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By K B Hemanth Raj

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CBCS Scheme

USN | | | | | | | | | | | 15MAT41

Fourth Semester B.E. Degree Examination, Dec.2017/Jan.2018
Engineering Mathematics – IV

Time: 3 hrs. Max. Marks: 80

Note: 1. Answer any FIVE full questions, choosing one full question from each module.
2. Use of statistical tables is permitted.

Module-1

1 a. Employ Taylor's series method to find y at x = 0.1. Correct to four decimal places given $\frac{dy}{dx} = 2y + 3e^x$; $y(0) = 0$. (05 Marks)

b. Using Runge Kutta method of order 4, find y(0.2) for $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0) = 1$, taking h = 0.2. (05 Marks)

c. If $y' = 2e^x - y$; $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ and $y(0.3) = 2.090$. Find y(0.4) using Milne's predictor corrector formula. Apply corrector formula twice. (06 Marks)

OR

2 a. Use Taylor's series method to find y(4.1) given that $(x^2 + y)y' = 1$ and $y(4) = 4$. (05 Marks)

b. Using modified Euler's method find y at x = 0.1, given $y' = 3x + \frac{y}{2}$ with y(0) = 1, h = 0.1. Perform two iterations. (05 Marks)

c. Find y at x = 0.4 given $y' + y + xy^2 = 0$ and $y_0 = 1$, $y_1 = 0.9008$, $y_2 = 0.8066$, $y_3 = 0.722$ taking h = 0.1 using Adams-Bashforth method. Apply corrector formula twice. (06 Marks)

Module-2

3 a. Given $y'' = xy'^2 - y^2$ find y at x = 0.2 correct to four decimal places, given y = 1 and y' = 0 when x = 0, using R-K method. (05 Marks)

b. If α and β are two distinct roots of $J_n(x) = 0$, then prove that $\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0$ if $\alpha \neq \beta$. (05 Marks)

c. If $x^3 + 2x^2 - x + 1 = ap_0(x) + bp_1(x) + cp_2(x) + dp_3(x)$ then, find the values of a, b, c, d. (06 Marks)

OR

4 a. Apply Milne's method to compute y(0.8) given that $y'' = 1 - 2yy'$ and the table.

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

Apply corrector formula twice. (05 Marks)

b. Show that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$. (05 Marks)

c. Derive Rodrigue's formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2 - 1)^n\right]$. (06 Marks)

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

Module-3

5 a. Define analytic function and obtain Cauchy Riemann equation in Cartesian form. (05 Marks)

b. Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$; c is the circle $|z| = 3$ by using theorem Cauchy's residue. (05 Marks)

c. Discuss the transformation $w = e^z$ with respect to straight line parallel to x and y axis. (06 Marks)

OR

6 a. Find the analytic function whose real part is $u = \frac{x^4 y^4 - 2x}{x^2 + y^2}$. (05 Marks)

b. State and prove Cauchy's integral formula. (05 Marks)

c. Find the bilinear transformation which maps the points $z = 1, i, -1$ into $w = 2, i, -2$. (06 Marks)

Module-4

7 a. Find the constant c, such that the function $f(x) = \begin{cases} cx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f. Also compute $p(1 < x < 2)$, $p(x \le 1)$, $p(x > 1)$. (05 Marks)

b. If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000 individuals, more than two will get a bad reaction. (05 Marks)

c. x and y are independent random variables, x take the values 1, 2 with probability 0.7; 0.3 and y take the values -2, 5, 8 with probabilities 0.3, 0.5, 0.2. Find the joint distribution of x and y hence find cov(x, y). (06 Marks)

OR

8 a. Obtain mean and variance of binomial distribution. (05 Marks)

b. The length of telephone conservation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth (i) ends less than 5 minutes, (ii) between 5 and 10 minutes. (05 Marks)

c. The joint distribution of two discrete variables x and y is $f(x, y) = k(2x + y)$ where x and y are integers such that $0 \le x \le 2$; $0 \le y \le 3$. Find: (i) The value of k; (ii) Marginal distributions of x and y; (iii) Are x and y independent? (06 Marks)

Module-5

9 a. Explain the terms: (i) Null hypothesis; (ii) Type I and type II errors; (iii) Significance level. (05 Marks)

b. A die thrown 9000 times and a throw of 3 or 4 was observed 3240 times. Is it reasonable to think that the die is an unbiased one? (05 Marks)

c. Find the unique fixed probability vector for the regular Stochastic matrix: (06 Marks)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

OR

10 a. A certain stimulus administered to each of the 12 patients resulted in the following change in blood pressure 5, 2, 8, −1, 3, 0, 6, −2, 1, 5, 0, 4. Can it be concluded that the stimulus will increase the blood pressure. ($t_{0.05}$ for 11 d.f = 2.201) (05 Marks)

b. It has been found that the mean breaking strength of a particular brand of thread is 275.6 gms with $\sigma = 39.7$ gms. A sample of 36 pieces of thread showed a mean breaking strength of 253.2 gms. Test the claim at 1+.. and 5–/. level of significance. (05 Marks)

c. A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarettes the next week with probability 0.2. One the other hand, if he smokes non filter cigarettes one week there is a probability of 0.7 that he will smoke non filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes? (06 Marks)

CBCS Scheme

Fourth Semester B.E. Degree Examination ①

Dec. 2017 / Jan. 2018

Engineering Mathematics-IV

15MAT41

Module-1

1) a) Employ Taylor's Series method to find y at $x=0.1$ Correct to four decimal places given $\frac{dy}{dx} = 2y+3e^x$; $y(0)=0$.

