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10MAT41

Fourth Semester B.E. Degree Examination, June/July 2016

Engineering Mathematics – IV

Time: 3 hrs. Max. Marks:100

Note: ***1. Answer any FIVE full questions, selecting atleast TWO questions from each part.***
2. Use of statistical tables permitted.

PART – A

1 a. Using Taylor's series method, solve $y' = x + y^2$, $y(0) = 1$ at $x = 0.1, 0.2$, considering upto 4^{th} degree term. **(06 Marks)**

b. Using modified Euler's method, find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0. Take h = 0.1. Perform two iterations in each stage. **(07 Marks)**

c. Using Adams-Bashforth method, obtain the solution of $\frac{dy}{dx} = x - y^2$ at x = 0.8 given that y(0) = 0, y(0.2) = 0.0200, y(0.4) = 0.0795, y(0.6) = 0.1762. Apply the corrector formula twice. **(07 Marks)**

2 a. Employing the Picard's method, obtain the second order approximate solution of the following problem at x = 0.2, $\frac{dy}{dx} = x + yz$, $\frac{dz}{dx} = y + zx$, $y(0) = 1$, $z(0) = -1$. **(06 Marks)**

b. Solve $\frac{dy}{dx} = 1 + xz$ and $\frac{dz}{dx} = -xy$ for x = 0.3 by applying Runge Kutta method given y(0) = 0 and z(0) = 1. Take h = 0.3. **(07 Marks)**

c. Using the Milne's method, obtain an approximate solution at the point x = 0.4 of the problem $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$ given that y(0) = 1, y(0.1) = 1.03995, y(0.2) = 1.138036, y(0.3) = 1.29865, $y'(0) = 0.1$, $y'(0.1) = 0.6955$, $y'(0.2) = 1.258$, $y'(0.3) = 1.873$. **(07 Marks)**

3 a. Define an analytic function and obtain Cauchy-Riemann equations in polar form. **(06 Marks)**

b. Show that $u = e^{2x}$ (x cos2y – y sin2y) is a harmonic function and determine the corresponding analytic function. **(07 Marks)**

c. If f(z) is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. **(07 Marks)**

4 a. Evaluate using Cauchy's integral formula $\int_c \frac{\cos \pi z}{z^2 - 1} dz$ around a rectangle with vertices $2 \pm i$, $-2 \pm i$. **(06 Marks)**

b. Find the bilinear transformation which maps 1, i, -1 to 2, i, -2 respectively. Also find the fixed points of the transformation. **(07 Marks)**

c. Discuss the conformal transformation of $w = z^2$. **(07 Marks)**

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

PART – B

5 a. Reduce the differential equation:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(k^2x^2-n^2)y=0$$ into Bessel form and write the complete solution in terms of $J_n(x)$ and $J_{-n}(x)$. (06 Marks)

b. Express $f(x)=x^3+2x^2-x-3$ in terms of Legendre polynomials. (07 Marks)

c. If α and β are the roots of $J_n(x)=0$ then prove that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx=\begin{cases}0, & \alpha\neq\beta\\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha=\beta\end{cases}$$ (07 Marks)

6 a. The probability that sushil will solve a problem is 1/4 and the probability that Ram will solve it is 2/3. If sushil and Ram work independently, what is the probability that the problem will be solved by (i) both of them; (ii) at least one of them? (06 Marks)

b. A committee consists of 9 students two of which are from first year, three from second year and four from third year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes; (ii) two belong to the same class and third to the different class; (iii) the three belong to the same class? (07 Marks)

c. The contents of three urns are: 1 white, 2 red, 3 green balls, 2 white, 1 red, 1 green balls and 4 white, 5 red, 3 green balls. Two balls are drawn from an urn chosen at random. These are found to be one white and one green. Find the probability that the balls so drawn came from the third urn. (07 Marks)

7 a. The probability mass function of a variate X is

x	0	1	2	3	4	5	6
p(x)	k	3k	5k	7k	9k	11k	13k

i) Find k

ii) Find $p(x<4)$, $p(x\geq 5)$, $p(3<x\leq 6)$, $p(x>1)$

iii) Find the mean. (06 Marks)

b. Derive the mean and variance of Poisson distribution. (07 Marks)

c. The mean height of 500 students is 151cm and the standard deviation is 15cm. Assuming that the heights are normally distributed, find how many students heights i) lie between 120 and 155cm. ii) more than 155cm. [Given A(2.07) = 0.4808 and A (0.27) = 0.1064, where A(z) is the area under the standard normal curve from 0 to z > 0]. (07 Marks)

8 a. The means of simple samples of sizes 1000 and 2000 are 67.5 and 68.0cm respectively. Can the samples be regarded as drawn from the same population of S.D 2.5cm [Given $z_{0.05}=1.96$]. (06 Marks)

b. A random sample of 10 boys had the following I.Q: 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do these data support the assumption of a population mean I.Q of 100? [Given $t_{0.05}$ for 9d.f = 2.26]. (07 Marks)

c. The following table gives the number of aircraft accidents that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week.

