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10MAT41

Fourth Semester B.E. Degree Examination, Dec.2013/Jan.2014
Engineering Mathematics – IV

Time: 3 hrs. Max. Marks:100

Note: ***Answer FIVE full questions, selecting at least TWO questions from each part.***

PART – A

1 a. Employ Taylor's series method to obtain the value of y at x = 0.1 and 0.2 for the differential equation $\frac{dy}{dx} = 2y + 3e^x$, y(0) = 0 considering upto fourth degree term. **(06 Marks)**

b. Determine the value of y when x = 0.1, given that y(0) = 1 and $y' = x^2 + y^2$ using modified Euler's formula. Take h = 0.05. **(07 Marks)**

c. Apply Adams-Bashforth method to solve the equation $\frac{dy}{dx} = x^2(1+y)$, given y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979. Evaluate y(1.4). **(07 Marks)**

2 a. Solve $\frac{dy}{dx} = 1 + zx$, $\frac{dz}{dx} = -xy$, y(0) = 0, z(0) = 1 at x = 0.3 by taking h = 0.3. Applying Runge-Kutta method of fourth order. **(06 Marks)**

b. Applying Picard's method to compute y(1.1) from the second approximation to the solution of the differential equation $y'' + y^2 y' = x^3$. Given that y(1) = 1, y'(1) = 1. **(07 Marks)**

c. Using the Mitni's method obtain an approximate solution at the point x = 0.8 of the problem $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$, give that y(0) = 0, y'(0) = 0, y(0.2) = 0.02, y'(0.2) = 0.1996, y(0.4) = 0.0795, y'(0.4) = 0.3937, y(0.6) = 0.1762, y'(0.6) = 0.5689. **(07 Marks)**

3 a. Derive Cauchy-Riemann equations in Cartesian form. **(06 Marks)**

b. Give $u + v = (x - y)(x^2 + 4xy + y^2)$ find the analytic function f(z) = u + iv. **(07 Marks)**

c. If f(z) = u + iv is an analytic function then prove that $\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2$ **(07 Marks)**

4 a. Find the image of the straight lines parallel to coordinate axes in z-plane under the transformation $w = z^2$. **(06 Marks)**

b. Find the bilinear transformation which maps the points z = 1, i, –1 on to the points w = 0, 1, ∞. **(07 Marks)**

c. Evaluate $\int_c \frac{e^{2z}}{(z+1)(z+2)}$, where c is the circle | z | = 3. **(07 Marks)**

PART – B

5 a. Find the solution of the Laplace equation in cylindrical system leading to Bessel's differential equation. (06 Marks)

b. If α and β are two distinct roots of $J_n(x) = 0$, then prove that $\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = 0$, if $m = n$. (07 Marks)

c. Express $f(x) = x^4 - 2x^3 + 3x^2 - 4x + 5$ in terms of legendre polynomial. (07 Marks)

6 a. A committee consists of 9 students, 2 from first year, 3 from second year and 4 from third year. 3 students are to be removed at random. What is the probability that (i) 3 students belongs to different class (ii) 2 belongs to the same class and third belongs to different class (iii) All the 3 belongs to the same class. (06 Marks)

b. State and prove Baye's theorem. (07 Marks)

c. The chance that a doctor will diagnose a disease correctly is 60%. The chance that a patient will die after correct diagnose is 40% and the chance of death after wrong diagnose is 70%. If a patient dies, what is the chance that disease was correctly diagnosed. (07 Marks)

7 a. The probability distribution of finite random variable x is given by the following table:

x :	0	1	2	3	4	5	6	7
p(x) :	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Find k, $p(x < 6)$, $p(x \geq 6)$, $p(3 < x \leq 6)$. (06 Marks)

b. Obtain the mean and variance of Poisson distribution. (07 Marks)

c. The life of an electric bulb is normally distributed with average life of 2000 hours and standard deviation of [illegible] hours. Out of 2500 bulbs, find the number of bulbs that are likely to last between 1900 and 2100 hours. Given that $p(0 < z < 1.67) = 0.4525$. (07 Marks)

8 a. Explain the following terms:
i) Null hypothesis (ii) Type I and Type II error (iii) Confidence limits. (06 Marks)

b. The weight of workers in a large factory are normally distributed with mean 68 kgs, and standard deviation 3 kgs. If 80 samples consisting of 35 workers each are chosen, how many of 80 samples will have the mean between 67 and 68.25 kgs. Given $p(0 < z < 2) = 0.4772$ and $p(0 \leq z \leq 0.5) = 0.1915$. (07 Marks)

c. Eleven students were given a test in statistics. They were provided additional coaching and then a second test of equal difficulty was held at the end of coaching. Marks scored by them in the two tests are given below.