>> $y' = 2y+3e^x$ and $y(0)=0 \Rightarrow x_0 = y_0 = 0$

Taylor's Series expansion is

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots \quad \text{—①}$$

Consider

$y' = 2y+3e^x$ $\qquad y'(0) = 2(0)+3e^0 = 3$

$y'' = 2y'+3e^x$ $\qquad y''(0) = 2(3)+3 = 9$

$y''' = 2y''+3e^x$ $\qquad y'''(0) = 2(9)+3 = 21$

from ①

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2}y''(0) + \frac{(0.1)^3}{6}y'''(0)$$

$$y(0.1) = 0 + 0.1(3) + \frac{0.01}{2}(9) + \frac{0.001}{6}(21)$$

$$y(0.1) = 0.3485$$

Thus $\boxed{y(0.1) = 0.3485}$

1) b) Using Runge Kutta method of order 4. find $y(0.2)$ for $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0)=1$ taking $h=0.2$ ②

>> By data $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

we shall first compute

k_1, k_2, k_3, k_4

$$k_1 = h f(x_0, y_0)$$
$$= 0.2\, f(0, 1)$$
$$= 0.2 \left[\frac{y-x}{y+x}\right] = 0.2\left[\frac{1-0}{1+0}\right] = 0.2$$

$$\boxed{k_1 = 0.2}$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.2\, f(0.1, 1.1)$$
$$= 0.2\left[\frac{y-x}{y+x}\right]$$
$$= 0.2\left[\frac{1.1-0.1}{1.1+0.1}\right]$$

$$\boxed{k_2 = 0.1667}$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.2\, f(0.1, 1.0835)$$
$$= 0.2\left[\frac{y-x}{y+x}\right]$$
$$= 0.2\left[\frac{1.0835-0.1}{1.0835+0.1}\right]$$

$$\boxed{k_3 = 0.1662}$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$
$$= 0.2\, f(0.2, 1.1662)$$
$$= 0.2\left[\frac{y-x}{y+x}\right]$$
$$= 0.2\left[\frac{1.1662-0.2}{1.1662+0.2}\right]$$

$$\boxed{k_4 = 0.1414}$$

$$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1 + \frac{1}{6}[0.2 + 2(0.1667) + 2(0.1662) + 0.1414]$$
$$= 1.1679$$

Thus *** $\boxed{y(0.2) = 1.1679}$ ***

1) c) If $y' = 2e^x - y$; $y(0) = 2$ $y(0.1) = 2.010$ $y(0.2) = 2.040$ (3)
$y(0.3) = 2.090$ Find $y(0.4)$ using Milne's predictor Corrector formula. Apply Corrector formula twice.

x	y	$y' = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^0 - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{0.1} - 2.01 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{0.2} - 2.04 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{0.3} - 2.09 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

By Milne's predictor - Corrector formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 2 + \frac{4(0.1)}{3}\left[2(0.2003) - 0.4028 + 2(0.6097)\right]$$

$$\boxed{y_4^{(P)} = 2.1623}$$

Now $y_4' = 2e^{x_4} - y_4 = 2e^{0.4} - 2.1623 = 0.8213$

we have milne's corrector formula,

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 2.04 + \frac{0.1}{3}\left[0.4028 + 4(0.6097) + 0.8213\right]$$

$$\boxed{y_4^{(c)} = 2.1621}$$

Now $y_4' = 2e^{0.4} - 2.1621 = 0.8215$

Applying the corrector formula again we have,

$$y_4^{(c)} = 2.04 + \frac{0.1}{3}\left[0.4028 + 4(0.6097) + 0.8215\right]$$

$$y_4^{(c)} = 2.1621$$

Thus ✱✱✱ $\boxed{y(0.4) = 2.1621}$ ✱✱✱

2) a) Use Taylor's Series (OR) method to find $y(4.1)$ given that $(x^2+y)y' = 1$ and $y(4)=4$ ④

>> Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots$$

By data $y' = \frac{1}{x^2+y}$ $x_0 = 4,\ y_0 = 4$

$$\therefore y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2!}y''(4) + \frac{(x-4)^3}{3!}y'''(4) \quad — ①$$

consider $y' = \frac{1}{x^2+y}$

$$y'(x^2+y) = 1$$

$$y'(4)[4^2+4] = 1 \Rightarrow y'(4) = \frac{1}{20} = 0.05$$

Diff. w.r.t x

$$y'(2x+y') + (x^2+y)y'' = 0$$

$$0.05[2(4)+0.05] + [4^2+4]y''(4) = 0$$

$$0.05[8+0.05] + 20y''(4) = 0$$

$$0.4025 + 20y''(4) = 0$$

$$\therefore y''(4) = -0.020125$$

We observe that the value of the derivative are small enough and the third degree term can also be neglected substituting these values in ① for computing $y(4.1)$ we have

$$y(4.1) = 4 + (4.1-4)\,0.05 + \frac{(4.1-4)^2}{2}(-0.020125)$$

Thus $y(4.1) = 4.0049$

Thus *** $\boxed{y(4.1) = 4.0049}$ ***

2) b) Using modified Euler's method find y at $x=0.1$ given $y' = 3x + \frac{y}{2}$ with $y(0)=1$, $h=0.1$ perform two iterations. ⑤

>> By data $x_0 = 0$, $y_0 = 1$, $h = 0.1$ $f(x_0, y_0) = \frac{3x_0}{2} + \frac{y_0}{2}$

$\therefore f(x_0, y_0) = 0.5$ $x_1 = x_0 + h$

$\boxed{x_1 = 0.1}$

$\boxed{y(x_1) = y(0.1) = ?}$

From Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(0)} = 1 + (0.1)(0.5) = 1.05$$

$$\boxed{y_1^{(0)} = 1.05}$$

modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.8 + \frac{1.05}{2}\right]$$

$$\boxed{y_1^{(1)} = 1.06625}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[0.8 + \frac{1.06625}{2}\right]$$

$$\boxed{y_1^{(2)} = 1.0667}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + 0.05\left[0.8 + \frac{1.0667}{2}\right]$$

$$\boxed{y_1^{(3)} = 1.0667}$$

Thus *** $\boxed{y(0.1) = 1.0667}$ ***

2) c) Find y at $x=0.4$ given $y'+y+xy^2=0$ and $y_0=1$, $y_1=0.9008$, $y_2=0.8066$, $y_3=0.722$ taking $h=0.1$ using Adams-Bashforth method. Apply corrector formula twice. (6)