Days :	Sun	Mon	Tue	Wed	Thur	Fri	Sat	Total
No. of accidents :	14	16	8	12	11	9	14	84

[Given $\psi^2_{0.05}$ 6d.f = 12.59] (07 Marks)

* * * * *

Engineering Mathematics - IV

- June / July 2016

① (a)

Soln: $y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots$

$$y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) \ldots$$

$y'(0) = 1,\quad y'' = 1 + 2yy' \quad y''(0) = 1 + 2(1)(1) = 3$

$y''' = 2yy'' + 2(y')^2 = 2 \cdot 3 + 2 = 8$

$y^{iv} = 2yy''' + 2y''y' + 2 \cdot 2y'y''$

$= 2 \cdot 8 + 2 \cdot 3 + 2 \cdot 2 \cdot 3 = 34$

$$y(x) = 1 + x \cdot 1 + \frac{x^2}{2} \cdot 3 + \frac{x^3}{6} \cdot 8 + \frac{x^4}{24} \cdot 34$$

$$y(x) = 1 + x + \frac{3}{2} x^2 + \frac{4}{3} x^3 + \frac{17}{12} x^4$$

$y(0.1) = 1.11647$

$y(0.2) = 1.27294$

(b) $y' = x + y$

I stage - $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 1 + 0.1 = 1.1$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

$$= 1 + 0.05\left[1 + (0.1 + 1.1)\right]$$

$$= 1.11$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[1 + (0.1 + 1.11)\right]$$

$$= 1.1105$$

$$\boxed{y(0.1) = 1.1105}$$

II stage : $x_0 = 0.1$, $y_0 = 1.1105$

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$= 1.1105 + 0.1(0.1 + 1.1105)$$

$$= 1.2315$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1.1105 + 0.05\left[1.2105 + (0.2 + 1.2315)\right]$$

$$= 1.2426$$

$$y_1^{(2)} = 1.2426$$

$$\boxed{y(0.2) = 1.2444}$$

(c) $\frac{dy}{dx} = x - y^2$

$y_0' = 0$, $y_1' = 0.1996$, $y_2' = 0.3937$, $y_3' = 0.5689$

$$y_4^{(p)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_4^{(p)} = 0.3049$$

$$y_4' = 0.707$$

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 0.3046$$

$$y_4' = 0.7072$$

$$y_4^{(c)} = 0.3046$$

$$\boxed{y(0.8) = 0.3046}$$

(2) (a)

$$\frac{dy}{dx} = x + yz, \quad \frac{dz}{dx} = y + zx,$$

$$y(0) = 1, \quad z(0) = -1$$

$$y = y_0 + \int_{x_0}^{x} f(x, y, z)\,dx$$

$$y = 1 + \int_0^x (x + yz)\,dx$$

$$z = z_0 + \int_0^x g(x, y, z)\,dx$$

$$= -1 + \int_0^x (y + zx)\,dx$$

$$y_1 = 1 + \int_0^x (x + y_0 z_0)\,dx$$

$$= 1 + \int_0^x x + 1(-1)\,dx = \int_0^x (x-1)\,dx + 1$$

$$= 1 + \left(\frac{x^2}{2} - x\right)_0^x \qquad z_1 = -1 + \int_0^x (1-x)\,dx$$

$$= 1 - x + \frac{x^2}{2} \qquad = -1 + x - \frac{x^2}{2}$$

$$y_2 = 1 + \int_0^x [x + (y_1 z_1)]\,dx$$

$$= 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2}\right)\left(1 - x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2}\right)^2\right] dx$$

$$= 1 + \int_0^x \left\{ x - \left[(1-x)^2 + \left(\frac{x^2}{2}\right)^2 + 2(1-x)\frac{x^2}{2}\right]\right\} dx$$

