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10MAT41

Fourth Semester B.E. Degree Examination, June 2012

Engineering Mathematics - IV

Time: 3 hrs. Max. Marks:100

Note: ***Answer FIVE full questions, selecting at least TWO questions from each part.***

PART – A

1 a. Using the Taylor's method, find the third order approximate solution at x = 0.4 of the problem $\frac{dy}{dx} = x^2y + 1$, with y(0) = 0. Consider terms upto fourth degree. **(06 Marks)**

b. Solve the differential equation $\frac{dy}{dx} = -xy^2$ under the initial condition y(0) = 2, by using the modified Euler's method, at the points x = 0.1 and x = 0.2. Take the step size h = 0.1 and carry out two modifications at each step. **(07 Marks)**

c. Given $\frac{dy}{dx} = xy + y^2$; y(0) = 1, y(0.1) = 1.1169, y(0.2) = 1.2773, y(0.3) = 1.5049, find y(0.4) correct to three decimal places, using the Milne's predictor-corrector method. Apply the corrector formula twice. **(07 Marks)**

2 a. Employing the Picard's method, obtain the second order approximate solution of the following problem at x = 0.2.

$$\frac{dy}{dx} = x + yz; \quad \frac{dz}{dx} = y + zx; \quad y(0) = 1, \quad z(0) = -1.$$ **(06 Marks)**

b. Using the Runge-Kutta method, solve the following differential equation at x = 0.1 under the given condition:

$$\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right), \qquad y(0) = 1, \qquad y'(0) = 0.5.$$

Take step length h = 0.1. **(07 Marks)**

c. Using the Milne's method, obtain an approximate solution at the point x = 0.4 of the problem $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$, y(0) = 1, y′(0) = 0.1. Given y(0.1) = 1.03995, y′(0.1) = 0.6955, y(0.2) = 1.138036, y′(0.2) = 1.258, y(0.3) = 1.29865, y′(0.3) = 1.873. **(07 Marks)**

3 a. Derive Cauchy-Riemann equations in polar form. **(06 Marks)**

b. If f(z) is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. **(07 Marks)**

c. If w = ϕ + i$\overset{\psi}{y}$ represents the complex potential for an electric field and $\overset{\psi}{y} = x^2 - y^2 + \frac{x}{x^2 + y^2}$ determine the function ϕ. Also find the complex potential as a function of z. **(07 Marks)**

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

4 a. Discuss the transformation of $w = z + \frac{k^2}{z}$. **(06 Marks)**

b. Find the bilinear transformation that transforms the points $z_1 = i$, $z_2 = 1$, $z_3 = -1$ on to the points $w_1 = 1$, $w_2 = 0$, $w_3 = \infty$ respectively. **(07 Marks)**

c. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ where c is the circle $|z| = 3$, using Cauchy's integral formula. **(07 Marks)**

PART – B

5 a. Obtain the solution of $x^2y'' + xy' + (x^2 - n^2)y = 0$ in terms of $J_n(x)$ and $J_{-n}(x)$. **(06 Marks)**

b. Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials. **(07 Marks)**

c. Prove that $\int_{-1}^{+1} P_m(x) \cdot P_n(x) dx = \frac{2}{2n+1}$, $m = n$. **(07 Marks)**

6 a. From five positive and seven negative numbers, five numbers are chosen at random and multiplied. What is the probability that the product is a (i) negative number and (ii) positive number? **(06 Marks)**

b. If A and B are two events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(A \cap B) = \frac{1}{4}$, find $P(A/B)$, $P(B/A)$, $P(\overline{A}/\overline{B})$, $P(\overline{B}/\overline{A})$ and $P(A/\overline{B})$. **(07 Marks)**

c. In a certain college, 4% of boy students and 1% of girl students are taller than 1.8 m. Furthermore, 60% of the students are girls. If a student is selected at random and is found taller than 1.8 m, what is the probability that the student is a girl? **(07 Marks)**

7 a. A random variable x has the density function $P(x) = \begin{cases} Kx^2, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$. Evaluate K, and find: i) $P(x \le 1)$, (ii) $P(1 \le x \le 2)$, (iii) $P(x \le 2)$, iv) $P(x > 1)$, (v) $P(x > 2)$. **(06 Marks)**

b. Obtain the mean and standard deviation of binomial distribution. **(07 Marks)**

c. In an examination 7% of students score less than 35% marks and 89% of students score less than 60% marks. Find the mean and standard deviation if the marks are normally distributed. It is given that $P(0 < z < 1.2263) = 0.39$ and $P(0 < z < 1.4757) = 0.43$. **(07 Marks)**

8 a. A random sample of 400 items chosen from an infinite population is found to have a mean of 82 and a standard deviation of 18. Find the 95% confidence limits for the mean of the population from which the sample is drawn. **(06 Marks)**

b. In the past, a machine has produced washers having a thickness of 0.50 mm. To determine whether the machine is in proper working order, a sample of 10 washers is chosen for which the mean thickness is found as 0.53 mm with standard deviation 0.03 mm. Test the hypothesis that the machine is in proper working order, using a level of significance of (i) 0.05 and (ii) 0.01. **(07 Marks)**

c. Genetic theory states that children having one parent of blood type M and the other of blood type N will always be one of the three types M, MN, N and that the proportions of these types will on an average be 1 : 2 : 1. A report states that out of 300 children having one M parent and one N parent, 30% were found to be of type M, 45% of type MN and the remainder of type N. Test the theory by χ^2 (Chi square) test. **(07 Marks)**