>> We prepare the table first.

x	y	$y'=-(y+xy^2)$
$x_0=0$	$y_0=1$	$y_0'=-1$
$x_1=0.1$	$y_1=0.9908$	$y_1'=-0.9819$
$x_2=0.2$	$y_2=0.8066$	$y_2'=-0.9367$
$x_3=0.3$	$y_3=0.722$	$y_3'=-0.8784$
$x_4=0.4$	$y_4=?$	

We have Predictor formula

$$y_4^{(P)}=y_3+\frac{h}{24}\left[55y_3'-59y_2'+37y_1'-9y_0'\right]$$

$$=0.722+\frac{0.1}{24}\left[55(-0.8784)-59(-0.9367)+37(-0.9819)-9(-1)\right]$$

$$\boxed{y_4^{(P)}=0.6371}$$

$$y_4^{(C)}=y_3+\frac{h}{24}\left[9y_4'+19y_3'-5y_2'+y_1'\right] \quad \text{——— (1)}$$

$$\boxed{y_4^{(C)}=0.6379}$$

Now $y_4'=f(x_4,y_4)=y_4'+y_4+x_4y_4^2=-0.8007$

(1) $\Rightarrow$ $\underline{y_4^{(C)}=0.6379}$

Thus *** $\boxed{y(0.4)=0.6379}$ ***

3) a) Given $y'' = xy'^2 - y^2$ find y at $x = 0.2$ correct to four decimal places, given $y = 1$ and $y' = 0$ when $x = 0$, using R-K method.

>> $y'' = xy'^2 - y^2$

Putting $y' = z$ and $y'' = z'$ The given eq^n

$z' = xz^2 - y^2$ with $y = 1$, $z = 0$ when $x = 0$

$y' = z$, $z' = xz^2 - y^2$ System of equations

$f(x, y, z) = z$ and $g(x, y, z) = xz^2 - y^2$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and $h = 0.2$

$k_1 = h f(x_0, y_0, z_0) = 0.2\, f(0, 1, 0) = 0.2(0) = 0$

$l_1 = h g(x_0, y_0, z_0) = 0.2\,[0 - 1^2] = -0.2$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$= 0.2\, f(0.1, 1, -0.1) = 0.2(-0.1) = -0.02$

$l_2 = h g(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = -0.1998$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$= 0.2\, f(0.1, 0.99, -0.0999) = -0.01998$

$l_3 = h g(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = -0.1958$

$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2\, f(0.2, 0.98002, -0.1958) = -0.03916$

$l_4 = 0.2\,[0.2 - (0.1958)^2 - (0.98002)^2] = -0.19065$

We have $y(x_0 + h) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4)$

Thus $\boxed{y(0.2) = 0.9801}$

3) b) If α and β are two distinct roots of $J_n(x)=0$, then prove that $\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$ if $\alpha \neq \beta$. (8)

$$x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0$$

if $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ the associated d.eq^{n}s

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \quad \text{---- (1)}$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \quad \text{--- (2)}$$

x^{ly} (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$

$$x v u'' + v u' + \alpha^2 u v x - \frac{n^2 u v}{x} = 0$$

and $$x u v'' + u v' + \beta^2 u v x - \frac{n^2 u v}{x} = 0$$

On subtracting we obtain

$$x(v u'' - u v'') + (v u' - u v') + (\alpha^2 - \beta^2) u v x = 0$$

ie $$\frac{d}{dx}\{x(v u' - u v')\} = (\beta^2 - \alpha^2) x u v$$

Integrating both sides w.r.t. x b/w 0 and 1

$$\left[x(v u' - u v')\right]_{x=0}^{1} = (\beta^2 - \alpha^2)\int_{x=0}^{1} x u v\,dx$$

ie $$(v u' - u v')_{x=1} - 0 = (\beta^2 - \alpha^2)\int_0^1 x u v\,dx \quad \text{— (3)}$$

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$ we have

$u' = \alpha J_n'(\alpha x)$, $v' = \beta J_n'(\beta x)$ and as a consequence of these (3) $\Rightarrow$

$$\left[J_n(\beta x)\alpha J_n'(\alpha x) - J_n(\alpha x)\beta J_n'(\beta x)\right]_{x=1} = (\beta^2 - \alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

Hence $$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{\beta^2 - \alpha^2}\{\alpha J_n \beta J_n' - \beta J_n J_n' \beta\} \quad \text{(4)}$$

Since α and β are distinct roots of $J_n(x) = 0$ we have $J_n(\alpha) = 0$ $J_n(\beta) = 0$ with the result the RHS of (4) $\Rightarrow$ zero provided $\beta^2 - \alpha^2 \neq 0$ or $\beta \neq \alpha$. Thus we've proved that if $\alpha \neq \beta$,

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$$

3) c) If $x^3+2x^2-x+1 = aP_0 + bP_1 + cP_2 + dP_3$ then, find the values of a, b, c, d

>> Let $f(x) = x^3+2x^2-x+1$

WbT $x^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1$, $x^2 = \frac{1}{3}P_0 + \frac{2}{3}P_2$

$x = P_1(x)$ and $1 = P_0(x)$

$\therefore f(x) = \frac{2}{5}P_3 + \frac{3}{5}P_1 + 2\left[\frac{P_0}{3} + \frac{2}{3}P_2\right] - P_1 + P_0$

$= \frac{2}{5}P_3 + \left(\frac{3}{5}-1\right)P_1 + \frac{4}{3}P_2 + \left(\frac{2}{3}+1\right)P_0$

$= \frac{2}{5}P_3 + \frac{4}{3}P_2 - \frac{2}{5}P_1 + \frac{5}{3}P_0$

hence $aP_0 + bP_1 + cP_2 + dP_3 = \frac{5}{3}P_0 - \frac{2}{5}P_1 + \frac{4}{3}P_2 + \frac{2}{5}P_3$