$$= 1 + \int_0^x x - \left(1 + x^2 - 2x + \frac{x^4}{4} + x^2 - x^3\right) dx$$

$$= 1 + \int_0^x \left(-1 + 3x - 2x^2 + x^3 - \frac{x^4}{4}\right) dx$$

$$y_2 = 1 - x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{20}$$

$$= 0.8 + (1.5)(0.04) - 0.66\,(0.008) + 0.0004$$
$$- 0.000016$$

$$= 0.7387$$

$$z_1 = -1 + \int_0^x (y_1 + z_1 x)\,dx$$

$$= -1 + \int_0^x \left(1 - x + \frac{x^2}{2} - x + x^2 - \frac{x^3}{2}\right) dx$$

$$= -1 + \int_0^x \left(1 - 2x + \frac{3}{2}x^2 - \frac{x^3}{2}\right) dx$$

$$= -1 + x - \frac{2x^2}{2} + \frac{3}{2}\frac{x^3}{3} - \frac{x^4}{8}$$

$$z_2 = -1 + x - x^2 + \frac{x^3}{2} - \frac{x^4}{8}$$

$$= 1.16 - 0.004 - 0.0002$$

when $x = 0.2$

$\boxed{y_2(0.2) = 0.8551}$, $\boxed{z_2(0.2) = -0.8362}$

(b) $y' = 1 + zx$, $z' = -xy$ $y(0) = 0$

$z(0) = 1$

$h = 0.3$

$x_0 = 0,\ y_0 = 0,\ z_0 = 1$

$k_1 = 0.3$ $l_1 = 0$

$k_2 = 0.345$ $l_2 = -0.00675$

$k_3 = 0.3448$ $l_3 = -0.0077$

$k_4 = 0.3893$ $l_4 = -0.03104$

$\boxed{y(0.3) = 0.3448}$ $\boxed{z(0.3) = 0.9899}$

(c) $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$ $y(0) = 1,\ y'(0) = 0.1$

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.03995$	$y_2 = 1.1396$	$y_3 = 1.29865$
$y' = z$	$z_0 = 0.1$	$z_1 = 0.6955$	$z_2 = 1.2582$	$z_3 = 1.7524$
$y'' = z'$	$z_0' = 6$	$z_1' = 6.0311$	$z_2' = 6.0827$	$z_3' = 6.2147$

By milne's predictor formula we have

$$y_4^{(p)} = y_0 + \frac{4h}{3}\{2z_1 - z_2 + 2z_3\} = 1.485$$

$$z_4^{(p)} = z_0 + \frac{4h}{3}\{2z_1' - z_2' + 2z_3'\} = 2.5545$$

Now $z_4' = 6y_4^{(p)} - 3z_4^{p} \times 0.4$

$= \underline{5.8446}$

By milne's corrector formulae we have

$$y_4^{(c)} = y_2 + \frac{h}{3}\{z_2 + 4z_3 + z_4\}$$

$$= y_2 + \frac{h}{3}\{z_2 + 4z_3 + z_4^{(p)}\}$$

$$= 1.5003$$

$$z_4^{(c)} = z_2 + \frac{h}{3}\{z_2' + 4z_3' + z_4'\}$$

$$= 2.8844.$$

Applying corrector formula again we get

$$y_4^{(c)} = y_2 + \frac{h}{3}\{z_2 + 4z_3 + z_4\}$$

$$= y_2 + \frac{h}{2}\{z_2 + 4z_3 + z_4^{(c)}\} = 1.498$$

$y(0.4) = 1.498$

(3) (a) A complex valued function $w = f(z)$ is said to be analytic at a point $z = z_0$ if

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists and

is unique at z_0 and in the neighbourhood of z_0. Further $f(z)$ is called analytic in a region if it is analytic every point of the region.

C-R equations in polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: let (r, θ) be the polar co-ordinates of the point whose cartesian co-ordinates are (x, y) Then we have

$x = r\cos\theta \quad y = r\sin\theta$

$z = x + iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

$\therefore u + iv = f(z) = f(re^{i\theta}) \quad \text{— (1)}$

Diff (1) partially w.r.t r, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta}$$

Diff (1) partially w.r.t θ we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, r.(ie^{i\theta})$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts weget

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

or $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(b) $$u = e^{2x}\left[x\cos 2y - y\sin 2y\right]$$

$$u_x = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)\,2e^{2x}$$

$$= e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$$

$$u_{xx} = e^{2x}(2\cos 2y) + (\cos 2y + 2x\cos 2y - 2y\sin 2y)\,2e^{2x}$$

$$u_y = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

$$u_{yy} = e^{2x}(-4x\cos 2y + 4y\sin 2y - 2\cos 2y - 2\cos 2y)$$

Consider $u_{xx} + u_{yy}$

$$= e^{2x}\left[4\cos 2y + 4x\cos 2y - 4y\sin y - 4x\cos 2y + 4y\sin 2y - 4\cos 2y\right]$$

$$= 0$$

Since u satisfies Laplace equation

$u_{xx} + u_{yy} = 0$

$\therefore$ u is harmonic.