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PART - A

1 (a) $$\frac{dy}{dx} = x^2y+1 \longrightarrow (1)$$

$$y(0)=0$$

Taylor's series expansion of $y(x)$ is given by

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data $x_0 = 0$ $y_0 = 0$

$$y' = x^2y+1$$

$$\therefore y(x) = y(0) + x\cdot y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0)$$

$$= 0 + x\cdot y'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^{iv}(0)$$

$$y(x) = x\,y'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^{iv}(0) \longrightarrow (2)$$

we need to compute $y'(0)$, $y''(0)$, $y'''(0)$, $y^{iv}(0)$

Consider $y' = x^2y+1$

$$y' = x^2y+1$$

$$y'(0) = 0^2\cdot y(0) + 1 = 1$$

$$\boxed{y'(0) = 1}$$

$$y' = x^2y + 1 \Rightarrow y'' = 2xy + x^2y'$$

$$\therefore y''(0) = 2(0).y(0) + 0^2.y'(0)$$

$$= 0.$$

$$\boxed{y''(0) = 0}$$

$$y'' = 2xy + x^2y'$$

$$\Rightarrow y''' = 2[y + xy'] + x^2y'' + 2x.y'$$

$$\therefore y'''(0) = 2[y(0) + 0.y'(0)] + 0.y''(0) + 2.0.y'(0)$$

$$= 2[0] + 0 = 0.$$

$$\boxed{\therefore y'''(0) = 0}$$

$$y''' = 2y + 2xy' + x^2y'' + 2xy'$$

$$y^{IV} = 2y' + 2[xy'' + y'] + x^2y''' + 2xy'' + 2xy'' + 2y'$$

$$y^{IV}(0) = 2y'(0) + 2[0.y''(0) + y'(0)] + 0.y'''(0) + 2.0.y''(0) + 2(0).y''(0) + 2y'(0)$$

$$= 2(1) + 2(0+1) + 0 + 0 + 0 + 2(1)$$

$$= 6$$

$$y^{IV}(0) = 6.$$

Substituting these values in eq (2)

$$y(x) = x(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(0) + \frac{x^4}{24}.(6)$$

$$\boxed{y(x) = x + \frac{x^4}{4}} \longrightarrow (3)$$

This is Taylor's series expansion upto 4^{th} degree.

Now we need to compute y at $x = 0.4$

ie $y(0.4)$.

$\therefore$ Put $x = 0.4$ in (3)

$$y(0.4) = 0.4 + \frac{(0.4)^4}{4}$$

$$\boxed{y(0.4) = 0.4064}$$

1(b) Given $\frac{dy}{dx} = -xy^2 \longrightarrow (1)$

$y(0) = 2 \longrightarrow (2)$

$h = 0.1$

Ist stage: By the data $x_0 = 0$ $y_0 = 2$

$f(x,y) = -xy^2$ $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

we need to find y at $x = 0.1$

ie y at x_1 ie $y(x_1)$

$\therefore y(0.1) = y_1 = ?$

we have Euler's formula $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$\therefore y_1^{(0)} = 2 + 0.1 \cdot f(x_0, y_0)$

$= 2 + 0.1 f(0, 2)$

$= 2 + 0.1 (0) = 2$

$y_1^{(0)} = 2.$

we have modified Euler's formula:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$f(x_0, y_0) = f(0, 2) = 0.$$

$$f(x_1, y_1^{(0)}) = f(0.1, 2) = -(0.1)(2)^2$$

$$= -0.4.$$

$$y_1^{(1)} = 2 + \frac{0.1}{2}\left[0 + (-0.4)\right] = 1.98$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 2 + \frac{0.1}{2}\left[0 + f(0.1, 1.98)\right]$$

$$= 2 + \frac{0.1}{2}\left[-0.39204\right]$$

$$y_1^{(2)} = 1.9803$$

$$\therefore y(0.1) = 1.9803.$$

II nd stage :

Now $x_0 = 0.1$ $y_0 = 1.9803$

we need to compute y at $x = 0.2$ by taking $h = 0.1$

ie y at $x_1 = x_0 + h = 0.1 + 0.1 = 0.2$

ie $y(x_1) = y_1 = ?$

we have Euler's formula $y_1^{(0)} = y_0 + h \cdot f(x_0, y_0)$

$$y_1^{(0)} = 1.9803 + 0.1 \cdot f(0.1, 1.9803)$$

$$f(x, y) = -xy^2$$

$$\therefore f(0.1, 1.9803) = -(0.1)(1.9803)^2 = -0.392$$

$$y_1^{(0)} = 1.9803 + (0.1)(-0.392)$$
$$= 1.9411$$

By Modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1.9803 + \frac{0.1}{2}\left[-0.392 + f(0.2, 1.9411)\right]$$

$$= 1.9803 + \frac{0.1}{2}\left[-1.1455\right] = 1.923$$

$$\therefore \quad y_1^{(1)} = 1.923$$

Now $\quad y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$$= 1.9803 + \frac{0.1}{2}\left[-0.392 + f(0.2, 1.923)\right]$$

$$= 1.9237$$

$$\therefore \quad \boxed{y(0.2) = 1.9237}$$

1 (c)

Given $\quad \frac{dy}{dx} = xy + y^2$

$y(0) = 1, \quad y(0.1) = 1.1169 \quad y(0.2) = 1.2773 \quad y(0.3) = 1.504$

$y(0.4) = ?$

x	y	$y' = xy + y^2$
$x_0 = 0$	$y_0 = 1$	$y_0' = 0 + 1^2 = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$y_1' = (0.1)(1.1169) + (1.1169)^2 = 1.359$
$x_2 = 0.2$	$y_2 = 1.2773$	$y_2' = 1.887$
$x_3 = 0.3$	$y_3 = 1.5049$	$y_3' = 2.716$
$x_4 = 0.4$	$y_4 = ?$	

we have the predictor formula

$$y_4^{(p)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$\therefore y_4^{(p)} = 1 + \frac{4(0.1)}{3}\left[2(1.359) - (1.887) + 2(2.716)\right]$$

$$y_4^{(p)} = 1.835$$

Now $y_4' = x_4 \cdot y_4 + y_4^2$

$$= (0.4)(1.835) + (1.835)^2 = 4.101$$

Next we have the corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$\therefore y_4^{(c)} = 1.2773 + \frac{0.1}{3}\left(1.887 + 4(2.716) + 4.101\right)$$

$$= 1.839$$

Now $y_4' = x_4 \cdot y_4 + y_4^2 = (0.4)(1.839) + (1.839)^2$

$$= 4.118$$

Again using corrector formula

$$y_4 = 1.2773 + \frac{0.1}{3}\left(1.887 + 4(2.716) + 4.118\right)$$

$$= 1.8402$$

⑤

2 (a)

