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10MAT41

Fourth Semester B.E. Degree Examination, Dec.2014/Jan.2015
Engineering Mathematics – IV

Time: 3 hrs. Max. Marks:100

Note: ***1. Answer any FIVE full questions, selecting atleast TWO questions from each part.***
2. Use of statistical table is permitted.

PART – A

1 a. Employ Taylor's series method to find an approximate solution to find y at x = 0.1 given $\frac{dy}{dx} = x - y^2$, y(0) = 1 by considering upto fourth degree term. **(06 Marks)**

b. Solve the following by Euler's modified method $\frac{dy}{dx} = \log(x+y)$, y(0) = 2 to find y(0.4) by taking h = 0.2. **(07 Marks)**

c. Given $\frac{dy}{dx} = x^2(1+y)$ and y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979. Evaluate y(1.4) by Adams-Bash forth method. Apply corrector formula twice. **(07 Marks)**

2 a. Solve $\frac{dy}{dx} = 1 + xz$ and $\frac{dz}{dx} = -xy$ for x = 0.3 by applying Runge Kutta method given y(0) = 0 and z(0) = 1. Take h = 0.3. **(06 Marks)**

b. Use Picard's method to obtain the second approximation to the solution of $\frac{d^2y}{dx^2} - x^3\frac{dy}{dx} - x^3y = 0$ given y(0) = 1, y′(0) = 0.5. Also find y(0.1). **(07 Marks)**

c. Apply Milne's method to compute y(0.4) given y″ + xy′ + y = 0, y(0) = 1, y′(0) = 0, y(0.1) = 0.995, y′(0.1) = -0.0995, y(0.2) = 0.9802, y′(0.2) = -0.196, y(0.3) = 0.956 and y′(0.3) = -0.2863. **(07 Marks)**

3 a. Derive Cauchy-Riemann equation in Cartesian form. **(06 Marks)**

b. Find an analytic function f(z) whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ and hence find its imaginary part. **(07 Marks)**

c. If f(z) is a holomorphic function of z, then show that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$. **(07 Marks)**

4 a. Discuss the transformation $w = z + \frac{1}{z}$. **(06 Marks)**

b. Find the BLT which maps the points z = 1, i, -1 to w = i, 0, -i. Find image of $|z| < 1$. **(07 Marks)**

c. Evaluate $\int_C \left\{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\right\} dz$ where 'C' is circle $|z| = 3$. **(07 Marks)**

PART – B

5 a. Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ interms of Legendre polynomials. **(06 Marks)**

b. Obtain the solution of $x^2y'' + xy' + (x^2 - x^2)\,y = 0$ interms of $J_n(x)$ and $J_{-n}(x)$. **(07 Marks)**

c. Derive Rodrique's formula $P_n(x) = \frac{1}{2^n \lfloor n} \frac{d}{dx^n}[(x^2-1)^x]$. **(07 Marks)**

6 a. State the axioms of probability. For any two events A and B, prove that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. **(06 Marks)**

b. A box 'A' contains 2 white and 4 black balls. Another box 'B' contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. Then a ball is drawn from the box B. Find the probability that it is white. **(07 Marks)**

c. In a certain college 4% of the boys and 1% of girls are taller than 1.8m. Further more 60% of the students are girls. If a student is selected at random and is found to be taller than 1.8m, what is the probability that the student is a girl? **(07 Marks)**

7 a. The probability density of a continuous random variable is given by $p(x) = y_0 e^{-|x|}, -10 < x < \infty$. Find y_0. Also find the mean. **(06 Marks)**

b. Obtain the mean and variance of binomial distribution. **(07 Marks)**

c. In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and SD of 60 hours. Estimate the number of bulbs likely to burn for.

i) More than 2150 hours.

ii) Less than 1950 hours.

iii) More than 1920 hours but less than 2160 hours.

Given A(1.5) = 0.4332, A(1.83) = 0.4664, A(2) = 0.4772. **(07 Marks)**

8 a. In a city 'A' 20% of a random sample of 900 school boys had a certain slight physical defect. In another city B, 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions is significant? Why? **(06 Marks)**

b. A manufacturer claimed that atleast 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipment revealed that 18 of them were faulty. Test his claim at a significance level of 1% and 5%. **(07 Marks)**

c. A set of five similar coins is tossed 320 times and the result is

No. of heads	0	1	2	3	4	5
Frequency	6	27	72	112	71	32

Test the hypothesis that the data follow a binomial distribution $[\chi^2_{0.05} = 11.07 \text{ for } 5\text{df}]$. **(07 Marks)**

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1 a. Here $x_0 = 0$ & $y(x_0) = y(0) = 1$.