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$

$u_y = -e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y)$

$f'(z) = u_x + iv_x$

$= u_x - iu_y$ (using C-R equation $v_x = -u_y$)

$f'(z) = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin y)$
$+ i e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y)$

putting $x = z$, $y = 0$ weget

$f'(z) = e^{2z}(1+2z)$

$\therefore f(z) = \int e^{2z}(1+2z)\,dz$

Integrating by parts weget

$$f(z) = (1+2z)\frac{e^{2z}}{2} - (2)\frac{e^{2z}}{4} + c$$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$$

$\therefore f(z) = u + iv = (x+iy)e^{2(x+iy)}$

$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$

$\therefore f(z) = e^{2x}(x\cos 2y - y\sin 2y) + ie^{2x}(x\sin 2y + y\cos 2y)$

(c) Let $f(z) = u+iv$ be analytic

$\therefore$ $|f(z)| = \sqrt{u^2+v^2}$

or $|f(z)|^2 = u^2+v^2 = \phi$

To prove that $\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}\phi = 4|f'(z)|^2$

i.e to prove $\phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

consider $\phi = u^2+v^2$

Diff partially w.r.t x we get

$\phi_x = 2uu_x + 2vv_x = 2[uu_x + vv_x]$

$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$ —(1)

Similarly diff $\phi = u+iv$ twice w.r.t y partially we get

$\phi_{yy} = 2[uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$ —(2)

Adding (1) and (2) we get

$$\phi_{xx} + \phi_{yy} = 2[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2]$$

Since $f(z) = u+iv$ is analytic, u and v are harmonic

$\therefore$ $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

Further by C-R equations we have

$v_y = u_x$ and $u_y = -v_x$

using these results

$$\phi_{xx} + \phi_{yy} = 2[u(0) + v(0) + u_x^2 + v_x^2 + (-v_x)^2 + u_x^2]$$

$$\phi_{xx} + \phi_{yy} = 2[2u_x^2 + 2v_x^2]$$

$$= 4[u_x^2 + v_x^2]$$

we have $f(z) = u + iv$

$$\Rightarrow f'(z) = u_x + iv_x$$

$$\therefore |f'(z)| = \sqrt{u_x^2 + v_x^2}$$

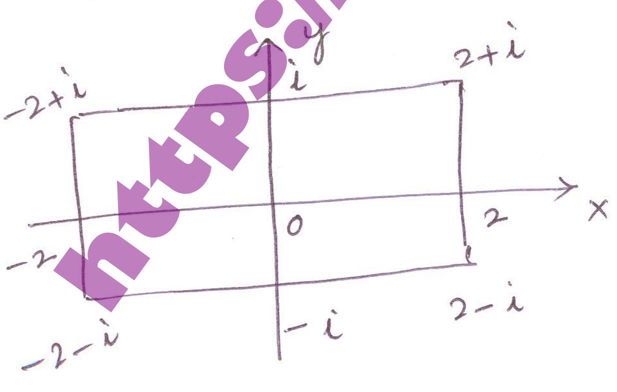
$$|f'(z)|^2 = u_x^2 + v_x^2$$

with this,

$$\underline{\underline{\phi_{xx} + \phi_{yy} = 4|f'(z)|^2}}$$

(4) (a) $\displaystyle\int \frac{\cos \pi z}{z^2 - 1} \, dz$

here $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and two singular points $a = 1$ and $a = -1$ lies inside this rectangle.



$$\therefore \int \frac{\cos \pi z}{z^2-1}\, dz = \int_C \frac{\cos \pi z}{(z-1)(z+1)}\, dz$$

$$= \frac{1}{2} \int_C \left(\frac{1}{z-1} - \frac{1}{z+1} \right) \cos \pi z \, dz$$

$$= \frac{1}{2} \left\{ \int_C \frac{\cos \pi z}{z-1}\, dz - \int \frac{\cos \pi z}{z+1}\, dz \right\}$$

$$= \frac{1}{2} \left\{ 2\pi i\, f(1) - 2\pi i\, f(-1) \right\} \quad -(1)$$

($\because$ using cauchy's Integral formula).