Test I	23	20	19	21	18	20	18	17	23	16	19
Test II	24	19	22	18	20	22	20	20	23	20	17

Do the marks give evidence that the student have benefited by extra coaching? Given $t_{0.05}(10) = 2.228$. Test the hypothesis at 5% level of significance. (07 Marks)

* * * * *

1.

a. Taylor's series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \dots \quad \text{--- (1)}$$

I To compute $y(0.1)$

Consider $y' = 2y + 3e^x$; $y'(0) = 2(0) + 3e^0 = 3$

$\therefore y'' = 2y' + 3e^x$; $y''(0) = 2(3) + 3 = 9$

$y''' = 2y'' + 3e^x$; $y'''(0) = 2(9) + 3 = 21$

From (1),

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2}y''(0) + \frac{(0.1)^3}{6}y'''(0)$$

$$= 0 + (0.1)3 + \frac{0.01}{2}(9) + \frac{0.001}{6}(21) = 0.3485$$

II To compute $y(0.2)$

Consider $y' = 2y + 3e^x$ & let $x_0 = 0.1$. $y_0 = 0.3485$

Now $y'(0.1) = 2y(0.1) + 3e^{0.1}$; $y'(0.1) = 4.0125$

$y'' = 2y' + 3e^x \Rightarrow y''(0.1) = 2(4.0125) + 3e^{0.1}$

$= 11.3405$

$y''' = 2y'' + 3e^x \Rightarrow y'''(0.1) = 25.9965$

$$y(0.2) = y(0.1) + (0.1)\, y'(0.1) + \frac{0.01}{2} y''(0.1) + \frac{0.001}{6} y'''(0.1)$$

$$= 0.8108$$

6. Given $x_0 = 0$, $y_0 = 1$, $f(x,y) = x^2+y^2$, $h = 0.05$

$f(x_0, y_0) = 1$, $x_1 = x_0 + h = 0.05$

To find $y(x_1) = y(0.05)$

By Euler's formula,

$$y_1^{(0)} = y_0 + h \cdot f(x_0, y_0) = 1 + (0.05)(1)$$

$$\therefore y_1^{(0)} = 1.05$$

By modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f(x_1, y_1^{(0)})\right\}$$

$$= 1.0513$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f(x_1, y_1^{(1)})\right\} = 1.0513$$

$$\therefore y(0.05) = 1.0513$$

To find $y(x_1) = y(0.1)$

Let $x_0 = 0.05$, $y_0 = 1.0513$, $h = 0.05$

$f(x,y) = x^2 + y^2 \Rightarrow f(x_0, y_0) = 1.0538$,

$x_1 = x_0 + h = 0.1$

Euler's formula,

b) $y_1^{(0)} = y_0 + h \cdot f(x_0, y_0) = 1.104$

By Modified Euler's formula.

$$y_1^{(1)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(0)})\}$$

$$= 1.1055$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\} = 1.1055$$

c)

x	y	$y' = f(x, y) = x^2(1+y)$
$x_0 = 1.0$	$y_0 = 1.000$	$y_0' = f(x_0, y_0) = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$y_1' = f(x_1, y_1) = 2.70193$
$x_2 = 1.2$	$y_2 = 1.548$	$y_2' = f(x_2, y_2) = 3.69912$
$x_3 = 1.3$	$y_3 = 1.979$	$y_3' = f(x_3, y_3) = 5.03451$
$x_4 = 1.4$	$y_4 = ?$	

By Adams-Bashforth predictor formula.

$$y_4^{(P)} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$= 1.979 + \frac{0.1}{24}\{55(5.03451) - 59(3.69912) + 37(2.70193) - 9(2)\}$$

$$= 2.5723$$

$y_4' = f(x_4, y_4) = x_4^2(1+y_4) = (1.4)^2(1+2.5723)$
$= 7.0017$

By Adam - Bashforth corrector formula

$y_4^{(c)} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1'] \quad — ①$

$y_4^{(c)} = 2.5749$

$y_4' = f(x_4, y_4) = (1.4)^2 + (1+2.5749) = 7.0068$

Substituting in ①.

$y_4^{(c)} = 1.979 + \frac{0.1}{24}\{9(7.0068) + 19(5.03451) -$
$5(3.66912) + 2.70193\}$

$= 2.575$

$y(1.4) = 2.575$

2

a. Here, $f(x,y,z) = 1+zx$, $g(x,y,z) = -xy$

$x_0 = 0$, $y_0 = 0$ & $z_0 = 1$, $h = 0.3$

$x_1 = x_0+h = 0+0.3 = 0.3$

$k_1 = h\, f(x_0, y_0, z_0) = 0.3\, f(0,0,1) = 0.3$

$l_1 = h \cdot g(x_0, y_0, z_0) = 0.3 \cdot g(0,0,1) = 0$

$k_2 = h \cdot f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right) = 0.3\, f(0.15, 0.15, 1)$
$= 0.345$

$l_2 = h\, g\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right) = 0.3 \cdot g(0.15, 0.15, 1)$
$= -0.00675$

$k_3 = h \cdot f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right) = 0.3\, f(0.15, 0.1725, 0.9966)$

$= 0.34485$

$l_3 = h\, g\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right) = 0.3\, g(0.3, 0.34485, 0.992$
$= -0.00776$