Hence, the Taylor's series solution of the problem in a neighbourhood of $x_0 = 0$ is

$$y(x) = y(0) + x\, y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + \cdots$$

Given $y' = x - y^2 \Rightarrow y'(0) = -1$

$\Rightarrow y'' = 1 - 2yy' \Rightarrow y''(0) = 3$

$y''' = -2\{yy'' + (y')^2\} \Rightarrow y'''(0) = -8$

$y^{iv} = -2\{yy''' + 3y'y''\} \Rightarrow y^{iv}(0) = 34$

$$\therefore y(x) = 1 + x(-1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(-8) + \frac{x^4}{4!}(34)$$

Taking $x = 0.1$,

$y(0.1) = 0.9137$

2 a. Here, $f(x, y, z) = 1 + xz$, $\phi(x, y, z) = -xy$

$x_0 = 0$, $y_0 = 0$, $z_0 = 1$

$k_1 = h f(x_0, y_0, z_0) = h(1 + z_0 x_0) = 0.3$

$l_1 = h \phi(x_0, y_0, z_0) = h(-x_0 y_0) = 0$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + l_1/2) = 0.345$

$l_2 = h \phi(x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + l_1/2) = -0.00675$

$k_3 = h f(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2) = 0.34485$

$$l_3 = h\,\phi(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + l_2/2) = -0.0077$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3,\ z_0 + l_3) = 0.3893$$

$$l_4 = h\,\phi(x_0 + h,\ y_0 + k_3,\ z_0 + l_3) = -0.03104$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.3448$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 0.9899$$

2b. Take $\dfrac{dy}{dx} = z$

Given eqn reduces to $\dfrac{dz}{dx} = x^3(y+z)$

Given conditions: $y_0 = 1$, $z_0 = 0.5$ with $x_0 = 0$

$f(x, y, z) = z$ & $\phi(x, y, z) = x^3(y+z)$

$$\frac{dy}{dx} = z \quad ; \quad \frac{dz}{dx} = x^3(y+z)$$

$$dy = z\,dx \quad ; \quad dz = x^3(y+z)\,dx$$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} z\,dx \quad ; \quad \int_{z_0}^{z} dz = \int_{x_0}^{x} x^3(y+z) \quad (1)$$

$$y = y_0 + \int_{x_0}^{x} z\,dx \quad ; \quad z = z_0 + \int_{x_0}^{x} x^3(y+z)$$

Put ~~$x = x_0$~~, $y = y_0$, $z = z_0$ in the RHS of (1),

$$y^{(1)} = 1 + \frac{1}{2}x \quad ; \quad z^{(1)} = \frac{1}{2} + \frac{3}{8}x^4$$

Put ~~$x = x_0$~~, $y = y_1$, $z = z_1$ in the RHS of (1)

$$y^{(2)} = 1 + \frac{x}{2} + \frac{3x^5}{40}$$

$$y \simeq 1 + \frac{x}{2} + \frac{3x^5}{40}$$

$$y(0.1) = 1 + \frac{0.1}{2} + \frac{3(0.1)^5}{40} = 1.05$$

3a. Suppose a complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in an open set S. Then $f(z)$ is differentiable at every point z of S so that

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists,}$$

with the limit being independent of the path along which $\delta z \to 0$.

Let $\delta z = \delta x + i\delta y$

Then $f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$

& $$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y}$$

$$= \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i\,\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y} \quad (1)$$

Let us consider the limit of the left hand side of ① as $\delta z \to 0$ along the path parallel to the x-axis

$$\lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z} = \lim_{\delta x \to 0} \frac{u(x+\delta x, y)-u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y)-v(x,y)}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—— ②}$$

Next, let us consider the limit of the left hand side of ① as $\delta z \to 0$ along the line parallel to the y-axis.

$$\lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z} = \lim_{\delta y \to 0} \frac{u(x, y+\delta y)-u(x,y)}{i\,\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y)-v(x,y)}{i\,\delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{—— ③}$$

From ② & ③

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

3 b. Let $f(z) = u + iv$ be the required analytic funcⁿ.

Let us employ the Milne – Thompson's method to find this analytic function.

Given $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{(2\cos 2x)(\cosh 2y - \cos 2x) - (\sin 2x)(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2} = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = -\frac{2(\sin 2x)(\sinh 2y)}{(\cosh 2y - \cos 2x)^2} = \phi_2(x,y)$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \phi_1(z,0) + i\,\phi_2(z,0)$$

$$= \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} - 0$$

$$= \frac{-2}{(1 - \cos 2z)^{\cancel{2}}} = \frac{-2}{2\sin^2 z}$$

$$= -\operatorname{cosec}^2 z = \frac{d}{dz}(\cot z)$$

$\Rightarrow f(z) = \cot z + c$ where c is a complex constant.