Thus $a = \frac{5}{3}$, $b = -\frac{2}{5}$, $c = \frac{4}{3}$, $d = \frac{2}{5}$

4) a) Apply Milne's method to compute $y(0.8)$ given that $y'' = 1 - 2yy'$ and the table

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

>> Putting $y' = \frac{dy}{dx} = z$ we obtain $y'' = z'$

The given Eqn becomes $z' = 1 - 2yy' = 1 - 2yz$

Now $z_0' = 1 - 2(0)(0) = 1$

$z_1' = 0.992$

$z_2' = 0.9374$

$z_3' = 0.7995$

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
$y' = z$	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3937$	$z_3 = 0.5689$
$y'' = z'$	$z_0' = 1$	$z_1' = 0.992$	$z_2' = 0.9374$	$z_3' = 0.7995$

We first Milne's predictor formulae

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$\therefore$ $y_4^{(P)} = 0.3049$ and $z_4^{(P)} = 0.7055$

Now Milne's corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4\right]$$

$$z_4^{(c)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

we have $z_4' = 1 - 2y_4^{(P)} z_4^{(P)}$

$= 1 - 2(0.3049)(0.7055)$

$$\boxed{z_4' = 0.5698}$$

$\therefore$ $\boxed{y_4^{(c)} = 0.3045}$ and $\boxed{z_4^{(c)} = 0.7074}$

Applying the corrector formula again

for y_4 we have

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.7074\right]$$

$$\boxed{y_4^{(c)} = 0.3046}$$

Thus the required $\underline{y(0.8) = 0.3046}$.

4) b) Show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

>> By the def^n

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \quad \text{---- (1)}$$

putting $n = 1/2$ in (1) we've

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{1/2+r} \frac{1}{\Gamma(r+3/2)\, r!}$$

$$= \sqrt{\frac{x}{2}} \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

on expanding we have

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)\, 1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)\, 2!} - \cdots\right] \quad \text{-- (2)}$$

WKT $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)\Gamma(n-1)$

putting $n = 3/2,\ 5/2,\ 7/2 \ldots$

$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$ $\quad \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$ $\quad \Gamma(7/2) = \frac{15\sqrt{\pi}}{8}$

substituting these values in the RHS of (2)

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4} \cdot \frac{4}{3\sqrt{\pi}} + \frac{x^4}{16} \cdot \frac{8}{15\sqrt{\pi}\, 2} - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}} \cdot \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right]$$

(we've $2/x$ taken as a common factor keeping in view of the desired result)

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

Thus $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

4) c) Derive Rodrigue's formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\{(x^2-1)^n\}$. (12)

Proof:- Let $u = (x^2-1)^n$

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ —— (1) Legendre's d.eqⁿ

diff. u. w.r.t. x we have

$u_1 = n(x^2-1)^{n-1} \cdot 2x$

(or) $(x^2-1)u_1 = 2nx(x^2-1)^n$

$(x^2-1)u_1 = 2nxu$

diff. w.r.t x again

$(x^2-1)u_2 + 2xu_1 = 2n(xu_1 + u)$

Apply Leibnitz rule we have

$D^n[(x^2-1)u_2] + 2D^n[xu_1] = 2nD^n[xu_1] + 2nD^n[u]$

$\left[(x^2-1)u_{n+2} + n \cdot 2x \cdot u_{n+1} + \frac{n(n-1)}{2} \cdot 2 \cdot u_n\right] + 2[xu_{n+1} + n \cdot u_n]$

$= 2n[xu_{n+1} + nu_n] + 2nu_n$

$(x^2-1)u_{n+2} + 2nxu_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n$

$= 2nxu_{n+1} + 2n^2u_n + 2nu_n$

$(x^2-1)u_{n+2} + 2xu_{n+1} - n^2u_n - nu_n = 0$

$(x^2-1)u_{n+2} + 2xu_{n+1} - nu_n(n+1) = 0$

$(x^2-1)u_{n+2} + 2xu_{n+1}$

(or) $(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$

This can be put in the form

$(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0$ —— (2)

Comparing (2) with (1) we conclude that u_n is a solⁿ of the Legendre's d.eqⁿ. It may be observed that u is a polynomial of degree $2n$ and hence u_n will be a polynomial of degree n.

Also $P_n(x)$ which satisfied Legendre's d.eqⁿ is also a polynomial of degree n. hence we must be the same as $P_n(x)$ but for some constant factor k

1

ie $P_n(x) = k u_n = k[(x^2-1)^n]_n$

$P_n(x) = k[(x-1)^n (x+1)^n]_n$

Apply Leibnitz theorem for the RHS we have

$$P_n(x) = k\Big[(x-1)^n \{(x+1)^n\}_n + n \cdot n (x-1)^{n-1} \{(x+1)^n\}_{n-1} + \frac{n(n-1)}{2!} n(n-1)(x-1)^{n-2} \{(x+1)^n\}_{n-2} + \cdots \{(x-1)^n\}_n (x+1)^n\Big] \quad (3)$$

It should be observed that if

$Z = (x-1)^n$ then

$Z_1 = n(x-1)^{n-1}$, $Z_2 = n(n-1)(x-1)^{n-2}$

$Z_n = n(n-1)(n-2) \cdots 2.1 (x-1)^{n-n}$

(or) $Z_n = n!(x-1)^0 = n!$

$\therefore \{(x-1)^n\}_n = n!$

We proceed to find k by choosing a suitable value for x.

Putting $x=1$ in (3) all the terms in RHS become zero except the last term which becomes.

$n!(1+1)^n = n! 2^n$

$P_n(1) = k \cdot n! 2^n$ and $P_n(1) = 1$

$\therefore 1 = k \cdot n! 2^n$

(or) $k = \frac{1}{n! 2^n}$

Since $P_n(x) = k u_n$

We have $P_n(x) = \frac{1}{n! 2^n} \{(x^2-1)^n\}_n$

Thus we have proved that

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Module-03

(14)

5) a) Define analytic function and obtain cauchy Riemann eqn in cartesian form.