$k_4 = h \cdot f(x_0+h, y_0+k_3, z_0+l_3) = 0.3 \cdot f(0.3, 0.34485, 0.9922)$
$= 0.3893$

$l_4 = h \cdot g(x_0+h, y_0+k_3, z_0+l_3) = 0.3 \cdot g(0.3, 0.34485, 0.9922)$
$= -0.031$

Now,

$$y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \frac{1}{6}(2.069) = 0.34483$$

$$z(x_1) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 1 + \frac{1}{6}(-0.0600) = 0.989997$$

b. Put $\frac{dy}{dx} = y' = z \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{dz}{dx}$

Given equation reduces to

$$\frac{dz}{dx} + y^2 \cdot z = x^3$$

$$\Rightarrow \frac{dz}{dx} = x^3 - y^2 z, \quad y(1) = 1, \quad z(1) = 1$$

Now the problem is to solve

$$y' = \frac{dy}{dx} = z, \quad \frac{dz}{dx} = x^3 - y^2 z \quad \text{with } y(1) = 1$$
$$\& \ z(1) = 1$$

Let $f(x, y, z) = z$, $\phi(x, y, z) = x^3 - y^2 z$

$x_0 = 1$, $y_0 = 1$, & $z_0 = 1$

Picard's iterative formulae gives

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\,dx \quad = 1 + \int_1^x z_{n-1}\,dx \quad \text{——} ①$$

$$z_n(x) = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\,dx = 1 + \int_1^x (x^3 - y_{n-1}^2 \cdot z_{n-1})\,dx$$

For $n=1$, ① & ② $\Rightarrow$ first approximate sol^n is

$$y_1(x) = 1 + \int_1^x z_0\,dx = 1 + \int_1^x (1)\,dx = 1 + (x)_1^x = x$$

$$z_1(x) = 1 + \int_1^x (x^3 - y_0^2 \cdot z_0)\,dx = \frac{x^4}{4} - x + 7/4$$

For $n=2$,

$$y_2(x) = 1 + \int_1^x z_1\,dx = 1 + \int_1^x \left(\frac{7}{4} - x + \frac{x^4}{4}\right)dx$$

$$= 1 + \left\{\frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}\right\}_1^x$$

$$= -\frac{3}{10} + \frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}$$

$z_2(x)$ is not required.

Thus, $y(x) = y_2(x) = -\frac{3}{10} + \frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}$ is the required second approximation. & $y(1.1) = y_2(1.1) = 1.1005$

3 C. Put $\frac{dy}{dx} = z \Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$

Thus, the given eqⁿ reduces to

$$\frac{dz}{dx} = 1-2yz \Rightarrow f(x,y,z) = 1-2yz$$

Now let us compute z' values

$$z_0' = 1-2y_0z_0 = 1$$

$$z_1' = 1-2y_1z_1 = 0.992$$

$$z_2' = 1-2y_2z_2 = 0.9374$$

$$z_3' = 1-2y_3z_3 = 0.7995$$

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
$y' = z$	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3937$	$z_3 = 0.5689$
$y'' = z'$	$z_0' = 1$	$z_1' = 0.992$	$z_2' = 0.9374$	$z_3' = 0.7995$

By Milne's Predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}\{2z_1 - z_2 + 2z_3\} = 0.3049$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\{2z_1' - z_2' + 2z_3'\} = 0.7055$$

$$z_4' = 1 - 2y_4^{(P)} z_4^{(P)} = 0.5698$$

By Milne's corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}\{z_2 + 4z_3 + z_4\} = y_2 + \frac{h}{3}\{z_2 + 4z_3 + z_4^{(P)}\}$$

$$= 0.3045$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\{z_2' + 4z_3' + z_4'\} = 0.7074$$

Applying corrector formula again for y_4, we get

$$y_4^{(C)} = y_2 + \frac{h}{3}\{z_2 + 4z_3 + z_4^{(C)}\} = 0.3046$$

Thus the required solⁿ is

$$y_4 = y(0.8) = 0.3046$$

3

a. Statement : "The necessary conditions that the func^n $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that, there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, and satisfy the equations; $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$"

Pf: Let $f(z)$ be analytic at a point $z = x+iy$ & hence by the definition. $f'(z) = \lim\limits_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$

exists & is unique.

$f(z) = u(x,y) + iv(x,y)$ & let δz be the increment in z corresponding to the increments $\delta x, \delta y$ in x, y.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \quad \text{---} (1)$$

Now, $\delta z = (z+\delta z) - z$ where $z = x+iy$

$\therefore \delta z = [(x+\delta x) + i(y+\delta y)] - [x+iy]$

$= \delta x + i\delta y$

Since δz tends to zero, we have the fol^g two possibilities

<u>case (i)</u>: Let $\delta y = 0 \Rightarrow \delta z = \delta x$ & $\delta z \to 0$ imply $\delta x \to 0$.

Now ① $\Rightarrow$

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

These limits from the basic definition are the partial derivatives of u & v wrt x.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow ①$$

<u>case (ii)</u>: Let $\delta x = 0$ so that $\delta z = i\delta y$ & $\delta z \to 0$ imply

$i\delta y \to 0$ or $\delta y \to 0$.