3 c. WKT $|f(z)|^2 = u^2+v^2$ ——— ①

& $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$ — ②

From ①, $|f(z)| = \sqrt{u^2+v^2}$

$$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}\left\{(u^2+v^2)^{1/2}\right\} = \frac{1}{2}(u^2+v^2)^{-1/2}(2uu_x + 2vv_x)$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)$$

so that $\left(\frac{\partial}{\partial x}|f(z)|\right)^2 = \frac{1}{u^2+v^2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2$

III'ly $\left(\frac{\partial}{\partial y}|f(z)|\right)^2 = \frac{1}{u^2+v^2}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2$

$\therefore$ $\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 =$

$$\frac{1}{u^2+v^2}\left\{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2\right\}$$

$$= \frac{1}{u^2+v^2}\left[u^2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + v^2\left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} + 2uv\left\{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right\}\right]$$

$$= \frac{1}{u^2+v^2}\left\{u^2|f'(z)|^2 + v^2|f'(z)|^2 + 0\right\}$$

$$= |f'(z)|^2$$

4 a. Consider the transformation

$$w = z + \frac{1}{z} \quad \text{——— ①}$$

$$\Rightarrow f'(z) = 1 - \frac{1}{z^2}$$

$\Rightarrow$ $f'(z)$ exists and not zero when $z \neq 0$ & $z^2 = 1$. $\therefore$ Transformation ① is conformal at at all points except at 0 & ± 1.

Taking $z = re^{i\theta}$ & $w = u + iv$ in ①

$$u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\therefore u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta \quad \text{——— ②}$$

From these we get,

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = \cos^2\theta + \sin^2\theta = 1 \quad \text{——— ③}$$

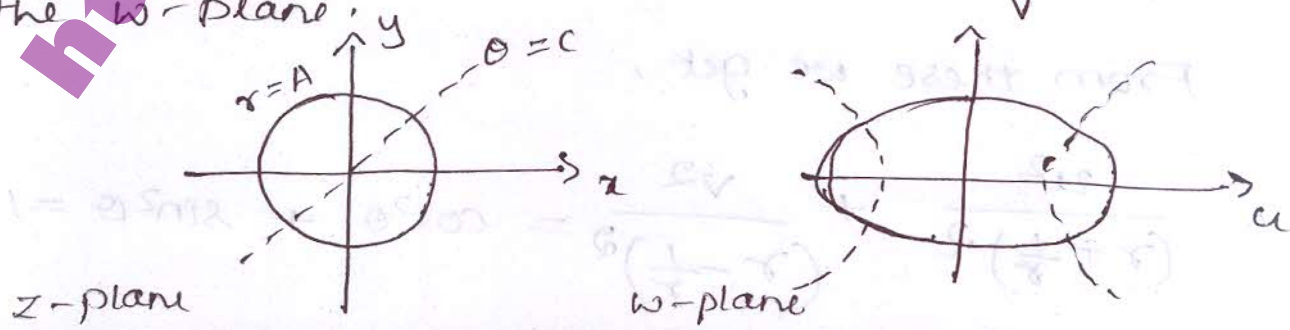
Consider the polar eq^n $r = A\ (\neq 1)$, a constant, which represents a circle centered at the origin in the z-plane. Then ③ represents an ~~circle~~ ellipse having centred at the origin of the w-plane and u-axis & v-axis as axes.

Thus, under the transformation ①, the circle $r = A$ centred at the origin in the z-plane is transformed into the ellipse ③ in the w-plane.

From ②,

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{a^2}{r}\right)^2 - \left(r - \frac{a^2}{r}\right)^2 = 4a^2 \quad \text{④}$$

For $\theta = c$, a constant eq^n ④ represents a hyperbola having centre at the origin of the w-plane & u-axis & v-axis as axes. Thus, under the transformation ①, the radial line $\theta = c$ in the z-plane is transformed to the hyperbola ④ in the w-plane.