A complex valued fun $w = f(z)$ is said to be analytic at a point $z = z_0$ if $\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique at z_0 and in the neighbourhood of z_0. Further $f(z)$ is said to be analytic in a region if it is analytic at every point of the region.

The necessary codns that the fun $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that, there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the eqns: are known as CR eqns.

$u_x = v_y$ and $v_x = -u_y$

Proof :- Let $f(z)$ be analytic at a point $z = x+iy$ and hence by the definition, $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$

In cartesian form $f(z) = u(x,y) + iv(x,y)$ and δz be the increment in z corresponding increments $\delta x, \delta y$ in x, y

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u[(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \rightarrow (1)$$

Now $\delta z = (z+\delta z) - z$

$$\boxed{\delta z = \delta x + i\delta y}$$

Case i) Let $\delta y = 0$ so that $\delta z = \delta x$ and $\delta x \to 0$
imply $\delta x \to 0$

Eqⁿ (1) ⟹

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

These limits from the basic defⁿ of pde

$$\therefore \boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \quad \text{——— (2)}$$

Case ii) Let $\delta x = 0$ so that $\delta z = i\delta y$ and $\delta z \to 0$

imply $i\delta y \to 0$ or $\delta y \to 0$

Now (1) ⟹

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

But $\frac{1}{i} = -i$

$$f'(z) = \lim_{\delta y \to 0} -i.\frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore \boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \quad \text{——— (3)}$$

Equating the RHS of (2) and (3) we've

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

equating real and imaginary parts we get

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

are C-R eqⁿs.

5) b) Evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}dz$: C is the circle $|z|=3$ by using Cauchy's residue theorem. (16)

Let $f(z)=\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}$; $C: |z|=3$

$z=1$ is a pole of order 2 and $z=2$ is a pole of order 1. Both of them lies within the circle $|z|=3$

Residue at $z=1$ be denoted by R_1 and we've

$$R_1 = \lim_{z\to 1}\frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-1)^2\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}\right\}$$

<u>bcz</u> $R[m,a]=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$

$$=\lim_{z\to 1}\frac{d}{dz}\left\{(\sin\pi z^2+\cos\pi z^2)\cdot\frac{1}{z-2}\right\}$$

$$=\lim_{z\to 1}(\sin\pi z^2+\cos\pi z^2)\cdot\frac{-1}{(z-2)^2}+\lim_{z\to 1}2\pi z(\cos\pi z^2-\sin\pi z^2)\cdot\frac{1}{z-2}$$

$\boxed{R_1=(1+2\pi)}$ $\qquad\because \sin\pi=0,\ \cos\pi=-1$

$$R_2=\lim_{z\to 2}(z-2)\cdot\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}$$

$$=\lim_{z\to 2}\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2}=1$$

$\therefore \boxed{R_2=1}$

Hence by Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i[R_1+R_2]$$
$$=2\pi i[1+2\pi+1]$$
$$=4\pi i[1+\pi]$$

Thus $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}dz = 4\pi i[1+\pi]$

5) c) Discuss the transformation $w = e^z$ with respect to straight line parallel to x and y axis. (17)

Consider $w = e^z$

ie $u + iv = e^{x+iy} = e^x e^{iy}$ or $e^x(\cos y + i\sin y)$

$\therefore u = e^x \cos y, \quad v = e^x \sin y$ —— (1)

We shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinates axes in the z-plane. That is x = constant and y = constant.

Let us eliminate x and y separately from (1)

Squaring and adding we get

$$u^2 + v^2 = e^{2x} \quad \text{—— (2)}$$

Also by dividing we get $\frac{v}{u} = \frac{e^x \sin y}{e^x \cos y}$

$$\frac{v}{u} = \tan y \quad \text{—— (3)}$$

case i) Let $x = c_1$ where c_1 is a constant.

Eqn (2) $\Rightarrow u^2 + v^2 = e^{2c_1}$ = constant $= r^2$

This represents a circle centre origin and radius r in the w-plane.

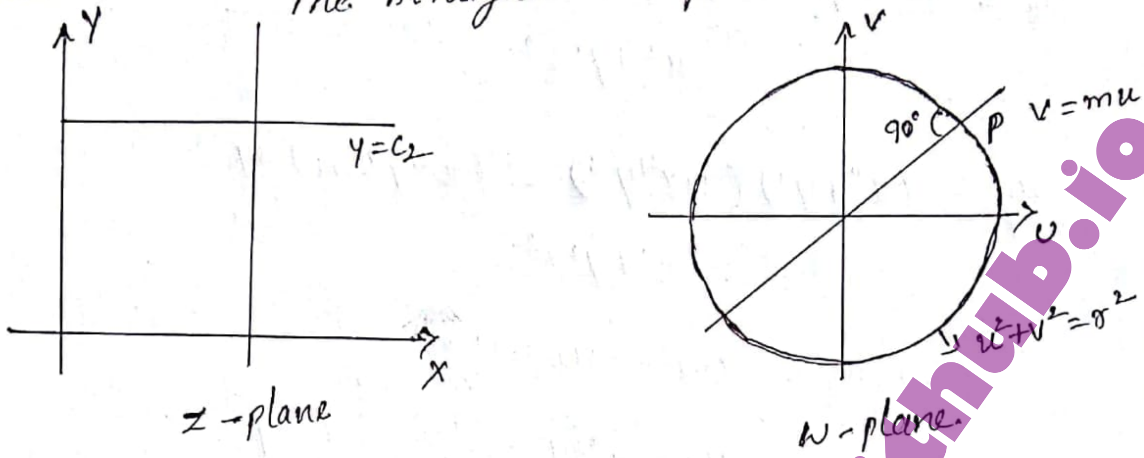
case-ii) Let $y = c_2$ where c_2 is a constant.