Now ① becomes

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

But $1/i = i/i^2 = i/-1 = -i$

and hence we have

$$f'(z) = \lim_{\delta y \to 0} -i \frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \longleftarrow (3)$$

From (2) & (3),

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Now, equating the real & imaginary parts.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Thus, we have established C-R equations.

3 b. Given $u-v = (x-y)(x^2+4xy+y^2)$

$\Rightarrow u-v = x^3+3x^2y-3xy^2-y^3$

diff ① partially wrt x & y

$\Rightarrow u_x - v_x = 3x^2+6xy-3y^2$ —— ①

$u_y - v_y = 3x^2-6xy-3y^2$

By C-R equations, $u_y = -v_x$ & $v_y = u_x$

$\therefore -v_x - u_x = 3x^2-6xy-3y^2$ —— ②

① + ② $\Rightarrow$ $-2v_x = 6(x^2-y^2)$

$\Rightarrow v_x = 3(y^2-x^2)$

① − ② $\Rightarrow$ $2u_x = 12xy \Rightarrow u_x = 6xy$

we have $f'(z) = u_x + iv_x$

$f'(z) = 6xy + i(3y^2-3x^2)$

Putting $x=z$ & $y=0$ we get

$f'(z) = 0 + i(0-3z^2)$

Integrating wrt z.

$f(z) = -3i\frac{z^3}{3} + C = -iz^3 + C$ is the required analytic funcn

3 C. Let $f(z) = u+iv$ be an analytic function.

$\therefore |f(z)| = \sqrt{u^2+v^2} = \phi$

we have to prove that $\left\{\frac{\partial\phi}{\partial x}\right\}^2 + \left\{\frac{\partial\phi}{\partial y}\right\}^2 = |f'(z)|^2$

To prove that $\phi_x^2 + \phi_y^2 = |f'(z)|^2$

where $\phi = \sqrt{u^2+v^2} \Rightarrow \phi^2 = u^2+v^2$

Diff partially wrt x,

$$2\phi\,\phi_x = 2u.u_x + 2v.v_x$$

$$\Rightarrow \phi\phi_x = uu_x + vv_x \quad \text{——— (1)}$$

lll^{ly} $\phi\phi_y = uu_y + vv_y$ ——— (2)

Squaring and adding (1) & (2).

$$\phi^2(\phi_x^2 + \phi_y^2) = (uu_x+vv_x)^2 + (uu_y+vv_y)^2$$

$$= u^2u_x^2 + v^2v_x^2 + 2uv.u_xv_x + u^2u_y^2 + v^2v_y^2 + 2uv.u_yv_y$$

Since $f(z) = u+iv$ is analytic,

by CR equations $u_y = -v_x$ & $v_y = u_x$

$$\phi^2(\phi_x^2+\phi_y^2) = (u^2 u_x^2 + v^2 v_x^2 + 2uv\cdot u_x\cdot v_x) + (u^2 v_x^2 + v^2 u_x^2 - 2uv\cdot u_x\cdot v_x)$$

$$= u^2(u_x^2+v_x^2) + v^2(u_x^2+v_x^2)$$

$$= (u_x^2+v_x^2)\cdot(u^2+v^2)$$

But $\phi^2 = u^2+v^2$

$$\therefore\ \phi^2(\phi_x^2+\phi_y^2) = \phi^2(u_x^2+v_x^2)$$

$$\Rightarrow \phi_x^2+\phi_y^2 = u_x^2+v_x^2 \quad \text{--- (3)}$$

But $f'(z) = u_x + iv_x$

$$\Rightarrow |f'(z)| = \sqrt{u_x^2+v_x^2}$$

$$\Rightarrow |f'(z)|^2 = u_x^2+v_x^2$$

(3) reduces to

$\phi_x^2+\phi_y^2 = |f'(z)|^2$ which is the required resul

4 a. Let $w = f(z) = z^2$ ——— ①

$\Rightarrow f'(z) = 2z$

Since $f'(z) \neq 0$, $\forall z \neq 0$ $\therefore$ $f(z)$ is conformal at all the points except at $z=0$.

Let $z = x+iy$ & $w = u+iv$ in ①

$u+iv = (x+iy)^2 = (x^2-y^2)+i(2xy)$

$\Rightarrow u = x^2-y^2$ & $v = 2xy$ ——— ②

Case (i): Consider a straight line $\parallel$ to y-axis in z-plane whose equation is $x = a$, where a is any real constant.