Let $z_1 = 1$, $z_2 = i$, $z_3 = -1$

& $w_1 = i$, $w_2 = 0$, $w_3 = -i$

Required bilinear transformation is :

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Substituting the given values,

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \frac{w-i}{w+i} = -\frac{(i+1)}{(i-1)} \cdot \frac{z-1}{z+1} = \frac{2i}{(-2)}\,\frac{z-1}{z+1} = i\,\frac{z-1}{z+1}$$

$$(w-i)(z+1) = i(w+i)(z-1)$$

$$\Rightarrow w\{(z+1) - i(z-1)\} = i(z+1) - (z-1)$$

$$w = \frac{z(i-1)+(1+i)}{z(1-i)+(1+i)} = \frac{2iz+2}{-2iz+2} = \frac{1+iz}{1-iz} \quad \text{—(1)}$$

This is the required bilinear transformation

To find the image of $|z|<1$, let us write (1) as

$$w(1-iz) = 1+iz \quad \Rightarrow -iz(w+1) = 1-w$$

$$z = \frac{1-w}{-i(w+1)} = i\,\frac{1-w}{1+w}$$

If $|z|<1$,

$$\Rightarrow |i| \left|\frac{1-w}{1+w}\right| < 1$$

$$\Rightarrow |1-w|^2 < |1+w|^2$$

$$\Rightarrow |1-(u+iv)^2| < |1+(u+iv)|^2$$

$$\Rightarrow |(1-u)+iv|^2 < |(1+u)+iv|^2$$

$$\Rightarrow 1+u^2-2u+v^2 < 1+u^2+2u+v^2$$

$$\Rightarrow -4u < 0$$

$$\Rightarrow u > 0$$

Thus, under the given transformation, the image of $|z|<1$ is $u>0$ (which is the right half of the w-plane).

c. Let us resolve $\frac{1}{(z-1)^2(z-2)}$ into partial fractions.

$$\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2} \quad \text{—} (1)$$

$A = -1$, $B = -1$, $C = 1$

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

X'ly ① by $f(z)$ & integrating wrt z over C,

$$I = \int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = -\int_C \frac{f(z)}{z-1}\,dz - \int_C \frac{f(z)}{(z-1)^2}\,dz + \int_C \frac{f(z)}{z-2}\,dz$$

$$= I_1 + I_2 + I_3 \text{ (say)} \quad \text{—②}$$

Let

$C: |z| = 3$

Points $z=1$ & $z=2$ both lie within C.

By Cauchy's integral formula,

$$I_1 = -2\pi i\, f(1) = -2\pi i\,(\sin\pi + \cos\pi) = 2\pi i$$

$$I_2 = -2\pi i\, f'(1) \quad \text{But, } f'(z) = 2\pi z\,[\cos\pi z^2 - \sin\pi z^2]$$

$$\Rightarrow I_2 = 4\pi^2 i$$

& $I_3 = 2\pi i\, f(2) = 2\pi i$

Hence from ②,

$$I = 2\pi i + 4\pi^2 i + 2\pi i = 4\pi i + 4\pi^2 i$$

2 c. Given

$y_0 = 1, \quad y_1 = 0.995, \quad y_2 = 0.9801, \quad y_3 = 0.956$

Let $y' = z$

$z_0 = 0, \quad z_1 = -0.0995, \quad z_2 = -0.196, \quad z_3 = -0.2867$

$y'' = z' \quad \Rightarrow \quad z' = -(xz + y)$

$z_1' = -0.985, \quad z_2' = -0.941, \quad z_3' = -0.87$

Consider Milne's predictor formula,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$\Rightarrow y_4^{(P)} = 0.9231 \quad \& \quad z_4^{(P)} = -0.3692$$

Next consider Milne's corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$z_4' = -(x_4 z_4^{(P)} + y_4^{(P)}) = -0.7754$$

Hence corrector formulae give

$$y_4^{(C)} = 0.9230 \quad \& \quad z_4^{(C)} = -0.3692$$

$$\therefore y(0.1) = 0.923$$

5 a. We have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$$

$\Rightarrow$

$$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

$$\therefore f(x) = x^4 + 3x^3 - x^2 + 5x - 2$$

$$= \left[\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)\right] + \left[\frac{6}{5}P_3(x) + \frac{9}{5}P_1(x)\right]$$

$$- \frac{1}{3}P_0(x) - \frac{2}{3}P_2(x) + 5P_1(x) - 2P_0(x)$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{32}{15}P_0(x)$$

5 b. The Bessel differential equation of order n is in the form,

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \quad \text{--- (1)}$$

where n is a non-negative real constant.