Eqn (3) $\Rightarrow \frac{v}{u} = \tan c_2 = m$ (say)

$\therefore v = mu$

This represents a straight line passing through the origin in the w-plane.

conclusion : The straight line parallel to



the x-axis $(y=c_2)$ in the z-plane maps onto a straight line passing through the origin in the w-plane. The straight line parallel to the y-axis $(x=c_1)$ in the z-plane maps onto a circle with origin and radius r where $r=e^{c_1}$ in the w-plane.

Suppose we draw a tangent at the point of intersection of these two curves in the w-plane the angle subtended is equal to 90°. hence these two curves can be regarded as orthogonal trajectories of each other.

(OR)

6) a) Find the analytic function whose real part is

$$u = \frac{x^4y^4 - 2x}{x^2+y^2}$$

WKT Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$u_x = \frac{(x^2+y^2)(4x^3y^4-2) - (x^4y^4-2x)2x}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(4x^4y^3) - (x^4y^4-2x)2y}{(x^2+y^2)^2}$$

Consider $f'(z) = u_x + i v_x$

But $v_x = -u_y$ by C-R eq^n

$\therefore f'(z) = u_x - i u_y$

putting $x = z,\ y = 0$ by Thomas method

$$f'(z) = \frac{z^2[0-2] - [0-2z]2z}{(z^2)^2} - i[0]$$

$$f'(z) = \frac{-2z^2 + 4z^2}{z^4}$$

$$f'(z) = \frac{4z^2 - 2z^2}{z^4}$$

$$f'(z) = \frac{2z^2}{z^4}$$

$$f'(z) = \frac{2}{z^2} \quad \text{integrate w.r.t. } z$$

$$f(z) = 2\int \frac{1}{z^2}\,dz$$

$$= 2\left[\frac{-1}{z}\right] + c$$

$$\boxed{f(z) = \frac{-2}{z} + c}$$

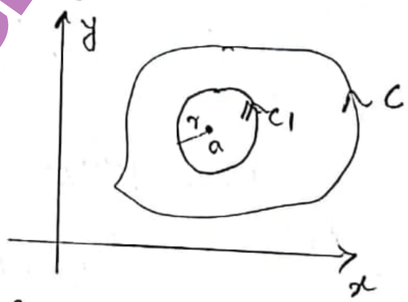
This is the required Analytic fu^n.

6) b) State and prove Cauchy's integral formula.

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is any point within C then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

Proof :- Since 'a' is a point within C, we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C.

The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region b/w C and C_1.



Now, as a consequence of Cauchy's theorem

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \longrightarrow (1)$$

The eqn of C_1 can be written as in the form

$|z-a| = r$, $\quad dz = ire^{i\theta}d\theta$

$\Rightarrow z-a = re^{i\theta}$ (or) $z = a+re^{i\theta}$

$\theta \rightarrow 0$ to 2π

$$(1) \Rightarrow \int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}d\theta = i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

$r>0$, however small, $r \rightarrow 0$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = if(a)\,[\theta]_0^{2\pi} = 2\pi i f(a)$$

Thus $\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$

Cauchy's integral formula

6) c) Find the bilinear transformation which maps the points $z = 1, i, -1$ into $w = 2, i, -2$

>> here $z_1 = 1, \quad z_2 = i, \quad z_3 = -1$

$w_1 = 2, \quad w_2 = i, \quad w_3 = -2$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-2)}{(w+2)} = \frac{(i-2)}{(i+2)} \frac{(i+1)}{(i-1)} \cdot \frac{(z-1)}{(z+1)}$$

$$= \frac{-3-i}{-3+i} \cdot \frac{(z-1)}{(z+1)}$$

$$= \frac{3+i}{3-i} \cdot \frac{z-1}{z+1} = p \text{ (say)}$$

This gives $\frac{w-2}{w+2} = p$

$$\Rightarrow (w-2) = p(w+2)$$

$$w - 2 - pw - 2p = 0$$

$$w(1-p) = 2(1+p)$$

$$w\left[1 - \frac{3+i}{3-i} \cdot \frac{z-1}{z+1}\right] = 2\left[1 + \frac{3+i}{3-i} \cdot \frac{z-1}{z+1}\right]$$

$$w\{(3-i)(z+1) - (3+i)(z-1)\} = 2\{(3-i)(z+1) + (3+i)(z-1)\}$$

$$w(6 - 2iz) = 2(6z - 2i)$$

$$\boxed{w = \frac{2(3z-i)}{3-iz}}$$

This is the required bilinear transformation

7) a) Find the constant C such that the fun $f(x)=\begin{cases} cx^2, & 0<x<3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f. Also compute

$P(1<x<2)$, $P(x\le 1)$, $P(x>1)$

>> $f(x)\ge 0$ if $c\ge 0$ Also we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

ie $\int_0^3 cx^2 dx = 1$

$\frac{cx^3}{3}\Big|_0^3 = 1 \Rightarrow c\frac{3^3}{3} = 1 \Rightarrow 9c = 1$

$\boxed{C = 1/9}$

i) $P(1<x<2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x^2}{9}dx = \frac{1}{9}\cdot\frac{x^3}{3}\Big|_1^2$

$= \frac{1}{27}[x^3]_1^2 = \frac{1}{27}[8-1] = \frac{1}{27}(7) = \frac{7}{27}$

Thus $P(1<x<2) = \frac{7}{27}$

ii) $P(x\le 1) = \int_0^1 f(x)dx = \int_0^1 \frac{1}{9}x^2 dx = \frac{1}{9}\cdot\frac{x^3}{3}\Big|_0^1$

$= \frac{1}{27}[1-0] = \frac{1}{27}$

Thus $P(x\le 1) = \frac{1}{27}$

iii) $P(x>1) = \int_1^3 f(x)dx = \int_1^3 \frac{x^2}{9}dx = \frac{1}{9}\cdot\frac{x^3}{3}\Big|_1^3$

$= \frac{1}{27}[3^3-1]$

$= \frac{1}{27}[27-1]$

$= \frac{26}{27}$

Thus $P(x>1) = \frac{26}{27}$

7) b) If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000 individuals, more than two will get a bad reaction. (23)

>> As the probability of bad reaction is very small, we've this follows poisson distribution and

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$Mean = m = np = 2000 \times 0.001 = 2$

we have to find $P(x>2)$

ie $P(x>2) = 1 - P(x \leq 2)$

$= 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!}\right]$

$= 1 - e^{-2}[1+2+2]$

$= 1 - 5e^{-2}$

$= \underline{\underline{0.3233}}$

7) c) x and y are independent random variables, x take the values 1, 2 with probability 0.7, 0.3 and y take the values −2, 5, 8 with probabilities 0.3, 0.5, 0.2. Find the joint distribution of x and y hence find cov (X, Y).