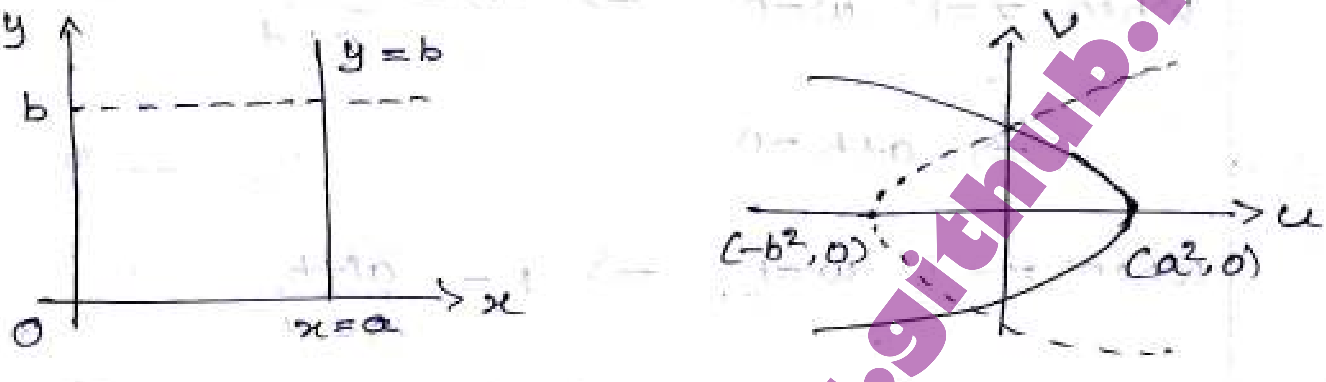
From ②, $u = a^2 - y^2$, $v = 2ay$

$\Rightarrow y^2 = a^2 - u$, $v^2 = 4a^2y^2$

$\therefore v^2 = 4a^2(a^2-u)$

or $v^2 = -4a^2(u-a^2)$ which is the equation of the parabola with $(a^2, 0)$ as vertex and focus at the origin.

$\therefore$ $w = z^2$ transforms a st line $\parallel$ to y-axis in z-plane to parabola with $-ve$ u-axis as its axis.



(z-plane) (w-plane)

case ii) Consider a st line $\parallel$ to x-axis in z-plane whose equation is $y = b$ where b is any real constant.

from (2), $u = x^2 - b^2$, $v = 2xb$

$$\Rightarrow x^2 = u + b^2, \quad v^2 = 4x^2b^2$$

$\therefore$ $v^2 = 4b^2(u + b^2)$ which is the equation of the parabola with $(-b^2, 0)$ as vertex and positive u-axis as its axis.

4 b. Let $w = \dfrac{az+b}{cz+d}$ be the required BLT.

When $z=1$, $w=0$ $\Rightarrow$ $0 = \dfrac{a+b}{c+d}$

$\Rightarrow$ $a+b=0$ —— ①

When $z=i$, $w=1$ $\Rightarrow$ $1 = \dfrac{ai+b}{ci+d}$

$\Rightarrow$ $ai+b-ci-d=0$ —— ②

~~When~~ we have $w = \dfrac{az+b}{cz+d}$ $\Rightarrow$ $\dfrac{1}{w} = \dfrac{cz+d}{az+b}$

When $z=-1$ & $w=\infty$ we get

$\dfrac{1}{\infty} = \dfrac{-c+d}{-a+b}$ $\Rightarrow$ $0 = \dfrac{-c+d}{-a+b}$

~~Adding~~ $-c+d=0$ —— ③

Adding ② & ③,

$ai+b-(i+1)c=0$ —— ④

① & ④ can be written as

$a+b+0.c=0$

$ia+b-(1+i)c=0$

Solving by the rule of cross multiplication.

$\dfrac{a}{-(1+i)-0} = \dfrac{-b}{-(1+i)-0} = \dfrac{c}{1-i}$

$$\Rightarrow \frac{a}{-(1+i)} = \frac{b}{1+i} = \frac{c}{1-i}$$

$\Rightarrow a = -(1+i),\ b = 1+i,\ c = 1-i$

From ③ we have $c = d$

$\therefore d = 1-i$

Substituting the values of a, b, c & d in $w = \frac{az+b}{cz+d}$

we get $$w = \frac{-(1+i)\cdot z + (1+i)}{(1-i)\cdot z + (1-i)} = \frac{(1+i)(1-z)}{(1-i)(1+z)}$$

x^n & dividing by $(1+i)$

$$w = \frac{(1+i)^2}{(1-i^2)} \cdot \left(\frac{1-z}{1+z}\right) = \frac{(1+i^2+2i)}{2} \cdot \left(\frac{1-z}{1+z}\right)$$

$\therefore w = i\left(\frac{1-z}{1+z}\right)$ which is the required BLT.

4c. Points $z = a = -1$, $z = a = -2$ being $(-1,0)$, $(-2,0)$ lies inside the circle $|z| = 3$.