We ~~Frobenius~~ use Frobenius method to solve this eq^n

co-eff^t of $y'' = x^2 = P_0(x)$, & $P_0(x) = 0$ at $x = 0$

We assume the series solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \text{——— ②}$$

$$\Rightarrow y' = \sum_{0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$\Rightarrow y'' = \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

① becomes

$$\sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{0}^{\infty} a_r (k+r) x^{k+r}$$
$$+ \sum_{0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{0}^{\infty} a_r x^{k+r} = 0$$

Collecting the first, second & fourth terms together we have,

$$\sum_{0}^{\infty} a_r x^{k+r} \left[(k+r)(k+r-1) + (k+r) - n^2\right] + \sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

$$\Rightarrow \sum_{0}^{\infty} a_r x^{k+r} \left[(k+r)^2 - n^2\right] + \sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

We shall equate the co-eff^t of the lowest degree term in x, ie x^k to zero.

$$\Rightarrow a_0 (k^2 - n^2) = 0$$

Setting $a_0 \neq 0$ we have $k^2 - n^2 = 0$

$$\Rightarrow k = \pm n$$

Also we need to independently equate the co-eff^t of x^{k+1} to zero.

ie $a_1\left[(k+1)^2 - n^2\right] = 0$

$\Rightarrow a_1 = 0$ ~~since~~

Next we shall equate the co-eff^t of x^{k+r} to zero.

$$a_r\left[(k+r)^2 - n^2\right] + a_{r-2} = 0$$

$$\Rightarrow a_r = \frac{-a_{r-2}}{\left[(k+r)^2 - n^2\right]} \quad (r \geq 2) \qquad \text{——} \ (3)$$

When $k = n$, (3) becomes

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = \frac{-a_{r-2}}{2nr + r^2}$$

Putting $r = 2, 3, 4 \ldots$ we obtain

$$a_2 = \frac{-a_0}{4n+4}, \quad a_3 = \frac{-a_1}{6n+9} = 0 \quad \text{since } a_1 = 0$$

|||ly $a_1 = a_3 = a_5 = a_7 = \cdots = 0$

Next $a_4 = \dfrac{-a_2}{8n+16} = \dfrac{-a_2}{8(n+2)} = \dfrac{a_0}{32(n+1)(n+2)}$ &

so on.

we substitute these values in the expanded form of (2)

$$y = x^k [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots]$$

Let $k = +n$ be denoted by y_1.

$$y_1 = x^n \left[a_0 - \frac{a_0}{4(n+1)} x^2 + \frac{a_0}{32(n+1)(n+2)} x^4 - \cdots \right]$$

$$\Rightarrow y_1 = a_0 x^n \left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5 (n+1)(n+2)} - \cdots \right] \quad (4)$$

Since we also have $k = -n$, let the sol^n for $k = -n$ be denoted by y_2.

Replacing n by $-n$ in (4) we have

$$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5 (-n+1)(-n+2)} - \cdots \right] \quad (5)$$

The complete solution of (1) is given by

$y = A y_1 + B y_2$ where A, B are arbitrary constants.

Now put $a_0 = \dfrac{1}{2^n \Gamma(n+1)}$ in (4) & let that sol^n be Y_1

$$Y_1 = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2) \cdot 2} - \cdots \right]$$

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1) \cdot 2} - \cdots \right]$$

We have a property of gamma functions

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

$$\therefore Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(n+3)\cdot 2} - \cdots\right]$$

This can further be put in the form

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma(n+1)\cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\cdot 2!}\left(\frac{x}{2}\right)^4 + \cdots\right.$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\cdot r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

This function is called the Bessel function of the first kind of order n denoted by $J_n(x)$.

Thus, $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$$

For $k=-n$, the solⁿ is denoted by $J_{-n}(x)$.

Hence the general solution of the Bessel's eqⁿ is given by

$$y = aJ_n(x) + bJ_{-n}(x)$$

where a and b are arbitrary constants and n is not an integer.

5 c. Let $u = (x^2-1)^n$

We shall first establish that the n^{th} derivative of u, that is u_n is a solution of the Legendre's diff eqn

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{——— } (1)$$

Differentiating u wrt x, we have

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} \cdot 2x$$

or $(x^2-1)u_1 = 2nx(x^2-1)^n$

ie $(x^2-1)u_1 = 2nxu$

Diff wrt x

$$(x^2-1)u_2 + 2xu_1 = 2n(xu_1 + u)$$

Diff this 'n' times by applying Leibnitz thm for the n^{th} derivative of a product given by

$$(UV)_n = UV_n + nU_1V_{n-1} + \frac{n(n-1)}{2!}U_2V_{n-2} + \cdots + U_nV$$

$$\therefore [(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$$

$$\left[(x^2-1)u_{n+2} + n\,2x\cdot u_{n+1} + \frac{n(n-1)}{2}\cdot 2\cdot u_n\right] + 2\left[xu_{n+1} + n\cdot 1\cdot u_n\right]$$
$$= 2n\left[xu_{n+1} + n\cdot 1\cdot u_n\right] + 2nu_n$$

$$(x^2-1)u_{n+2} + 2nxu_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n$$
$$= 2nxu_{n+1} + 2n^2u_n + 2nu_n$$

$$\Rightarrow (x^2-1)u_{n+2} + 2xu_{n+1} - n^2u_n - nu_n = 0$$

$$(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$$

$$\Rightarrow (1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \quad \text{—— ②}$$