Given $\frac{dy}{dx} = x + yz \quad y(0) = 1 \longrightarrow (1)$

$\frac{dz}{dx} = y + zx \quad z(0) = -1 \longrightarrow (2)$

(1) $\Rightarrow$ $dy = (x+yz)\,dx \quad \therefore y = 1 \quad x = 0$

Now $\int_{1}^{y} dy = \int_0^x (x+yz)\,dx$

$\therefore y = 1 + \int_0^x (x+yz)\,dx \longrightarrow (3)$

(2) $\Rightarrow$ $dz = (y+zx)\,dx \quad z = -1,\ x = 0$

$\int_{-1}^{z} dz = \int_0^x (y+zx)\,dx$

$z = -1 + \int_0^x (y+zx)\,dx \longrightarrow (4)$

Ist approximation

$y_1 = 1 + \int_0^x (x + y_0 z_0)\,dx$

$= 1 + \int_0^x (x + 1\cdot(-1))\,dx = 1 + \int_0^x (x-1)\,dx$

$y_1 = 1 + \frac{x^2}{2} - x$

$$z_1 = -1 + \int_0^x (y_0 + z_0 x)\, dx \qquad = -1 + \int_0^x (1 - x)\, dx$$

$$z_1 = -1 + x - \frac{x^2}{2}$$

IInd approximation :

$$y_2 = 1 + \int_0^x (x + y_1 z_1)\, dx = 1 + \int_0^x x + \left(1 - x + \frac{x^2}{2}\right)\left(-1 + x - \frac{x^2}{2}\right)$$

$$= 1 - x + \frac{3}{2}x^2 - \frac{2}{3}\cdot x^3 + \frac{x^4}{4} - \frac{x^5}{20}$$

now
$$z_2 = -1 + \int_0^x (y_1 + z_1 x)\, dx$$

$$= -1 + \int_0^x \left(1 - x + \frac{x^2}{2}\right) + x\left(-1 + x - \frac{x^2}{2}\right) dx$$

$$= -1 + x - x^2 + \frac{x^3}{2} - \frac{x^4}{8}.$$

Now y at $x = 0.2$ ie $y(0.2)$ and $z(0.2)$

$$y(0.2) = 1 - (0.2) + \frac{3}{2}(0.2)^2 - \frac{2}{3}(0.2)^3 + \frac{(0.2)^4}{4} - \frac{(0.2)^5}{20}$$

$$y(0.2) = 0.855 = 0.855$$

$$z(0.2) = -0.836$$

②⑤ Given $\frac{d^2y}{dx^2} = x^3\left(y+\frac{dy}{dx}\right)$ $y(0)=1$ $y'(0)=0.5$

$h = 0.1$

$$\frac{d^2y}{dx^2} - x^3\frac{dy}{dx} - x^3y = 0 \longrightarrow (1)$$

putting $\frac{dy}{dx} = z \Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx}$

$\therefore$ (1) $\Rightarrow$ $\frac{dz}{dx} - x^3z - x^3y = 0$

Hence we have a system of equations

$\frac{dy}{dx} = z$ $\frac{dz}{dx} = x^3(y+z)$ where $y=1$, $z=0.5$ when $x=0$

$x_0=0$, $y_0=1$, $z_0=0.5$ and $h=0.1$

we shall first compute the following

$K_1 = h\,f(x_0,y_0,z_0) = 0.1\,f(0,1,0.5) = 0.1\times0.5$

$= 0.05$

$l_1 = h.g(x_0,y_0,z_0) = 0.1\,g(0,1,0.5)$

$= 0(1+0.5) = 0.$

$$k_2 = h \cdot f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 \cdot f(0.05,\ 1.025,\ 0.5)$$

$$= (0.1)(0.5) = 0.05$$

$$l_2 = h \cdot g\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 \cdot g(0.05,\ 1.025,\ 0.5) = 0.1\left[(0.05)^3 (1.025 + 0.5\right.$$

$$l_2 = 0.000019$$

$$k_3 = h \cdot f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$= 0.1 \cdot f(0.05,\ 1.025,\ 0.500009)$$

$$= 0.05$$

$$l_3 = 0.1 \cdot g(0.05,\ 1.025,\ 0.500009)$$

$$= 0.1\left[(0.05)^3 (1.025 + 0.500009)\right]$$

$$= 0.00001906 \simeq 0.00002$$

$$k_4 = h \cdot f(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$= 0.1\ f(0.1,\ 1.05,\ 0.500019) = 0.1(0.500019) = 0.0500002$$

$l_4 = h \cdot g(x_0+h,\ y_0+k_3\ \ z_0+l_3)$

$= h \cdot g(0.1,\ 1.05,\ 0.500019)$

$= 0.1 \times (0.1)^3 (1.05 + 0.500019)$

$= 0.000155$

Now $y(x_0+h) = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$y(0.1) = 1 + \frac{1}{6}[0.05+0.1+0.1+0.050002]$

$y(0.1) = 1.05$

2 c

Given $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$; $y(0)=1,\ y'(0)=0.1$

Putting $y' = \frac{dy}{dx} = z$ we obtain $y'' = \frac{d^2y}{dx^2} = z'$

The given equation becomes ~~$z' = \ldots$~~

$z' + 3xz - 6y = 0$

$z' = 6y - 3xz$

Now $z_0' = \ldots$

$z_1' = z'(0.1) = 6\,y(0.1) - 3\times 0.1 \times z(0.1)$

$= 6 \times 1.03995 - 0.3 \times 0.6955 = 6.03105$

$$z_2' = z'(0.2) = 6\,y(0.2) - 3\times 0.2\times z(0.2)$$
$$= 6\times 1.138036 - 0.6\times 1.258 = 6.073416.$$

$$z_3' = z'(0.3) = 6y(0.3) - 3\times 0.3\times z(0.3)$$
$$= 6\times(1.29865) - (0.9\times 1.873) = 6.1062$$

Now we find that $z_4^{(p)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$

$$= 0.1 + \frac{0.4}{3}\left\{2(6.03105) - 6.073416 + 2(6.1062)\right\}$$
$$= 2.5268$$

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$
$$= 1 + \frac{4(0.1)}{3}\left[2(\overset{0.6955}{\cancel{1.03995}}) - 1.258 + 2(1.873)\right]$$
$$= \cancel{3.0688} = 1.5172$$

Next we consider Milne's corrector formulae

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4\right]$$
$$z_4^{(c)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4' = 6y_4^{(p)} - 3z_4^{(p)}x_4 = 6(\overset{1.5172}{\cancel{3.0688}}) - 3(2.5268)(0.4)$$
$$= 6.07104.$$

Now $y_4^{(c)} = \cancel{1.1380} + \frac{0.1}{3}\left[1.258 + 4(1.873) + \overset{2.5268}{\cancel{6.0710}}\right]$

$$= \underline{1.5139}$$

3 (a) Stat: If $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and satisfy the equations $\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}$.

proof: Let $f(z)$ be analytic at a point $z = re^{i\theta}$

$\therefore f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique.