>> Since X and Y are independent, the joint distribution J(x,y) is obtained by using the defn $J_{ij} = f(x_i)g(y_j)$

X \ Y	$y_1 = -2$	$y_2 = 5$	$y_3 = 8$	$f(x_i)$
$x_1 = 1$	J_{11}	J_{12}	J_{13}	0.7
$x_2 = 2$	J_{21}	J_{22}	J_{23}	0.3
$g(y_j)$	0.3	0.5	0.2	1

The joint distribution table is as follows

X \ Y	−2	5	8	$f(x_i)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$	0.3	0.5	0.2	1

we have $cov(X,Y) = E(XY) - \mu_x \mu_y$ ——— (1)

$$\mu_x = E(X) = \sum x f(x)$$

$$= 1(0.7) + 2(0.3) = 1.3$$

$$\mu_y = E(Y) = \sum y g(y)$$

$$= -2(0.3) + 5(0.5) + 8(0.2) = 3.5$$

$$E(XY) = \sum x y J_{ij}$$

$$= 1(-2)(0.21) + 1(5)(0.35) + 1(8)(0.14)$$

$$+ 2(-2)(0.09) + 2(5)(0.15) + 2(8)(0.06)$$

$$= -0.42 + 1.75 + 1.12 - 0.36 + 1.5 + 0.96$$

$$= 4.55$$

(1) ⇒ Hence $cov(X,Y) = 4.55 - 1.3(3.5) = 0$

Thus $cov(X,Y) = 0$ for independent random variables X and Y is verified.

(OR)

8) a) <u>Obtain mean and variance of binomial distribution.</u> (25)

$$\text{Mean } (m) = \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_0^n x \cdot {}^nC_x\, p^x q^{n-x}$$

$$= \sum_0^n x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_0^n \frac{n(n-1)!}{(x-1)!(n-x)!} p \cdot p^{x-1} q^{n-x}$$

$$= np \sum_1^n \frac{(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_1^n {}^{(n-1)}C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np + (q+p)^{n-1} = np$$

$$\boxed{\text{Mean } (m) = np}$$

$$\text{Variance } (V) = \sum_{x=0}^{n} x^2 p(x) - m^2 \quad \text{--- (1)}$$

$$\text{Now } \sum_0^n x^2 p(x) = \sum_0^n [x(x-1)+x]\, p(x)$$

$$= \sum_0^n x(x-1)\, p(x) + \sum_0^n x\, p(x)$$

$$= \sum_0^n x(x-1)\, {}^nC_x\, p^x q^{n-x} + np$$

$$= \sum_0^n x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + np$$

$$= n(n-1)p^2 \sum_2^n \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!} p^{x-2} q^{n-2-(x-2)} + np$$

$$= n(n-1)p^2\, {}^{(n-2)}C_{(x-2)}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

$$V = \{n(n-1)p^2 + np\} - (np)^2 = npq$$

<u>$\text{Variance} = npq$</u>

8) b) The length of telephone conservation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth

i) ends less than 5 minutes.

ii) between 5 and 10 minutes. (26)

>> we have $f(x) = \alpha e^{-\alpha x}, \; x > 0$ mean $= 1/\alpha$

By data $1/\alpha = 5$ $\therefore \alpha = 1/5$

hence $f(x) = \frac{1}{5} e^{-x/5}$ is the p.d.f

i) $$P(x<5) = \int_0^5 f(x)\,dx$$

$$= \int_0^5 \frac{1}{5} e^{-x/5}\,dx$$

$$= \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^5$$

$$= -e^{-x/5}\Big]_0^5$$

$$= -\left[e^{-1} - 1\right]$$

$$= 1 - 1/e$$

Thus $\boxed{P(x<5) = 0.6321}$

ii) $$P(5<x<10) = \int_5^{10} f(x)\,dx = \int_5^{10} \frac{1}{5} e^{-x/5}\,dx$$

$$= \frac{1}{5} \frac{e^{-x/5}}{-1/5}\Big|_5^{10} = -\left[e^{-x/5}\right]_5^{10}$$

$$= \frac{1}{e} - \frac{1}{e^2} = 0.2325$$

$\therefore \boxed{P(5<x<10) = 0.2325}$

8) c) The joint distribution of two discrete variables X and y is $f(x, y) = k(2x+y)$ where x and y are integers such that $0 \le x \le 2$; $0 \le y \le 3$.
Find i) The value of k ii) marginal distributions of x and y iii) Are x and y independent?

>> $X = \{x_i\} = \{0, 1, 2\}$ and

$Y = \{y_j\} = \{0, 1, 2, 3\}$

$f(x, y) = k(2x+y)$ and the joint probability distribution table is formed as follows.