Comparing ② & ①, we conclude that u_n is a solution of the Legendre's DE. It may be observed that u is a polynomial of degree $2n$ and hence u_n will be a polynomial of degree n.

$$P_n(x) = ku_n = k\left[(x^2-1)^n\right]_n$$

$$\Rightarrow P_n(x) = k\left[(x-1)^n(x+1)^n\right]_n$$

Applying Leibnitz thm for RHS we have

$$P_n(x) = k\Big[(x-1)^n\{(x+1)^n\}_n + n\cdot n(x-1)^{n-1}\{(x+1)^n\}_{n-1}$$

$$+ \frac{n(n-1)}{2!}n(n-1)(x-1)^{n-2}\{(x+1)^n\}_{n-2} + \cdots$$

$$\cdots + \{(x-1)^n\}_n(x+1)^n\Big] \quad \text{—— ③}$$

If $Z = (x-1)^n$ then

$$Z_n = n(n-1)(n-2)\cdots 2\cdot 1\cdot(x-1)^{n-n}$$

$$= n!\,(x-1)^0$$

$$= n!$$

$$\therefore \{(x-1)^n\}_n = n!$$

To find k, put $x=1$ in (3)

$P_n(1) = k\, n!\, 2^n$ (all terms except last term becomes zero).

$\Rightarrow k\, n!\, 2^n = 1$

$\Rightarrow k = \dfrac{1}{n!\, 2^n}$

Since $P_n(x) = k\, u_n$, we have $P_n(x) = \dfrac{1}{n!\, 2^n} \left\{ (x^2-1)^n \right\}_n$

Thus $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}\left[(x^2-1)^n\right]$

a. Axioms of Probability:

~~R~~ ~~RECD~~ If S is the sample space & E is the set of all events then to each event A in E we associate a unique real number $P = P(A)$ known as the probability of the event A, if the foll axioms are satisfied.

1. $P(S) = 1$

2. For every event A in E $0 \le P(A) \le 1$

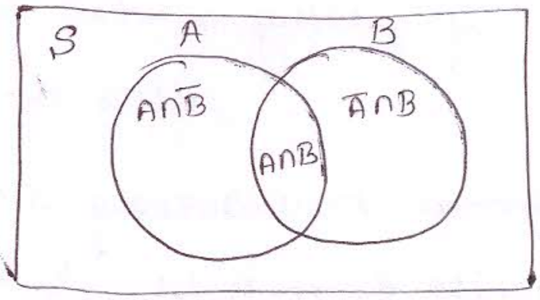
3. If $A_1, A_2, A_3, \ldots A_n$ are mutually exclusive events of E then $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$

Addition Rule:

If A and B are any two events of S which are not mutually exclusive then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Pf: We prove the result using the following Venn diagram:



From the fig.

$$A = (A\cap\bar{B}) \cup (A\cap B)$$

$$B = (\bar{A}\cap B) \cup (A\cap B)$$

$$\Rightarrow P(A) = P(A\cap\bar{B}) + P(A\cap B)$$ since $A\cap\bar{B}$ & $A\cap B$ are disjoint.

$$P(B) = P(\bar{A}\cap B) + P(A\cap B)$$

$P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 & A_2 are mutually exclusive.

Now, $P(A) + P(B) = [P(A\cap\bar{B}) + P(A\cap B) + P(\bar{A}\cap B)] + P(A\cap B)$

ie $P(A) + P(B) = [P(A\cup B)] + P(A\cap B)$

Thus, $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

6 b. Total number of balls in urn A = 2W + 4B = 6

Total number of balls in urn B = 5W + 7B = 12

(a) Suppose the transferred ball is white.

Probability of the transfer of a white ball is $2/6 = 1/3$

Then urn B will have $6W + 7B = 13$ balls.

Hence prob of getting a white ball from B after the transfer is $6/13$.

$\therefore$ Prob of ~~getting~~ transferring a white ball & getting white from B is $\frac{1}{3} \times \frac{6}{13} = \frac{2}{13}$

(b) Suppose the transferred ball is black.