We shall resolve $\frac{1}{(z+1)(z+2)}$ into partial fractions

$$\frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$1 = A(z+2) + B(z+1)$

when $z=-2$, $B=-1$

when $z=-1$, $A=1$

$$\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\int_C \frac{e^{2z}}{(z+1)(z+2)} \cdot dz = \int_C \frac{e^{2z}}{z+1} \cdot dz - \int_C \frac{e^{2z}}{(z+2)} \cdot dz \quad \text{——} ①$$

By Cauchy Integral formula,

$$\int_C \frac{f(z)}{z-a} \cdot dz = 2\pi i \, f(a)$$

Taking $f(z) = e^{2z}$, $a=-1,-2$ respectively in the above formula & substituting in ①,

$$\int_C \frac{e^{2z}}{(z+1)(z+2)} dz = 2\pi i \cdot f(-1) - 2\pi i \, f(-2)$$

$$= 2\pi i \left\{ e^{-2} - e^{-4} \right\}$$

$$= 2\pi i \left\{ \frac{1}{e^2} - \frac{1}{e^4} \right\}$$

PART-B

Sol 5(A) The co-ordinates (ρ, ϕ, z) are called cylinder co-ordinates. and the relationship with the cartesian co-ordinates (x, y, z) is given by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z.$$

The laplace equation $\nabla^2 u = 0$ in cylinderical system is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{---(1)}$$

We shall solve this by the method of seperation of variables

Let $u = R.H.Z$ be the sol^n of (1) where

$R = R(\rho)$, $H = H(\phi)$, $Z = Z(z)$.

Substituting this in (1) we get

$$\frac{\partial^2 (RHZ)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial (RHZ)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 (RHZ)}{\partial \phi^2} + \frac{\partial^2 (RHZ)}{\partial z^2} = 0$$

$$\Rightarrow HZ\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho} HZ\frac{\partial R}{\partial \rho} + \frac{1}{\rho^2} RH\frac{\partial^2 H}{\partial \phi^2} + RZ\frac{d^2 Z}{dz^2} = 0$$

Dividing throughout by RHZ we get

$$\Rightarrow \frac{1}{R}\frac{d^2 R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 H}\frac{d^2 H}{d\phi^2} = -\frac{1}{Z}\frac{d^2 Z}{dz^2} \quad \text{---(2)}$$

The LHS is a function of ρ and ϕ and RHS is a function of z. $\therefore$ They must be equal to a constant

Let us take $\frac{1}{Z}\frac{d^2Z}{dz^2} = 1$, so that (2) becomes

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 H}\frac{d^2H}{d\phi^2} = -1$$

Multiplying by ρ^2 we get

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \rho^2 = -\frac{1}{H}\frac{d^2H}{d\phi^2} \quad -③$$

Again LHS is a f^n of ρ and RHS is a f^n of ϕ. Therefore they must be equal to a constant

Now setting $-\frac{1}{H}\frac{d^2H}{d\phi^2} = n^2$, ③ becomes

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \rho^2 = n^2$$

$$\Rightarrow \rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (\rho^2 - n^2)R = 0$$

The equation can be written in the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

This is Bessel's diff. eq^n of order n in standard form originating from the Laplace eq^n in cylindrical systems

(12)

Sol 5(b) We know that $J_n(\lambda x)$ is sol^n of bessel equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-n^2)y=0$$

If $u=J_n(\alpha x)$, $v=J_n(\beta x)$ then the associated diff. equation are

$$x^2u''+xu'+(\alpha^2x^2-n^2)u=0 \quad -(1)$$

$$x^2v''+xv'+(\beta^2x^2-n^2)v=0 \quad -(2)$$

Multiplying eq^n (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$, we get

$$xvu''+vu'+\alpha^2uvx-n^2\frac{uv}{x}=0$$

$$xuv''+uv'+\beta^2uvx-n^2\frac{uv}{x}=0.$$

On substracting we get

$$x(u''v-v''u)+(uv'+vu')+(\alpha^2-\beta^2)uvx=0$$

ie $\frac{d}{dx}\{x(u'v-uv')\}=(\beta^2-\alpha^2)uvx$

Integrating b.s. by w.r.t x from 0 to 1,

$$x\{vu'-uv'\}\Big]_{x=0}^{1}=(\beta^2-\alpha^2)\int_0^1 uvx\,dx$$

$$\{(u'v-v'u)\}_{x=1}=\beta^2-\alpha^2\int_0^1 uvx\,dx \quad -(3)$$

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$.

$u' = \alpha J_n'(\alpha x)$, $v' = \beta J_n'(\beta x)$.

$\therefore$ (3) reduces to.

$$\{J_n(\beta x) \cdot \alpha J_n'(\alpha x) - J_n(\alpha x)\, \beta J_n'(\beta x)\}_{x=1}$$
$$= (\beta^2 - \alpha^2) \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx$$

Hence.

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \frac{1}{\beta^2 - \alpha^2}\{\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\} \quad \text{--- (4)}$$

Since α and β are distinct roots of $J_n(x) = 0$

$\Rightarrow J_n(\alpha) = 0$, $J_n(\beta) = 0$

$\therefore$ (4) results becomes zero provided $\beta \neq \alpha$.