In the polar form $f(z) = u(r,\theta) + iv(r,\theta)$ and let δz be the increment in z corresponding to the increments δr, $\delta\theta$ in r,θ.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(r+\delta r, \theta+\delta\theta) - v(r,\theta)]}{\delta z}$$

consider $z = re^{i\theta}$. Since z is a function of two variables r,θ, we have $\delta z = \frac{\partial z}{\partial r}\cdot\delta r + \frac{\partial z}{\partial \theta}\cdot\delta\theta$

$$= \frac{\partial}{\partial r}(re^{i\theta})\cdot\delta r + \frac{\partial}{\partial \theta}(re^{i\theta})\cdot\delta\theta$$

$$\text{ie } \delta z = e^{i\theta}\delta r + ire^{i\theta}\cdot\delta\theta$$

Since δz tends to zero, we have the following two possibilities.

Case (i): Let $\delta\theta = 0$ so that $\delta z = e^{i\theta} \cdot \delta r$ and $\delta z \to 0$ imply $\delta r \to 0$.

Now ① becomes
$$f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta} \cdot \delta r} + i \lim_{\delta r \to 0} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta} \cdot \delta r}$$

ie
$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \longrightarrow ②$$

Case (ii): Let $\delta r = 0$ so that $\delta z = i r e^{i\theta} \cdot \delta\theta$ and $\delta z \to 0$ imply $\delta\theta \to 0$.

Now ① becomes
$$f'(z) = \lim_{\delta\theta \to 0} \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{i r e^{i\theta} \cdot \delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r, \theta+\delta\theta) - v(r,\theta)}{i r e^{i\theta} \cdot \delta\theta}$$

$$= \frac{1}{i r e^{i\theta}}\left[\lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{\delta\theta}\right]$$

$$f'(z) = \frac{1}{i r e^{i\theta}}\left[\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right] = \frac{1}{r e^{i\theta}}\left[\frac{1}{i}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]$$

But $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

$$\therefore f'(z) = \frac{1}{r e^{i\theta}}\left[-i\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right] = e^{-i\theta}\left[\frac{-i}{r} \cdot \frac{\partial u}{\partial\theta} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right]$$

ie $f'(z) = e^{-i\theta}\left[\frac{1}{r}\cdot\frac{\partial v}{\partial\theta} - \frac{i}{r}\frac{\partial u}{\partial\theta}\right] \longrightarrow (3)$

Equating RHS of eq (2) & (3) we have

$$\cancel{e^{-i\theta}}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right] = \cancel{e^{-i\theta}}\left[\frac{1}{r}\cdot\frac{\partial v}{\partial\theta} - \frac{i}{r}\frac{\partial u}{\partial\theta}\right]$$

equating real and imaginary parts on both sides

we have $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}$

These are CR equations in polar form.

(3) (b)

If $f(z)$ is analytic, we need to show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Let $f(z) = u+iv$ be analytic.

$\therefore |f(z)| = \sqrt{u^2+v^2} \Rightarrow |f(z)|^2 = u^2+v^2 = \phi$ (say)

To prove that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\phi = 4|f'(z)|^2$

ie to prove that $\phi_{xx}+\phi_{yy} = 4|f'(z)|^2$

Consider $\phi = u^2+v^2$

Differentiate w.r.t 'x' partially

$\therefore \phi_x = 2uu_x + 2vv_x = 2[uu_x + vv_x]$

Differentiating w.r.t 'x' again we get

$$\phi_{xx} = 2\left[u u_{xx} + u_x^2 + v v_{xx} + v_x^2\right] \longrightarrow (1)$$

Similarly we can also get

$$\phi_{yy} = 2\left[u u_{yy} + u_y^2 + v v_{yy} + v_y^2\right] \longrightarrow (2)$$

Adding (1) and (2) we have

$$\phi_{xx} + \phi_{yy} = 2\left[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2\right] \longrightarrow (3)$$

Since $f(z)$ is analytic u & v are harmonic.

Hence $u_{xx} + v_{yy} = 0$, $v_{xx} + v_{yy} = 0$. Further we also have CR equations $v_y = u_x$, $u_y = -v_x$

$$\therefore (3) \Rightarrow \phi_{xx} + \phi_{yy} = 2\left[u \cdot 0 + v \cdot 0 + u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2\right]$$

ie $\phi_{xx} + \phi_{yy} = 2\left[2u_x^2 + 2v_x^2\right] = 4\left[u_x^2 + v_x^2\right] \longrightarrow (4)$

But $f'(z) = u_x + i v_x \Rightarrow |f'(z)| = \sqrt{u_x^2 + v_x^2}$

$$\Rightarrow |f'(z)|^2 = u_x^2 + v_x^2$$

$$\therefore (4) \Rightarrow \phi_{xx} + \phi_{yy} = 4|f'(z)|^2 //$$

3 c

Given $\psi = (x^2-y^2) + \dfrac{x}{x^2+y^2}$

$$\psi_x = \frac{(x^2+y^2)1 - x\cdot 2x}{(x^2+y^2)^2} + 2x = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\psi_y = \frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2} - 2y = -2y - \frac{2xy}{(x^2+y^2)^2}$$

consider $f'(z) = \phi_x + i\psi_x$ But $\phi_x = \psi_y$

ie $f'(z) = \psi_y + i\psi_x$

Putting $x = z, y = 0$ we have

$$f'(z) = [\psi_y]_{(z,0)} + i[\psi_x]_{(z,0)}$$

$$= 0 + i\left(2z + \frac{-z^2}{(z^2)^2}\right) = i\left(2z - \frac{1}{z^2}\right)$$

$$\therefore f(z) = i\int\left(2z - \frac{1}{z^2}\right)dz + c = i\left(z^2 + \frac{1}{z}\right) + c$$

$$f(z) = i\left(z^2 + \frac{1}{z}\right) + c$$

To find ϕ we shall separate the RHS into real and imaginary parts

$$\phi + i\psi = i\left\{(x+iy)^2 + \frac{1}{x+iy}\right\} + c$$

$$= i\left\{(x^2+i^2y^2+2xiy) + \frac{x-iy}{(x+iy)(x-iy)}\right\} + c$$

$$= i\left\{(x^2-y^2)+2xiy\right\} + i\left\{\frac{x-iy}{x^2+y^2}\right\} + c$$

$$= i(x^2-y^2) - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + c$$

$$\therefore \phi + i\psi = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2-y^2+\frac{x}{x^2+y^2}\right)$$