Prob of transffer is $4/6 = 2/3$.

Then urn B will have $5W + 8B = 13$ balls.

Hence prob of getting a white ball after the transffer is $5/13$.

$\therefore$ Prob of transferring a black ball & getting a white from B is $\frac{2}{3} \times \frac{5}{13} = \frac{10}{39}$

Thus, by addition theorem, required prob =

$$\frac{2}{13} + \frac{10}{39} = \frac{16}{39}$$

6 c. Let $P(A)$ = prob of selecting a girl student from among all students.

$P(B)$ = prob of selecting a boy student from among all students.

$P(E/A)$ = prob of selecting a student who is taller than 1.8 m, from among the girl students

$P(E/B)$ = prob of selecting a student who is taller than 1.8M from among the boy students.

Then from what is given, we have

$P(A) = 60\% = 0.6$,

$P(B) = 40\% = 0.4$

$P(E/A) = 1\% = 0.01$

$P(E/B) = 4\% = 0.04$

we are required to find $P(A/E)$.

Using Baye's theorem,

$$P(A/E) = \frac{P(A)\,P(E/A)}{P(A)\,P(E/A) + P(B)\,P(E/B)}$$

$$= 0.2727$$

7.a. We must have $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$$\Rightarrow \int_{-10}^{\infty} y_0 e^{-|x|} \cdot dx = 1$$

$$\Rightarrow \int_{-10}^{0} y_0 e^{x} \cdot dx + \int_{0}^{\infty} y_0 e^{-x} \cdot dx = 1$$

$$\Rightarrow y_0 \left\{ \left[e^{x} \right]_{-10}^{0} - \left[e^{-x} \right]_{0}^{\infty} \right\} = 1$$

$$\Rightarrow y_0 \left[1 - e^{-10} - 0 + 1 \right] = 1$$

$$\Rightarrow y_0 = \frac{1}{2 - e^{-10}}$$

$$\text{Mean } \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-10}^{\infty} x \, \frac{1}{2-e^{-10}} \, e^{-|x|} \, dx$$

$$= \frac{1}{2-e^{-10}} \left\{ \int_{-10}^{0} x e^{x} \, dx + \int_{0}^{\infty} x e^{-x} \, dx \right]$$

$$= \frac{1}{2-e^{-10}} \left\{ \left[x e^{x} - e^{x} \right]_{-10}^{0} + \left[-x e^{-x} - e^{-x} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{2-e^{-10}} \left\{ \left[-10\, e^{-10} \cancel{-1} + e^{-10} \cancel{+1} \right] \right.$$

$$= \frac{e^{-10}}{2-e^{-10}} \left[-9 \right]$$

$$= \frac{-9}{2e^{10}-1} = \frac{9}{1-2e^{10}}$$

7 b. If p is the probability of success & q is the probability of failure, the probability of x successes out of n trials is given by

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$\text{mean } (\mu) = \sum_{x=0}^{n} x\, P(x) = \sum_{x=0}^{n} x \cdot {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^{x} q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{(n-1)}C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(p+q)^{n-1} = np$$

$$\text{Variance } (V) = \sum_{x=0}^{n} x^2 P(x) - \mu^2 \quad \text{--- (1)}$$

$$\sum_{x=0}^{n} x^2 P(x) = \sum_{x=0}^{n} [x(x-1)+x]\, P(x)$$

$$= \sum_{x=0}^{n} x(x-1) P(x) + \sum_{x=0}^{n} x P(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_{x}\, p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} \frac{n(n-1).\,(n-2)!}{(x-2)!\,(n-x)!} p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{(n-2)}C_{(x-2)}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2\,(q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

& $\mu = np$

substituting these in ①

$$V = n(n-1)p^2 + np - (np)^2$$

$$= np(1-p) = npq$$

7 c. Here, $\mu = 2040$ & $\sigma = 60$.

∴ if x is the life of a bulb, the corresponding standard normal variate is

$$z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$$

(i) For $x = 2150$, we find $z \approx 1.83$.