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0.$$

5(c) By Rodrigue's formula, we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

Putting $n = 0, 1, 2, 3, 4$ we get

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

$P_2(x)$ can be written as

$$x^2 = \frac{2P_2(x) + 1}{3}$$

$$\Rightarrow x^2 = \frac{1}{3}\left(2P_2(x) + P_0(x)\right)$$

$P_3(x)$ can be written as

$$x^3 = \frac{1}{5}\left(2P_3(x) + 3x\right)$$

$$= \frac{1}{5}\left(2P_3(x) + 3P_1(x)\right)$$

For $n=4$, $P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

$$\Rightarrow x^4 = \frac{1}{35}\left[8P_4(x) + 30x^2 - 3\right]$$

$$x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

Now $f(x) = x^4 - 2x^3 + 3x^2 - 4x + 5$

$$= \frac{8}{35} P_4(x) - \frac{4}{5} P_3(x) + \frac{18}{7} P_2(x) - \frac{26}{5} P_1(x) + \frac{31}{5} P_0(x).$$

Sol 6 (a) Any 3 students out of 9 students of a committee are removed, in $9C_3$ ways

∴ No. of exhaustive cases $= n = 9C_3 = 84$

Let A_n, B_n and C_n represents the events of selecting n students from first year, 2nd yr and Third year

(i) P(The 3 students belong to different classes)

$$= P(A_1 \cap B_1 \cap C_1) = \frac{m}{n} = \frac{2C_1 \times 3C_1 \times 4C_1}{9C_3}$$

$$= \frac{24}{84} = 0.2857$$

(ii) P(The 2 students belong to same class and third belong to diff class)

$= P(A_2B_1$ or A_2C_1 or B_2A_1 or B_2C_1 or C_2A_1 or $C_2B_1)$

$$= \frac{55}{84} = 0.6548$$

(14)

(iii) P(All the 3 students belongs to the same class)

$= P(B_3 \text{ or } C_3)$

$= P(B_3) + P(C_3)$

$= \frac{9C_3}{9C_3} + \frac{4C_3}{9C_3} \approx 0.0595$

Sol 6(b) Let $E_1, E_2, \ldots, E_n$ are mutually disjoint events with $P(E_i) \neq 0$, $i = 1, \ldots, n$, then for any arbitrary event A which is a subset of $S = \bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, then

$$P(E_i | A) = \frac{P(E_i)\, P(A|E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A|E_i)}$$

Proof:- Since $A \subset \left(\bigcup_{i=1}^{n} E_i\right) = S$

we have $A = A \cap S$

$= A \cap \left(\sum_{i=1}^{n} E_i\right)$

$= A \cap (E_1 \cup E_2 \ldots \cup E_n)$

$A = (A \cap E_1) \cup (A \cap E_2) \ldots \cup (A \cap E_n)$

$A = \sum_{i=1}^{n} (A \cap E_i)$

$$P(A) = P\left(\bigcup_{i=1}^{n} (A \cap E_i)\right)$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$= \sum_{i=1}^{n} P(E_i)\, P(A/E_i) \quad \text{--- (1)}$$

($\because$ product rule of probability)

We know that

$$P(A \cap E_i) = P(E_i)\, P(A/E_i) \quad \text{--- (2)}$$

We have

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} \quad \text{--- (3)}$$

Substituting (1), (2) in (3) we get

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$$

Ques Sol 6(c) Let E_1 = doctor diagnoses the disease correctly

E_2 = Doctor diagnoses the disease wrongly

A = Death of patient (or Patient dies)

given $P(E_1) = 60\% = 0.6$, $P(E_2) = 1 - P(E_1) = 0.4$.

$P(A/E_1) = 40\% = 0.4$, $P(A/E_2) = 70\% = 0.7$

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Here we have to find

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{\sum_{i=1}^{2} P(E_i)\,P(A/E_i)}$$

$$= \frac{0.6\times 0.4}{0.6\times 0.4 + 0.4\times 0.7} = 0.4615$$

7(a) We have $\sum_x P(x) = 1$

$\Rightarrow 0+K+2K+2K+3K+K^2+2K^2+7K^2+7K = 1$

$\Rightarrow 10K^2+9K-1 = 0$

$\Rightarrow K = -1,\ K = \frac{1}{10}$

Since $K \neq -1$ $\therefore$ $\boxed{K = \frac{1}{10}}$

Consider

$P(X<6) = P(X=0,1,2,3,4,5)$

$= P(0)+P(1)+P(2)+P(3)+P(4)+P(5)$

$= 8K+K^2$

$= \frac{80+1}{100} = 0.81$

$P(X<6) = 0.81$

$P(X\geq 6) = 1-P(X<6) = \frac{19}{100} = 0.19$

$P(3 < x \leq 6) = P(x=4, x=5, x=6)$

$= P(4) + P(5) + P(6)$

$= 3K + K^2 = \frac{33}{100} = 0.33$

7(b) Let X be a Poisson variate with parameter λ, then

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2 \ldots \infty$$

$$\text{Mean} = E(X) = \sum_{i=0}^{\infty} x\, P(x)$$

$$= \sum_{i=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda \cdot \lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \cdot \lambda \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} \ldots\right)$$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$= \lambda e^{0} = \lambda$$

$\therefore \boxed{\text{Mean} = \lambda}$

Consider $E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)P(x)$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} = \lambda^2$$