here $f(z) = \cos \pi z$

$f(1) = \cos \pi = -1$

$f(-1) = \cos(-\pi) = \cos \pi = -1$

$\therefore$ (1) reduces to

$$\int_C \frac{\cos \pi z}{z^2-1}\, dz = \frac{1}{2} \left\{ 2\pi i\, (-1) - 2\pi i (-1) \right\}$$

$$= \frac{1}{2} \left\{ -2\pi i + 2\pi i \right\} = 0$$

$$\therefore \int \frac{\cos \pi z}{z^2-1}\, dz = 0 .$$

(b) let $w = \dfrac{az+b}{cz+d}$ be the required BLT

when $z=1$, $w=2$ we get $2 = \dfrac{a+b}{c+d}$

$a + b - 2c - 2d = 0 \quad -(1)$

when $z = i$, $w = i$ weget $i = \dfrac{ia+b}{ic+d}$

$\Rightarrow \quad ia+b+c-di=0 \qquad -(2)$

when $z=-1$, $w=-2$ we get $-2=\dfrac{-a+b}{-c+d}$

$\Rightarrow \quad -a+b-2c+2d=0 \qquad -(3)$

Adding (1) and (3)

$2b-4c=0 \Rightarrow b-2c=0$

which can be written as

$b-2c+0.d=0 \qquad -(4)$

considering $(2)+(3)i$ we get

$(1+i)b+(1-2i)c+id=0 \qquad -(5)$

solving (4) and (5) by the rule of cross multiplication we get

$$\frac{b}{-2i-0}=\frac{-c}{i-0}=\frac{d}{(1-2i)+2(1+i)}$$

$$\Rightarrow \frac{b}{-2i}=\frac{c}{-i}=\frac{d}{3}$$

$\therefore \quad b=-2i, \quad c=-i \quad d=3$

Substituting these in (1) we get

$\therefore \quad a-2i+2i-6=0$

$\Rightarrow a=6$

$w=\dfrac{az+b}{cz+d}, \quad \Rightarrow w=\dfrac{6z-2i}{-iz+3}$ which is the required BCT.

(c) $w = z^2$

let $w = f(z) = z^2$ — (1)

$\Rightarrow f'(z) = 2z$

Since $f'(z) \neq 0$, $\forall z \neq 0$ $\therefore$ $f(z)$ is conformal to all the points except at $z = 0$

let $z = x+iy$ and $w = u+iv$ in (1)

$u+iv = (x+iy)^2 = (x^2-y^2) + i\,2xy$

$\Rightarrow u = x^2-y^2$ and $v = 2xy$. — (2)

<u>case (i)</u> consider a st-line parallel to y axis in z plane whose equation is $x = a$ where a is any real constant

from (2) $u = a^2-y^2$, $v = 2ay$

$\Rightarrow y^2 = a^2-u$, $v^2 = 4a^2y^2$

$\therefore v^2 = 4a^2(a^2-u)$

$\therefore v^2 = -4a^2(u-a^2)$

which is the equation of the parabola with $(a^2, 0)$ as vertex and focus at the origin.

$\therefore w = z^2$ transforms a st. line parallel to y-axis in z plane to parabola with negative u-axis as its axis.

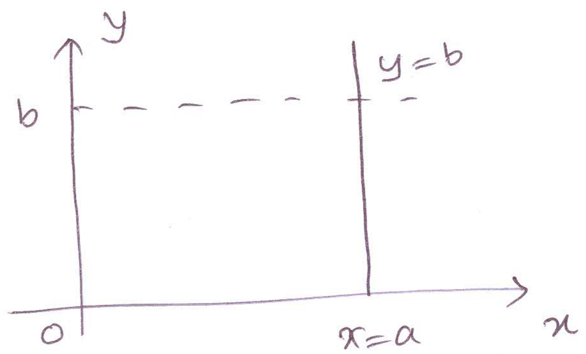
<u>case (ii)</u> consider a st-line parallel to x-axis in z plane whose equation is $y = b$, where b is any real constant.

from (2), $u = x^2 - b^2$, $v = 2xb$

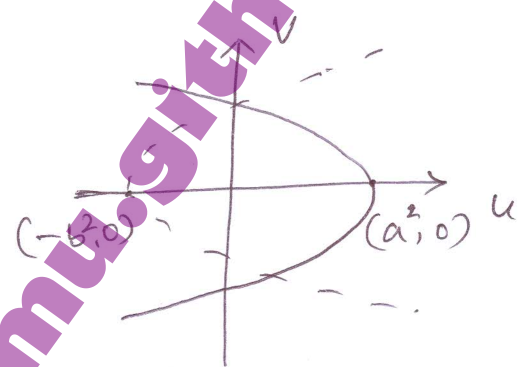
$$\Rightarrow x^2 = u + b^2, \quad v^2 = 4x^2b^2$$

$$\therefore v^2 = 4b^2(u + b^2)$$

which is the equation of the parabola with $(-b^2, 0)$ as vertex and positive u-axis as its axis.