$$\boxed{\therefore \phi = -2xy + \frac{y}{x^2+y^2}}$$

4 (a) Discussion of $\omega = z + \frac{k^2}{z}$

Putting $z = re^{i\theta}$, we have

$$u + iv = re^{i\theta} + \left(\frac{k^2}{r}\right) e^{-i\theta} = r(\cos\theta + i\sin\theta) + \left(\frac{k^2}{r}\right)(\cos\theta - i\sin\theta)$$

ie $$u + iv = \left[r + \frac{k^2}{r}\right]\cos\theta + i\left[r - \frac{k^2}{r}\right]\sin\theta.$$

$$\Rightarrow \quad u = \left(r + \frac{k^2}{r}\right)\cos\theta \quad \text{and} \quad v = \left(r - \frac{k^2}{r}\right)\sin\theta \longrightarrow (1)$$

we shall eliminate r and θ separately from (1)

To eliminate θ let us put (1) in the form

$$\frac{u}{r + \frac{k^2}{r}} = \cos\theta \; ; \quad \frac{v}{\left(r - \frac{k^2}{r}\right)} = \sin\theta$$

Squaring and adding we obtain

$$\frac{u^2}{\boxed{\left[r + \frac{k^2}{r}\right]^2}} + \frac{v^2}{\left(r - \frac{k^2}{r}\right)^2} = 1, \quad r \neq k \longrightarrow (2)$$

To eliminate r, let us put (1) in the form

$$\frac{u}{\cos\theta} = \left(r + \frac{k^2}{r}\right) \; ; \quad \frac{v}{\sin\theta} = \left(r - \frac{k^2}{r}\right)$$

Squaring and subtracting we obtain

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left[r + \frac{k^2}{r}\right]^2 - \left(r - \frac{k^2}{r}\right)^2 = 4k^2$$

$$\& \quad \frac{u^2}{(2k\cos\theta)^2} - \frac{v^2}{(2k\sin\theta)^2} = 1 \longrightarrow (3)$$

Since $z = re^{i\theta}$, $|z| = r$ and $\operatorname{amp} z = \theta$

$$|z| = r \Rightarrow x^2+y^2 = r^2.$$

This represents a circle with centre origin and radius r in the z-plane when r is a constant.

$$\operatorname{amp} z = \theta \Rightarrow \tan^{-1}(y/x) = \theta \Rightarrow y/x = \tan\theta$$

This represents a straight line in the z-plane when θ is a constant.

We shall discuss the image in the w-plane, corresponding to r = constant (circle) and θ = constant (straight line) in the z-plane.

Case (i): Let r = constant

Eq (2) is of the form $\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$ where $A = r + \frac{k^2}{r}$, $B = r - \frac{k^2}{r}$

This represents an ellipse in the w-plane with foci

$$\left(\pm\sqrt{A^2-B^2}, 0\right) = (\pm 2k, 0)$$

Hence we conclude that the circle $|z| = r$ = constant in the z-plane maps onto an ellipse in the w-plane with foci $(\pm 2k, 0)$

<u>Case (ii)</u> : let $\theta =$ constant.

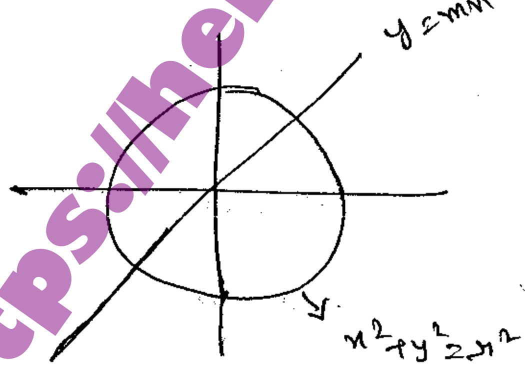
eq (8) is of the form $\dfrac{u^2}{A^2} - \dfrac{v^2}{B^2} = 1$ where $A = 2k\cos\theta$, $B = 2k\sin\theta$

This represents a hyperbola in the w-plane with foci

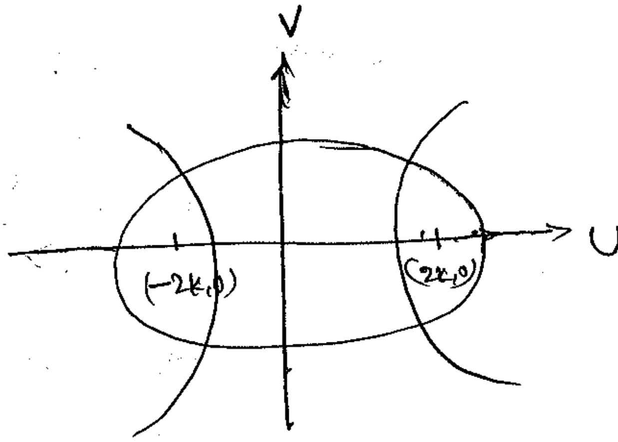
$$\left(\pm\sqrt{A^2+B^2}, 0\right) = (\pm 2k, 0)$$

Hence we conclude that the straight line passing through the origin in the z-plane maps onto a hyperbola in the w-plane with foci $(\pm 2k, 0)$.

Since both these conics (ellipse and hyperbola) have the same foci independent of r, θ they are called confocal conics.



z-plane



w-plane

4 (b) Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation

$z_1 = i \quad w_1 = 1$

$\therefore 1 = \dfrac{ai+b}{ci+d} \Rightarrow ai+b-ci-d=0 \longrightarrow (1)$

$z_2 = 1, \quad w_2 = 0$

$\therefore 0 = \dfrac{a+b}{c+d} \Rightarrow a+b=0 \longrightarrow (2)$

$z_3 = -1 \quad w_3 = \infty \Rightarrow \frac{1}{w} = 0.$

Consider $\dfrac{1}{w} = \dfrac{cz+d}{az+b}$

When $z=-1 \quad \frac{1}{w} = 0.$

$\therefore 0 = \dfrac{cz+d}{az+b} = \dfrac{-c+d}{-a+b}$

$\Rightarrow -c+d=0 \longrightarrow (3)$

$(1)+(3) \Rightarrow ai+b-(1+i)c=0 \longrightarrow (4)$

Let us solve (1) & (4)