Hence the prob that a ~~lamp~~ bulb is selected at random will burn for more than 2150 hours is

$$P(x > 2150) = P(z > 1.83)$$

$$= P(z \geq 0) - P(0 \leq z < 1.83)$$

$$= 0.5 - A(1.83)$$

$$= 0.0336.$$

∴ Expected number $= 0.336 \times 2000 = 67$

(ii) For $x = 1950$, we find $z = -1.5$

$$P(x < 1950) = P(z < -1.5) = P(z > 1.5)$$

$$= P(z \geq 0) - P(0 < z < 1.5) = 0.5 - A(1.5)$$

$$= 0.5 - 0.4332 = 0.0668$$

∴ Required number $= 0.0668 \times 2000 = 134$

(iii) $P(1920 < x < 2160) = P(-2 < z < 2) = 2P(0 < z < 2)$

$= 2A(2)$

$= 2 \times 0.4772$

$= 0.9544$

∴ Required number $= 0.9544 \times 2000 = 1909$

8 a. Here $N_1 = 900$, $P_1 = 20\% = 0.2$

$N_2 = 1600$, $P_2 = 18.5\% = 0.185$

Let us make the hypothesis that there is no difference b/w the proportion of boys with defect in the two cities.

Then $\mu(P_1 - P_2) = 0$

$$\sigma_{P_1-P_2} = \sqrt{\frac{(0.2)\times(0.8)}{900} + \frac{0.185 \times 0.815}{1600}} = 0.0165$$

Corresponding z-score is

$$z = \frac{(P_1 - P_2) - \mu(P_1 - P_2)}{\sigma_{P_1-P_2}} = \frac{0.2 - 0.185}{0.0165}$$

$= 0.909$

This z-score is less than $z_c = 1.96$ & $z_c = 2.58$.

∴ we can't reject the hypothesis. we infer that there is no significant difference b/w the two cities in the matter considered.

8 c. Prob that x number of fair coins out of 5 show a head in a single toss is given by the binomial func

$$b(5, \tfrac{1}{2}, x) = {}^5C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{5-x} = \tfrac{1}{2^5}\left({}^5C_x\right)$$

$$= \tfrac{1}{32}\left({}^5C_x\right) = b(x), \text{ say.}$$

Accordingly, in 320 tosses the expected number of tosses in which x no of coins show a head is $320 \times b(x)$. Hence the expected frequencies are

$$e_1 = 320 \times b(0) = 320 \times \tfrac{1}{32} \times {}^5C_0 = 10$$

$$e_2 = 320 \times b(1) = 50$$

$$e_3 = 320 \times b(2) = 100$$

$$e_4 = 320 \times b(3) = 100$$

$$e_5 = 320 \times b(4) = 50$$

$$e_6 = 320 \times b(5) = 10$$

Corresponding observed freq are:

$f_1 = 6$, $f_2 = 27$, $f_3 = 72$, $f_4 = 112$, $f_5 = 71$, $f_6 = 32$

$$\chi^2 = \frac{(6-10)^2}{10} + \frac{(27-50)^2}{50} + \frac{(72-100)^2}{100} + \frac{(112-100)^2}{100} + \frac{(71-50)^2}{50} + \frac{(32-10)^2}{10}$$

$$= 78.68$$

we note that the number of degrees of freedom is $6-1 = 5$

Given $\chi^2_{0.05}(5) = 11.07$

we observe that $\chi^2 = 78.68$ is very much greater than $\chi^2_{0.05}(5)$. $\therefore$ we reject the hypothesis that the observed data follows a binomial distribution.

8.6 Let p be the prob of success which being the prob of the equipment supplied to the factory conformal to the specifications.

By data $p = 0.95$ & $q = 0.05$

$H_0 : p = 0.95$ & the claim is correct.

$H_1 : p < 0.95$ & the claim is false.

we choose the one claimed test to determine whether the supply is conformal to the specification

$\mu = np = 200 \times 0.95 = 190$

$\sigma = \sqrt{npq} = \sqrt{200 \times 0.95 \times 0.05} = 3.082$

Expected no of equipments according to specification $= 190$

Actual no $= 182$ since 18 out of 200 were faulty.

$\therefore$ difference $= 190 - 182 = 8$

Now $Z = \frac{x - np}{\sqrt{npq}} = \frac{8}{3.082} = 2.6$

The value of Z is greater than the critical value 1.645 at 5% level & 2.33 at 1% level of significance. Claim of the manufacturer is rejected at 5% as well as at 1% level of significance in accordance with the one tailed test.

16. $f(x,y) = \log(x \cdot y)$, $x_0 = 0$, $y_0 = 2$, $h = 0.2$

$x_1 = x_0 + h = 0.2$

$y_1^{(0)} = y_0 + h f(x_0, y_0) = 2 + 0.2 f(0,2) = 2.0602$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 2.0655$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 2.0656$

$x_0 = 0.2$, $y_0 = 2.0656$, $x_1 = 0.4$

$y_1^{(0)} = y_0 + h f(x_0, y_0) = 2.1366$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 2.1415$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 2.1416$

$\therefore y(0.4) = 2.1416$