(z-plane)



(w-plane)

<u>case (iii)</u> let $z = re^{i\theta} \Rightarrow w = z^2 = r^2 e^{i2\theta}$

$w = Re^{i\phi}$ where $R = r^2$ and $\phi = 2\theta$

clearly as θ varies from 0 to $\frac{\pi}{2}$ in z plane ϕ varies from 0 to π in w plane.

Similarly as θ varies from 0 to π in z plane ϕ varies from 0 to 2π in w-plane.

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PART B

(5) (a) Put $\lambda x = t \Rightarrow \lambda = \frac{t}{x}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \lambda$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda \cdot \frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(\lambda \cdot \frac{dy}{dt}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\lambda \cdot \frac{dy}{dt}\right)\lambda$$

$$\frac{d^2y}{dx^2} = \lambda^2 \frac{d^2}{dt^2}$$

Substituting the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ along with $x = \frac{t}{\lambda}$ in ~~x^2~~ $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = 0$

we get

$$\frac{t^2}{\lambda^2} \cdot \lambda^2 \frac{d^2y}{dt^2} + \frac{t}{\lambda} \cdot \lambda \frac{dy}{dt} + (t^2 - n^2)y = 0$$

$$\Rightarrow t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$$

This is in the form of Bessel's differential equation whose solution is given by

$$y = aJ_n(t) + bJ_{-n}(t)$$

Thus $y = aJ_n(kt) + bJ_{-n}(kt)$ is the solution.

(b) $f(x) = x^3 + 2x^2 - x - 3$

$P_0(x) = 1 \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_1(x) = x$

$\Rightarrow \quad x^2 = \frac{1}{3}(2P_2(x) + P_0(x))$

$P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad \Rightarrow \quad x^3 = \frac{1}{5}(2P_3(x) + 3P_1(x))$

substituting the values of x, x^2, x^3 in $f(x)$

we get

$$f(x) = \frac{1}{5}\{2P_3(x) + 3P_1(x)\} + \frac{2}{3}\{2P_2(x) + P_0(x)\} - P_1(x) - 3P_0(x)$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) - \frac{7}{3}P_0(x).$$

(c) Proof: we know that $J_n(\lambda x)$ is the solution of the equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0$

If $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ then the associated differential equations are

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad -(1)$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad -(2)$$

Multiplying (1) by $\frac{v}{x}$ and 2 by $\frac{u}{x}$ we get

$$xvu'' + vu' + \alpha^2uvx - \frac{n^2uv}{x} = 0$$

on substracting we get

$$x(vu'' - uv'') + (vu' - uv') + (\alpha^2 - \beta^2) uvx = 0$$

i.e, $\frac{d}{dx}\{x(vu' - uv')\} = (\beta^2 - \alpha^2) uvx$

Integrating both sides w.r.t x between 0 and 1 we get

$$xvu'' + vu' + \alpha^2 uvx - \frac{n^2 uv}{x} = 0$$

and $$xuv'' + uv' + \beta^2 uvx - \frac{n^2 uv}{x} = 0$$

on subtracting we get

$$x(vu'' - uv'') + (vu' - uv') + (\alpha^2 - \beta^2) uvx = 0$$

i.e $\frac{d}{dx}\{x(vu' - uv')\} = (\beta^2 - \alpha^2) uvx$

Integrating both sides w.r.t x between 0 and 1

we get $$\{x(vu' - uv')\}_{x=0}^{1} = (\beta^2 - \alpha^2)\int_{x=0}^{1} xuv\,dx$$

$$(vu' - uv')_{x=1} - 0 = (\beta^2 - \alpha^2)\int_{x=0}^{1} xuv\,dx \quad -(3)$$

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$

$\Rightarrow u' = \alpha J_n'(\alpha x)$ $\quad v' = \beta J_n'(\beta x)$

As a consequence of these (3) reduces to

$$\{J_n(\beta x)\alpha J_n'(\alpha x) - J_n(\alpha x)\beta J_n'(\beta x)\}_{x=1} = (\beta^2 - \alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

hence $$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{\beta^2 - \alpha^2}\{\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\} \quad -(4)$$

