1/4)$

$\therefore \boxed{E[Y] = 1}$

Thus ; $\boxed{\mu_X = E[X] = 3}$, $\boxed{\mu_Y = E[Y] = 1}$

(b) $E[XY]$

WKT; $E[XY] = \sum x_i y_j J_{ij}$

$= x_1 y_1 J_{11} + x_1 y_2 J_{12} + x_1 y_3 J_{13} + x_2 y_1 J_{21} + x_2 y_2 J_{22} + x_3 y_3 J_{23}$

$= 1(-4)(1/8) + (1)(2)(1/4) + 1(7)(1/8) + 5(-4)(1/4) + 5(2)(1/8) + 5(7)(1/8)$

$= -1/2 + 1/2 + 7/8 - 5 + 5/4 + 35/8 = 3/2$.

$\therefore \boxed{E[XY] = 3/2}$

c) σ_x & σ_y.

WKT; $\sigma_x^2 = E[X^2] - (E[X])^2$, $\sigma_y^2 = E[Y^2] - (E[Y])^2$

Consider; $E[X^2] = \sum x_i^2 f(x_i)$

$E[X^2] = x_1 f(x_1) + x_2 f(x_2)$
$= (1)(1/2) + (25)(1/2)$

$\therefore \boxed{E[X^2] = 13}$

$E[Y^2] = \sum y_j^2 g(y_j)$
$= (16)(3/8) + 4(3/8) + 49(1/4)$

$\boxed{E[Y^2] = 79/4}$

Consider; $\sigma_x^2 = 13 - (3)^2$

$\sigma_x^2 = 4$

$\boxed{\sigma_x = 2}$

$\sigma_y^2 = 79/4 - (1)^2$

$\sigma_y = \sqrt{75/4} = \boxed{4.33 = \sigma_y}$

d) <u>COV (X, Y)</u>

WKT; $COV(X, Y) = E[XY] - \mu_x \mu_y$
$= (3/2) - (3)(1) = -3/2$

$\therefore \boxed{COV(X, Y) = -3/2}$

e) <u>$\rho(X, Y)$</u>

WKT; $\rho(X, Y) = \dfrac{COV[X, Y]}{\sigma_x \sigma_y}$

$= \dfrac{-3/2}{(2)\sqrt{75/4}} = \dfrac{-3}{2\sqrt{75}} = \boxed{-0.1732 = \rho(X, Y)}$

2) The joint probability distributions ~~of~~ table for 2 random variables X & Y is as follows;

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0.

Determine marginal probability distributions of X & Y

Also Compute; (a) Expectations of X, Y & XY.

(b) SD's of X, Y.

(c) Covariance of X & Y

(d) Correlation of X & Y.

Soln:-

∵ WKT; Marginal distributions of X & Y are got by adding all respective row entries & respective column entries.

Distribution of X.

	x_1	x_2
x_i	1	2
$f(x_i)$	$f(x_1)$ 0.6	$f(x_2)$ 0.4

$f(x_1) = 0.1+0.2+0.3$

$\boxed{f(x_1)=0.6}$

$f(x_2) = 0.2+0.1+0.1$

$\boxed{f(x_2)=0.4}$

Distribution of Y.

	y_1	y_2	y_3	y_4
y_j	-2	-1	4	5.
$g(y_j)$	0.3 $g(y_1)$	0.3 $g(y_2)$	0.1 $g(y_3)$	0.3. $g(y_4)$.

$g(y_1) = 0.1+0.2 = \boxed{0.3 = g(y_1)}$

$g(y_2) = 0.2+0.1 = \boxed{g(y_2) = 0.3}$

$g(y_3) = 0+0.1 = \boxed{g(y_3) = 0.1}$

$g(y_4) = 0.3+0 \Rightarrow \boxed{g(y_4) = 0.3.}$

(a) $\mu_x = E[X] = \sum_i x_i f(x_i)$

$= x_1 f(x_1) + x_2 f(x_2)$

$= 1(0.6) + 2(0.4)$

$\boxed{\mu_x = E[X] = 1.4}$

$\mu_y = E[Y] = \sum_j y_j g(y_j)$

$= y_1 g(y_1) + y_2 g(y_2) + y_3 g(y_3) + y_4 g(y_4)$

$\boxed{\mu_y = E[Y] = 1}$

$E[XY] = \sum_{ij} x_i y_j J_{ij}$

$= x_1 y_1 J_{11} + x_1 y_2 J_{12} + x_1 y_3 J_{13} + x_1 y_4 J_{14} + x_2 y_1 J_{21} + x_2 y_2 J_{22}$

$+ x_2 y_3 J_{23} + x_2 y_4 J_{24}$

$\therefore \boxed{E[XY] = 0.9}$

$\sigma_X^2 = E[X^2] - (E[X])^2$

$E[X^2] = \sum_i x_i^2 f(x_i)$

$\boxed{E[X^2] = 2.2.}$

$\sigma_Y^2 = E[Y]^2 - (E[Y])^2$

$E[Y^2] = \sum_j y_j^2 g(y_j)$

$\boxed{E[Y^2] = 10.6}$

$\therefore \sigma_X^2 = 2.2 - (1.4)^2$

$\sigma_X^2 = 0.24$

$\boxed{\sigma_X = 0.49.}$

$\sigma_Y^2 = 10.6 - (1)^2$

$\sigma_Y^2 = 9.6$

$\boxed{\sigma_Y = 3.1}$

(c) $Cov(X,Y) = E[XY] - E[X].E[Y]$

$= 0.9 - (1.4)(1) = -0.5$

$\boxed{Cov(X,Y) = -0.5.}$

(d) Correlation of X & Y $= \rho(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}.$

$\rho(X,Y) = \dfrac{-0.5}{(0.49)(3.1)} = \boxed{-0.3 = \rho(X,Y)}$

(3). Given the following joint distribution of random variables X & Y, find corresponding marginal distribution, Also compute Covariance & correlation of random variables X & Y.

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12.

Soln :- The marginal distributions of X & Y is ;

x_i	2	4	6
$f(x_i)$	1/4	1/2	1/4

y_j	1	3	9
$g(y_j)$	1/2	1/3	1/6.

$\therefore$ $\text{Cov}(X,Y) = E[XY] - \mu_X \mu_Y$

$E[X] = \sum_i x_i f(x_i)$

$= 2(1/4) + 4(1/2) + 6(1/4) \Rightarrow \boxed{E[X] = 4}$

$E[Y] = \sum_j y_j g(y_j)$

$= 1(1/2) + 3(1/3) + 9(1/6) = \boxed{3 = E[Y]}$

$E[XY] = \sum_{i,j} x_i y_j J_{ij}$

$E[XY] = x_1 y_1 J_{11} + x_1 y_2 J_{12} + x_1 y_3 J_{13} + x_2 y_1 J_{21} + x_2 y_2 J_{22} + x_2 y_3 J_{23}$
$+ x_3 y_1 J_{31} + x_3 y_2 J_{32} + x_3 y_3 J_{33}.$

$\boxed{E[XY] = 12}$

we have ; $\text{Cov}[X,Y] = E[XY] - E[X]E[Y]$

$= 12 - 4(3) = 0.$

$\therefore \boxed{\text{Cov}(X,Y) = 0.}$

Correlation of X & Y :— $\rho(X,Y) = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}.$

$\therefore \boxed{\rho(X,Y) = 0.}$