X \ Y	0	1	2	3	Sum
0	0	k	2k	3k	6k
1	2k	3k	4k	5k	14k
2	4k	5k	6k	7k	22k
Sum	6k	9k	12k	15k	42k

a) we must have $42k = 1$ $\therefore k = 1/42$

b) marginal probability distribution is as follows

x_i	0	1	2
$f(x_i)$	$6/42 = 1/7$	$14/42 = 1/3$	$22/42 = 11/21$

y_j	0	1	2	3
$g(y_j)$	$1/7$	$3/14$	$2/7$	$5/14$

c) It can be easily seen that

$f(x_i)\, g(y_j) \neq J_{ij}$

$\therefore$ x and y are not independent

9) a) Explain the terms

i) Null hypothesis;

The hypothesis formulated for the purpose of its rejection under the assumption that it is true is called null hypothesis.

ii) Type I error

If a hypothesis is rejected while it should have been accepted is known as Type I error.

Type II error

If a hypothesis is accepted while it should have been rejected is known as Type II error.

iii) significance level:

The probability level below which leads to rejection of the hypothesis is known as significance level.

9) b) A die thrown 9000 times and a throw of 3 or 4 was observed 3240 times. Is it reasonable to think that the die is an unbiased one? (29)

>> Probability of getting 3 or 4 in a single throw is $p = 2/6 = 1/3$ and $q = 1 - p = 2/3$

$\therefore$ expected number of successes $= 1/3 \times 9000 = 3000$

Observed number of successes $= 3240$

The difference $= 3240 - 3000 = 240$

Consider $Z = \frac{x - np}{\sqrt{npq}}$

$= \frac{240}{\sqrt{9000 \times 1/3 \times 2/3}} = \frac{240}{\sqrt{2000}} = 5.37$

Since $Z = 5.37 > 2.58$ we conclude that the die is biased.

9) c) Find the unique fixed probability vector for the regular stochastic matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

>> we have to find $v = (x, y, z)$ where $x + y + z = 1$ such that $vA = v$

$$\Rightarrow [x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

ie $\left[\frac{y}{6}, x+\frac{y}{2}+\frac{2z}{3}, \frac{y}{3}+\frac{z}{3}\right] = [x, y, z]$

$\Rightarrow \frac{y}{6} = x, \quad x+\frac{y}{2}+\frac{2z}{3} = y, \quad \frac{y}{3}+\frac{z}{3} = z$

ie $y = 6x, \quad 6x+3y+4z = 6y, \quad y+z = 3z$

$y = 6x, \quad 6x-3y+4z = 0, \quad y-2z = 0$

using $y = 6x$ and $z = 1-x-y$
$= 1-x-6x$
$= 1-7x$

in $6x-3y+4z=0$ we have

$6x-18x+4-28x = 0$

$\therefore x = 1/10$

Hence $y = 6/10, \quad z = 3/10$

Thus the required unique fixed probability vector v is given by $v = (1/10, 6/10, 3/10)$

(OR)

10) a) A certain stimulus administered to each of the 12 patients resulted in the following change in blood pressure 5, 2, 8, −1, 3, 0, 6, −2, 1, 5, 0, 4. Can it be concluded that the stimulus will increase the blood pressure. ($t_{.05}$ for 11 df = 2.201)

$\bar{x} = \frac{\Sigma x}{n} = \frac{31}{12} = 2.5833$

$s^2 = \frac{1}{n-1}\Sigma(x-\bar{x})^2 = \frac{1}{n-1}\left\{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2\right\}$

$= \frac{1}{11}\left\{185 - \frac{1}{12}(31)^2\right\}$

$s^2 = 9.538$

$\therefore \boxed{s = 3.088}$

we have $t = \frac{\bar{x}-\mu}{s}\sqrt{n}$

Let us suppose that the stimulus administration is not accompanied with increase in blood pressure, we can take $\mu = 0$

$$\therefore t = \frac{2.5833-0}{3.088}\sqrt{12}$$
$$= 2.8979$$
$$\simeq 2.9$$

$$\therefore t = 2.9 > 2.201$$

Hence the hypothesis is rejected at 5% level of significance. we conclude with 95% confidence that the stimulus in general is accompanied with increase in blood pressure.

10) b) It has been found that the mean breaking strength of a particular brand of thread is 275.6 gms with $\sigma = 39.7$ gms. A sample of 36 pieces of thread showed a mean breaking strength of 253.2 gms. Test the claim 1% and 5% level of significance.

>> we have to decide b/w the two hypothesis

$H_0: \mu = 275.6$ gms, mean breaking strength

$H_1: \mu < 275.6$ gms, inferior in breaking strength

we choose the one tailed test

Mean breaking strength of a sample of 36 pieces = 253.2

$\therefore$ difference $= 275.6 - 253.2 = 22.4$

$n = 36$

$$Z = \frac{\text{difference}}{(\sigma/\sqrt{n})}$$

$$= \frac{22.4}{39.7/6} = 3.38$$

The value of Z is greater than the critical value of $Z = 1.645$ at 5% level and 2.33 at 1% level of significance.

Under the hypothesis H_1 that the thread has become inferior is accepted at both 5% and 1% levels in accordance with one tailed test.

10) c) A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarattes the next week with probability 0.2 one the other hand, if he smokes non filter cigarattes one week there is a probability of 0.7 that he will smoke non filter cigarattes the next week as well. In the long run how often does he smoke filter cigarattes?

>> The state space of the system is {A, B}

where

A : Smoking filter cigarattes.

B : Smoking non filter cigarattes

The associated transition matrix is as follows

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array} = \begin{bmatrix} 8/10 & 2/10 \\ 3/10 & 7/10 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix}$$

we have to find the unique fixed probability vector, $v = (x, y)$ such that $vp = v$

where $x+y=1$

ie $[x, y] \cdot \frac{1}{10}\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix} = [x, y]$

$[8x+3y, 2x+7y] = [10x, 10y]$

$\Rightarrow 8x+3y=10x, \quad 2x+7y=10y$

Using $y=1-x$ in the first eq^n, we get

$8x + 3(1-x) = 10x$

$\boxed{x = 3/5} \qquad \boxed{y = 2/5}$

Hence $v = (x, y) = (3/5, 2/5) = (P_A, P_B)$

In the long run, he will smoke filter cigarettes 3/5 or 60% of the time.

End