Since α and β are distinct roots of $J_n(x)=0$

$\Rightarrow$ $J_n(\alpha)=0$ and $J_n(\beta)=0$

with this result RHS of (4) becomes zero provided $(\beta^2-\alpha^2)\neq 0$ or $\beta\neq\alpha$. Thus we have proved that if $\alpha\neq\beta$

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0 \quad -(5)$$

Now we shall discuss the case when $\alpha=\beta$

The RHS of (4) becomes an indeterminate form of the type $\frac{0}{0}$ when $\alpha=\beta$. we shall evaluate by taking limits on both sides as $\beta\to\alpha$ keeping α fixed by applying L' Hospital's rule.

i.e $\lim\limits_{\beta\to\alpha}\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$

$$= \lim_{\beta\to\alpha}\frac{1}{\beta^2-\alpha^2}\left\{\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right\}$$

Since α is fixed we must have $J_n(\alpha)=0$

($\because$ α is the root of the equation $J_n(x)=0$)

$\therefore$ $\lim\limits_{\beta\to\alpha}\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$

$$= \lim_{\beta\to\alpha}\frac{1}{(\beta^2-\alpha^2)}\left\{\alpha J_n(\beta) J_n'(\alpha)\right\}$$

$$= \lim_{\beta\to\alpha}\frac{1}{2\beta}\left\{\alpha J_n'(\beta) J_n'(\alpha)\right\}$$

$$= \frac{1}{2\alpha}\,\alpha\, J_n'(\alpha) J_n'(\alpha)$$

$$\Rightarrow \int_0^1 x\{J_n(\alpha x)\}^2 dx = \frac{1}{2}\{J_n'(\alpha)\}^2 \quad \text{— (6)}$$

Further from Recurrence relations we have

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$\therefore \quad J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha)$$

Since $J_n(\alpha)=0$ we get

$$J_n'(\alpha) = -J_{n+1}(\alpha)$$

$\therefore$ (6) reduces to

$$\int_0^1 x\{J_n(\alpha x)\}^2 dx = \frac{1}{2}\{J_{n+1}(\alpha)\}^2$$

The above result is known as Lommel Integral formula. Thus we have proved that

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2}[J_{n+1}(\alpha)]^2 & \text{if } \alpha = \beta \end{cases}$$

which is the required orthogonal property of Bessel's functions.

(6) (a)

$$P(S) = \frac{1}{4} \qquad P(\bar{S}) = \frac{3}{4}$$

$$P(R) = \frac{2}{3} \qquad P(\bar{R}) = \frac{1}{3}$$

$$P(S \cap R) = P(S)\,P(\bar{R}) = \frac{1}{4} \times \frac{2}{3} = \frac{2}{12} = \frac{1}{6}$$

$$P(\text{Atleast one}) = 1 - [P(\bar{S}).P(\bar{R})]$$

$$= 1 - \frac{3}{4} \times \frac{1}{3} = \frac{3}{4}$$

(b) First year - 9 students

Second year - 3 students

Third year - 4 students

(i) different classes $= \dfrac{2C_1 \times 3C_1 \times 4C_1}{9C_3}$

$= \underline{0.2857}$

(ii) 2 same class and 1 different class

$= \dfrac{2C_2 \times 7C_1}{9C_3} + \dfrac{3C_2 \times 6C_1}{9C_3} + \dfrac{4C_2 \times 5C_1}{9C_3}$

$= \underline{0.6547}$

(iii) 3 same classes $- \dfrac{3C_3}{9C_3} + \dfrac{4C_3}{9C_3}$

$= \dfrac{5}{84} = \underline{0.0595}$

(c) First urn - 1W, 2R, 3G   third urn - 4W, 5R, 3G

second urn - 2W, 1R, 1G $P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$

$P(E/U_1) = \dfrac{1C_1 \times 3C_1}{6C_2} = \dfrac{3}{15} = \underline{\underline{0.2}}$

$P(E|U_2) = \dfrac{2C_1 \times 1C_1}{4C_2} = \dfrac{2}{6} = \dfrac{1}{3} = \underline{\underline{0.333}}$

$P(E|U_3) = \dfrac{4C_1 \times 3C_1}{12C_2} = \dfrac{12}{66} = \underline{\underline{0.1818}}$

$$P(u_3|E) = \frac{P(u_3)\,P(E|u_3)}{P(u_1)\,P(E|u_1) + P(u_2)\,P(E|u_2) + P(u_3)\,P(E|u_3)}$$

$$= \frac{\frac{2}{11}\cdot\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{11}} = \frac{\frac{2}{33}}{\frac{1}{15}+\frac{1}{9}+\frac{2}{33}}$$

$$= \underline{\underline{0.2543}}.$$

(7)