$$1a + 1b + 0 \cdot c = 0 \longrightarrow (1)$$

$$ia + 1b - (1+i)c = 0 \longrightarrow (4)$$

Applying rule of cross multiplication, we have

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -(1+i) \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 0 \\ i & -(1+i) \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 1 \\ i & 1 \end{vmatrix}}$$

ie $$\frac{a}{-(1+i)} = \frac{-b}{-(1+i)} = \frac{c}{1-i}$$

$$a = -(1+i) \quad b = (1+i) \quad c = 1-i = d.$$

Substituting these values in the assumed BLT we have

$$w = \frac{-(1+i)z + (1+i)}{(1-i)z + (1-i)}$$

ie $$w = \frac{(1+i)}{(1-i)}\left(\frac{1-z}{1+z}\right)$$

4 (c)

we shall first resolve $\frac{1}{(z-1)^2(z-2)}$ into partial fractions

let $\frac{1}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{z-2} \rightarrow (1)$

$$1 = A(z-1) + B(z-2) + C(z-1)^2$$

put $z=1, \Rightarrow B=-1$

$z=2, \Rightarrow C=1$

equating the coefficient of z^2 on both sides we have

$$A+C=0 \Rightarrow -C=A \Rightarrow A=-1$$

let $f(z) = \sin\pi z^2 + \cos\pi z^2$

multiplying eq (1) by $f(z)$ and integrating w.r.t 'z' over C by using the value of the constants obtained we have

$$\int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = -\int_C \frac{f(z)}{z-1}\,dz - \int_C \frac{f(z)}{(z-1)^2} + \int_C \frac{f(z)}{z-2}\,dz \rightarrow (2)$$

$$I = I_1 + I_2 + I_3 \text{ (say)}$$

Also $C : |z| = 3$

The points $z=1$ and $z=2$ both lies with in C.

Hence by Cauchy's Integral formula,

$$I_1 = -[2\pi i\, f(1)] = -2\pi i(\sin\pi + \cos\pi) = -2\pi i(0-1) = 2$$

$$I_2 = -[2\pi i\, f'(1)] \quad \text{But } f'(z) = 2\pi z(\cos\pi z^2 - \sin\pi z^2)$$

$$\text{Hence } I_2 = -[2\pi i \cdot 2\pi(\cos\pi - \sin\pi)] = 4\pi^2 i$$

$$I_3 = 2\pi i\, f(2) = 2\pi i(\sin 4\pi + \cos 4\pi) = 2\pi i(0+1) = 2\pi$$

Hence from ② $I = 2\pi i + 4\pi^2 i + 2\pi i = 4\pi i + 4\pi^2 i$

$$\therefore \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\, dz = 4\pi i(1+\pi)$$

5 (a) The Bessel D.E of order 'n' is in the form

$$x^2y'' + xy' + (x^2-n^2)\,y = 0 \longrightarrow (1)$$ where n is a nonneg real constant

We employ Frobenius method to solve this equation.

We assume the series solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \longrightarrow (2)$$

$$\therefore y' = \sum_0^{\infty} a_r\,(k+r)\,x^{k+r-1}$$

$$y'' = \sum_0^{\infty} a_r (k+r)(k+r-1)\,x^{k+r-2}$$

Now (1) $\Rightarrow$ $$\sum_0^{\infty} a_r(k+r)(k+r-1)\,x^{k+r} + \sum_0^{\infty} a_r(k+r)\,x^{k+r} + \sum_0^{\infty} a_r\,x^{k+r+2}$$

$$-n^2\sum_0^{\infty} a_r\,x^{k+r} = 0$$

collecting first, second, and fourth terms together

$$\sum_0^{\infty} a_r x^{k+r}\left[(k+r)(k+r-1)+(k+r)-n^2\right] + \sum_0^{\infty} a_r\,x^{k+r+2} = 0$$

ie $$\sum_0^{\infty} a_r\,x^{k+r}\left[(k+r)\{k+r-1+1\}\right] + \sum_0^{\infty} a_r\,x^{k+r+2} = 0$$

ie $$\sum_0^{\infty} a_r\,x^{k+r}\left[(k+r)^2-n^2\right] + \sum_0^{\infty} a_r\,x^{k+r+2} = 0$$

We shall equate the coefficient of the lowest degree term in 'x'

ie x^k to zero. ie $a_0(k^2-n^2)=0$

Setting $a_0 \neq 0$ we have $k^2-n^2=0 \Rightarrow k=\pm n$

Now equate the coefficient of x^{k+1} to zero.

ie $a_1[(k+1)^2 - n^2] = 0.$

$\Rightarrow a_1 = 0 \quad [\because (k+1)^2 - n^2 \neq 0$ because $k = \pm n]$

Next we shall equate the coefficient of x^{k+r} $(r \geq 2)$ to zero.

ie $a_r[(k+r)^2 - n^2] + a_{r-2} = 0$

$$a_r = \frac{-a_{r-2}}{[(k+r)^2 - n^2]} \quad r \geq 2 \longrightarrow (3)$$

when $k = n$ (2) $\Rightarrow a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = \frac{-a_{r-2}}{2nr + r^2}$

putting $r = 2, 3, 4, \ldots$ we obtain

$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}$; $a_3 = \frac{-a_1}{6n+9} = 0$ since $a_1 = 0$

Similarly $a_5, a_7, \ldots$ are all equal to zero.

ie $a_1 = a_3 = a_5 = a_7 = 0 \ldots$

Next $a_4 = \frac{-a_2}{8n+16} = \frac{-a_2}{8(n+2)} = \frac{a_0}{32(n+1)(n+2)}$ and so on.

we substitute these values in the expanded form of (2)

$y = x^k(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots)$

Also let the solution for $k = +n$ be denoted by y_1.

$$\therefore y_1 = x^n\left[a_0 - \frac{a_0}{4(n+1)}x^2 + \frac{a_0}{32(n+1)(n+2)}x^4 - \cdots\right]$$

ie $$y_1 = a_0 x^n\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} \cdots\right]$$

Since we also have $k = -n$, let the solution for $k = -n$ be denoted by y_2. Replacing n by $-n$ in (4), we have

$$y_2 = a_0 x^{-n}\left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots\right] \rightarrow (5)$$

The complete solution of (1) is given by

$y = Ay_1 + By_2$ where A, B are arbitrary constants.