$\therefore E(X(X-1)) = \lambda^2$

$E(X^2) = E(X(X-1)) + E(X)$

$= \lambda^2 + \lambda$

Variance $= E(X^2) - [E(X)]^2$

$= (\lambda^2+\lambda) - (\lambda)^2$

$= \lambda$

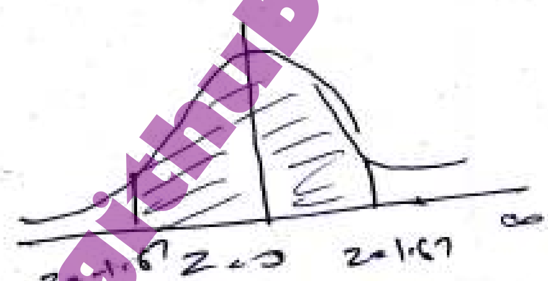
Variance $= \lambda$

$$\boxed{\text{Mean} = \text{variance} = \lambda}$$

Sol 7(c) Let X: life time of electric bulb in hours

Given $X \sim N(\mu = 2000, \sigma = 60)$

then $Z = \frac{X - 2000}{60}$



Consider $P[1900 < X < 2100]$

$= P\left[\frac{1900-2000}{60} < \frac{X-2000}{60} < \frac{2100-2000}{60}\right]$

$= P(-1.67 < Z < 1.67)$

$=$ Area from -1.67 to 1.67

$= 2 \times$ Area from 0 to 1.67

$= 2 \times 0.4525$

$= 0.905$

N: No. of bulbs $= 2500$

No. of bulbs likely to last b/w 1900 & 2100 hr

$= 2500 \times 0.905$

$= 2262.5$

$\simeq \underline{2263}$

(17)

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Sol 8 (a)

(i) Null hypothesis :- It is the hypothesis which is tested for possible rejection under the assumption that is true.

The Null hypothesis asserts that there is no real difference between sample statistic and the corresponding population parameter or between two independent sample statistic.

The null hypothesis is usually denoted by H_0.

(ii) Type (I) and Type II errors.

The decision to accept or reject the null hypothesis H_0 is made on the basis of the information supplied by the sample data. There is always a chance of making error. There are two possible types of errors in the testing of hypothesis.

The error of rejecting H_0 when H_0 is true is known as type-I error and the error of accepting H_0 when H_0 is not true is known a type-II error.

Confidence Interval

(ii) Confidence limits :- is the limit within which the true value of the parameter is expected to lie and are constructed on sample statistic.

Let θ be the unknown population parameter t_1 and t_2 are the two constants computed from the sample observations drawn from the given population such that θ lies between t_1 and t_2 then the interval (t_1, t_2) is called confidence limit intervals

t_1 and t_2 are called confidence limits

Sol 8 (b) Let X: weight of workers in a factory in kgs

Given $X \sim N(\text{mean} = 68, SD = 3)$

$$SNV = Z = \frac{X - \text{mean}}{S.D/\sqrt{n}} \sim N(0,1)$$

mean = 68, S.D = 3, n = 35, N = 80

$$SNV = Z = \frac{X - 68}{3/\sqrt{35}}$$

Consider $P[67 < X < 68.25]$

$= \cancel{P\left(\frac{67-68}{3/\sqrt{35}} <\right.}$

$= P[-1.97 < Z \leq 0.493]$

or $P(-2 < Z < 0.5)$

$= P[-2 < Z < 0] + P(0 < Z < 0.5)$

$= P(0 < Z < 2) + P(0 < Z < 0.5)$

$= 0.4772 + 0.1915$

$= 0.6687.$

$\therefore$ out of 80 samples. No of samples will have mean weight between 67 and 68.25

$= N \cdot P(67 < \bar{x} < 68.25)$

$= 80 \times 0.6687$

$= 53.496 \simeq 53$

8 (f) Let H_0: students have not benefited by extra coaching.

H_1: Students have benefited by extra coaching.

Let us construct the following table

Boy	Marks in 1st test	Marks in 2nd test x_2	$d = x_2 - x_1$	$(d-\bar{d})$	$(d-\bar{d})^2$
1	23	24	1	0	0
2	20	19	-1	-2	4
3	19	22	3	2	4
4	21	18	-3	-4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	0	-1	1
10	16	20	4	3	9
11	19	17	-2	-3	9
Total			11		50

$$\bar{d} = \frac{\Sigma d}{n} = \frac{11}{11} = 1$$

$$\Sigma(d-\bar{d})^2 = 50$$

$$S_d = \sqrt{\frac{1}{n}\Sigma(d-\bar{d})^2} = \sqrt{\frac{50}{11}} = 2.132$$

Test statistic for H_0: $t = \frac{\bar{d}\sqrt{(n-1)}}{S_d} = \frac{\sqrt{10}}{2.132} = 1.483 = t_{cal}$

For $(n-1) = (11-1)$ df $t_{0.05} = 2.228$

Since $\chi^2_{cal} = 1.48 < \chi^2_{tab} = 2.228$

$\therefore H_0$ may be accepted

$\therefore$ The students have not benefited by extra coaching