X	0	1	2	3	4	5	6
P(X)	K	3K	5K	7K	9K	11K	13K

we have $\sum P(X) = 1$

$\Rightarrow K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$49K = 1$

$\Rightarrow K = \frac{1}{49}$

$P(X<4) = P(X=0,1,2,3)$

$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49}$

$= \frac{16}{49} = \underline{\underline{0.3265}}$

$P(X\geq 5) = \frac{24}{49} = \underline{\underline{0.4897}}$

$P(3 < X \leq 6) = \frac{33}{49} = \underline{\underline{0.6734}}$

$P(X>1) = P(X=2,3,4,5\ 6)$

$= \frac{45}{49} = \underline{\underline{0.9183}}$

$$\text{Mean} = \sum x p(x) = \underline{\underline{4.1428}} = \frac{203}{49}$$

(b) Let x be a Poisson variate with parameter m then

$$p(x) = \frac{e^{-m} m^{x}}{r!} \quad , \quad r = 0, 1, 2, 3, \ldots, \infty$$

we have

$$\text{Mean} = E(x) = \sum x\, p(x)$$

$$= \sum_{r=0}^{\infty} r \frac{e^{-m} m^{r}}{r!}$$

$$= e^{-m} \sum_{r=1}^{\infty} \frac{m \cdot m^{r-1}}{(r-1)!}$$

$$= e^{-m} \cdot m \left\{ 1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right\}$$

$$= m e^{-m} e^{m} = m e^{0} = m.$$

$\underline{\text{Mean} = E(x) = m}$

consider

$$\sum r(r-1)\, p(r)$$

$$= \sum_{r=0}^{\infty} r(r-1) \frac{e^{-m} m^{r}}{r!}$$

$$= m^2 e^{-m} \sum_{r=2}^{\infty} \frac{m^{r-2}}{(r-2)!}$$

$$= m^2 e^{-m}\left\{1+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+\cdots\right\}$$

$$= m^2 e^{-m} e^{m} = m^2 e^0 = \underline{m^2}$$

$\underline{\text{Mean} = \text{Variance} = m}$

Standard $\underline{\text{deviation} = \sqrt{\text{variance}} = \sqrt{m}}$

(c) $Z = \dfrac{X-\mu}{\sigma} = \dfrac{x-151}{15}$

$$P(120 < x < 155) = P(-2.07 < Z < 0.2666)$$

$$= \phi(2.07) + \phi(0.2666)$$

$$= 0.4808 + 0.1064$$

$$= \underline{\underline{0.5872}}$$

$$P(X > 155) = P(Z > 0.2666)$$

$$= 0.5 - \phi(0.2666)$$

$$= 0.5 - 0.1064$$

$$= \underline{\underline{0.3936}}$$

between 120 and 155

No. of students $= 0.5872 \times 500$

$= 294$

greater than 155, No. of

students $= 0.3936 \times 500$

$= 197.$

(8) (a)

$\bar{x}_1 = 67.5$, $\bar{x}_2 = 68$ cm $n_1 = 1000$, $n_2 = 2000$

$\sigma = 2.5$

$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$

under H_0, $Z = \dfrac{|\bar{x}_1 - \bar{x}_2|}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \underline{5.16}$

$$\underline{Z_{0.05} = 1.96}$$

Conclusion : ∵ calculated value is greater than the table value reject H_0. In other words the sample cannot be regarded as drawn from the same population.

(5) $H_0 : \mu = 100$, $H_1 : \mu \neq 100$

$\bar{x} = \dfrac{\sum x}{n} = 97.2$ $n = 10$

$\sigma_{\bar{x}}^2 = \dfrac{\sum (x-\bar{x})^2}{n} = 183.36$

$$\underline{\sigma_x = 13.54}$$

under H_0, $t = \dfrac{|\bar{x} - \mu|}{\sigma_x/\sqrt{n-1}} = 0.62$.

$t_{0.05}$ for 9 d.f $= 2.26$

Conclusion : Calculated value is less than table we have to accept H_0. In other words the population mean I.Q $= 100$.

(c) $H_0 : P = \frac{1}{7}$ $H_1 : P \neq \frac{1}{7}$ $(i = 1, 2, \ldots, 7)$

$$f_i = NP_i = \frac{84}{7} = 12$$

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 4.35$$

$\chi^2 = 0.05$ for 6 df $= 12.59$

Conclusion : $\because$ calculated value is less than table value we have to accept H_0. In other words the accidents are uniformly distributed over the week.