We shall now standardize the solution as in (4) by choosing

$a_0 = \frac{1}{2^n \Gamma(n+1)}$ and the same be denoted by Y_1.

$$Y_1 = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \left(\frac{x}{2}\right)^2 \cdot \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{(n+1)(n+2)\cdot 2} - \right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\cdot\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot\Gamma(n+1)\cdot 2}\right.$$

CMR

5(b) Given $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$

we have $P_0(x) = 1$ $\quad P_1(x) = x$ $\quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ $\quad P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

Using the above formulae, we have

$1 = P_0(x)$

$x = P_1(x)$, $\quad x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)$

$x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$

$x^4 = \frac{1}{35}\left[8 P_4(x) + 30x^2 - 3\right]$

$= \frac{1}{35}\left[8P_4(x) + 30\left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)\right] - 3 P_0(x)\right]$

$= \frac{8}{35} P_4(x) + \frac{10}{35} P_0(x) + \frac{20}{35} P_2(x) - \frac{3}{35} P_0(x)$

$x^4 = \frac{8}{35} P_4(x) + \frac{2}{7} P_0(x) + \frac{4}{7} P_2(x) - \frac{3}{35} P_0(x)$

Now $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$

$= \frac{8}{35} P_4(x) + \frac{2}{7} P_0(x) + \frac{4}{7} P_2(x) - \frac{3}{35} P_0(x) + 3\Big[$

$\frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)\Big] - \left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)\right] + 5P_1(x)$

$- 2 P_0(x)$

$\therefore f(x) = \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) + \frac{196}{105} P_0(x)$

$$y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \cdot \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{\Gamma(n+3)\cdot 2} - \cdots\right]$$

This can further be put in the form

$$y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma(n+1)\cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\cdot 2!}\left(\frac{x}{2}\right)^4 \cdots\right]$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\cdot r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$$

This function is called the Bessel function of the first kind of order 'n' denoted by $J_n(x)$

Thus $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$

Further the solution for $k=-n$ be denoted by $J_{-n}(x)$

Hence the general solution of the Bessel's equation is given by $y = a J_n(x) + b J_{-n}(x)$, where a, b are arbitrary constants and n is not an integer

6 (a) There are five positive numbers and seven negative numbers.

Out of these 12 numbers 5 numbers are choosen at random. This can be done in $^{12}C_5$ ways.

(i) The product is negative if

a) one number is negative

b) 3 numbers are negative.

c) all 5 numbers are negative.

$\therefore$ The Required probability is equal to

$$P(\text{getting } -ve \text{ number}) = \frac{^7C_1 \times {}^5C_4}{^{12}C_5} + \frac{^7C_3 \times {}^5C_2}{^{12}C_5} + \frac{^7C_5 \times {}^5C_0}{^{12}C_5}$$

$$= 0.5126$$

(ii) The product is positive if all are positive numbers (ie no negative numbers) (or) 2 negative numbers (or) 4 negative n

$\therefore$ The Required probability is equal to

$$P(\text{getting } +ve \text{ number}) = \frac{^7C_0 \times {}^5C_5}{^{12}C_5} + \frac{^7C_2 \times {}^5C_3}{^{12}C_5} + \frac{^7C_4 \times {}^5C_1}{^{12}C_5}$$

$$= 0.4873$$

6 (b)

Given $P(A) = \frac{1}{2}$ $P(B) = \frac{1}{3}$ $P(A \cap B) = \frac{1}{4}$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}.$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P(\overline{A}|\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} \longrightarrow (1)$$

But $P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\therefore P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4}] = \frac{5}{12}$$

$$\therefore (1) \Rightarrow P(\overline{A}|\overline{B}) = \frac{\frac{5}{12}}{\frac{2}{3}} = \frac{5}{8}.$$

Also $P(\overline{B}|\overline{A}) = \dfrac{P(\overline{B} \cap \overline{A})}{P(\overline{A})} = \dfrac{\frac{5}{12}}{\frac{1}{2}} = \dfrac{5}{6}$

$$\left[\because P(\overline{A}) = 1 - P(A) = 1 - \tfrac{1}{2} = \tfrac{1}{2}.\right]$$

6 c

Let A : Selecting a girl student

B : Selecting a boy student.

E : Selecting a student taller than 1.8 m

∴ from the data $P(A) = 0.6$

$P(B) = 0.4$

$P(E/A) = 0.01$

$P(E|B) = 0.04$

Now we need to find $P(A|E) = ?$

By Baye's theorem

$$P(A|E) = \frac{P(A)\cdot P(E/A)}{P(A)\cdot P(E/A) + P(B)\cdot P(E|B)}$$

$$= \frac{0.6 \times 0.01}{(0.6\times 0.01) + (0.4)(0.04)}$$

$$= \underline{\underline{0.2727}}$$

7 (a) Given $P(X) = \begin{cases} Kx^2 & 0 \le x \le 3 \\ 0 & \text{else where} \end{cases}$

In order that $P(X)$ may be a p.d.f, the two conditions to be satisfied are : $P(X) \ge 0$ and $\int_{-\infty}^{\infty} P(X)\,dx = 1$

The given function satisfies the first condition if $K \ge 0$.

The second condition is satisfied if

$$\int_0^3 p(x)\,dx = 1 \Rightarrow \int_0^3 Kx^2\,dx = 1$$

$$\Rightarrow K\left[\frac{x^3}{3}\right]_0^3 = 1 \Rightarrow \boxed{K = \frac{1}{9}}$$

i) $P(X \le 1) = \int_{-\infty}^{1} P(X)\,dx = \frac{1}{9} \cdot \int_0^1 x^2\,dx = \frac{1}{27}$

ii) $P(1 \le X \le 2) = \int_1^2 P(X)\,dx = \frac{1}{9} \cdot \int_1^2 x^2\,dx = \frac{7}{27}$

iii) $P(X \le 2) = \int_{-\infty}^{2} P(X)\,dx = \frac{1}{9} \cdot \int_0^2 x^2\,dx = \frac{8}{27}$

iv) $P(X > 1) = 1 - P(X \le 1) = 1 - \frac{1}{27} = \frac{26}{27}$

v) $P(X > 2) = 1 - P(X \le 2) = 1 - \frac{8}{27} = \frac{19}{27}$.

7(b) Mean and S.D of the Binomial distribution

$$\text{Mean } (\mu) = \sum_{x=0}^{n} x\, p(x) = \sum_{x=0}^{n} x \cdot {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!} p^{x} \cdot q^{n-x} = \sum_{x=0}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!} p \cdot p^{x-1} q^{n-x}$$

$$= np \cdot \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!} \cdot p^{x-1} \cdot q^{(n-1)-(x-1)}$$

$$\mu = np \cdot \sum_{x=1}^{n} {}^{(n-1)}C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(q+p)^{n-1} = np.$$

$$\boxed{\text{Mean } (\mu) = np}$$

$$\text{variance } (V) = \sum_{x=0}^{n} x^2 \cdot p(x) - \mu^2 \longrightarrow (1)$$

$$\text{Now } \sum_{x=0}^{n} x^2 \cdot p(x) = \cancel{\sum_{x=0}^{n}} \; \cancel{x^2 (p(x)) +}$$

$$= \sum_{x=0}^{n} [x(x-1) + x]\, p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, p(x) + \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x(x-1) \cdot {}^{n}C_{x}\, p^{x} q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x \cdot q^{n-x} + np$$

$$= \sum_{x=0}^{n} \frac{n(n-1)\cdot(n-2)!}{(x-2)!\,(n-x)!} p^2 \cdot p^{x-2} \cdot q^{n-x} + np$$

$$= n(n-1)\cdot p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!} p^{x-2} \cdot q^{(n-2)-(x-2)} +$$

$$= n(n-1)\, p^2 \sum_{x=2}^{n} (n-2)C_{(x-2)}\, p^{x-2} \cdot q^{(n-2)-(x-2)} + np$$

$$= n(n-1)\cdot p^2 \,(q+p)^{n-2} + np$$

$$\therefore \sum x^2 p(x) = n(n-1)p^2 + np.$$

Now ① $\Rightarrow$ $V = \{ n(n-1)\cdot p^2 + np \} - (np)^2$

$$= n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

$$V = npq$$

$$S.D\ (\sigma) = \sqrt{\text{Variance}} = \sqrt{npq}$$

7 c Let μ and σ be the mean and S.D of the normal distribution

By data we have $P(x<35) = 0.07$

$P(x<60) = 0.89$

we have standard normal variable $z = \frac{x-\mu}{\sigma}$

when $x = 35$ $\quad z = \frac{35-\mu}{\sigma} = z_1$ (say)

$x = 60$ $\quad z = \frac{60-\mu}{\sigma} = z_2$ (say)

Hence we have $P(z<z_1) = 0.07$ & $P(z<z_2) = 0.89$

ie $0.5 + \phi(z_1) = 0.07$ & $0.5 + \phi(z_2) = 0.89$

$\therefore$ $\phi(z_1) = -0.43$ & $\phi(z_2) = 0.39$

using the given data in the RHS of these we have

$\phi(z_1) = -\phi(1.4757)$ and $\phi(z_2) = \phi(1.2263)$

$\Rightarrow z_1 = -1.4757$ & $z_2 = 1.2263$

ie $\frac{35-\mu}{\sigma} = -1.4757$ & $\frac{60-\mu}{\sigma} = 1.2263$

$\mu - 1.4757\sigma = 35$ and $\mu + 1.2263\sigma = 60$

By solving we get $\mu = 48.65$

$\sigma = 9.25$

8 (a) Here the sample mean is $\bar{X} = 82$ and the sample standard deviation is $s = 18$. Further the sample size $N = 400$

$\therefore$ The confidence limits for the population mean are

$$\bar{X} \pm z_c \sigma_{\bar{X}} = \bar{X} \pm z_c \frac{s}{\sqrt{N}}$$

$$\bar{X} \pm z_c \cdot \frac{s}{\sqrt{N}} = 82 \pm z_c \cdot \frac{18}{\sqrt{400}} = 82 \pm 0.9 z_c$$

For 95% confidence level we have $z_c = 1.96$ (See table)

Accordingly, the required confidence limits are

$$82 \pm 0.9(1.96) = 82 \pm 1.764 = 83.764, 80.236$$

This means that with 95% confidence we can say that the population mean lies in the interval $(80.236, 83.764)$.

8 (b) Here the sample size is $N = 10$ (so that $\nu = N-1 = 9$)

Sample mean is $\bar{X} = 0.53$ and sample standard deviation $s = 0.03$

Let us make the hypothesis

H : μ = population mean $= 0.50$ and the machine is in proper working order.

Under this hypothesis, the t-score is

$$t = \frac{\bar{X} - \mu}{s}\sqrt{\nu} = \frac{(0.53 - 0.50)}{0.03}\sqrt{9} = 3.00$$

i) For $\nu = 9$ we find $t_{0.05} = 2.26$

Here the confidence interval is $(-t_{0.05}, t_{0.05}) = (-2.26,$

The value $t = 3$ lies outside this interval. Accordingl

we reject the hypothesis H.

ie at 0.05 level of significance, it is unlikely tha

the machine is in proper working order.

ii) For $\nu = 9$ we find $t_{0.01} = 3.25$

The confidence interval is $(-t_{0.01}, t_{0.01}) = (-3.25, 3.2$

The value $t = 3$ lies inside the interval. Accordingly

we do not reject the Hypothesis H.

ie at 0.01 level of significance, it is likely tha

the machine is in proper working order.

5 (c) According to the given hypothesis of the genetic theory the children with blood types M, MN and N are in proportions 1 : 2 : 1. This means that one child in four will have blood type M, two children in four will have blood type MN and one child in four will have blood type N.

$\therefore$ Out of 300 children, the expected number of children having blood type M is $\frac{1}{4} \times 300 = 75 = e_1$

blood type MN is $\frac{2}{4} \times 300 = 150 = e_2$

blood type N is $\frac{1}{4} \times 300 = 75 = e_3$

According to the report, these frequencies are

$f_1 = \frac{30}{100} \times 300 = 90$ $\quad f_2 = \frac{45}{100} \times 300 = 135$

$f_3 = \frac{25}{100} \times 300 = 75$

$\therefore$ The corresponding $\chi^2 = \frac{(90-75)^2}{75} + \frac{(135-150)^2}{150} + \frac{(75-75)^2}{75} = 3 + \frac{3}{2} + 0$

$= 4.5$

We note that the number of degrees of freedom is $3-1=2$. For this degree of freedom we have

$\chi^2_{0.05} = 5.99$ $\chi^2_{0.01} = 9.21$

Since $\chi^2 = 4.5$ is less than both of $\chi^2_{0.05}(2)$ and $\chi^2_{0.01}(2)$, we do not reject the hypothesis.

ie The genetic theory seems to be correct.