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By K B Hemanth Raj

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CBCS Scheme

USN | | | | | | | | | | | **15MAT41**

Fourth Semester B.E. Degree Examination, June/July 2017
Engineering Mathematics–IV

Time: 3 hrs. Max. Marks: 80

Note: ***1. Answer FIVE full questions, choosing one full question from each module.***
2. Use of statistical tables are permitted.

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

Module-1

1 a. Find by Taylor's series method the value of y at x = 0.1 from $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$ (upto 4^{th} degree term). **(05 Marks)**

b. The following table gives the solution of $5xy' + y^2 - 2 = 0$. Find the value of y at x = 4.5 using Milne's predictor and corrector formulae. **(05 Marks)**

x	4	4.1	4.2	4.3	4.4
y	1	1.0049	1.0097	1.0143	1.0187

c. Using Euler's modified method. Obtain a solution of the equation $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$, with initial conditions y = 1 at x = 0, for the range $0 \le x \le 0.4$ in steps of 0.2. **(06 Marks)**

OR

2 a. Using modified Euler's method find y(20.2) and y(20.4) given that $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ with y(20) = 5 taking h = 0.2. **(05 Marks)**

b. Given $\frac{dy}{dx} = x^2(1+y)$ and y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979. Evaluate y(1.4) by Adams-Bashforth method. **(05 Marks)**

c. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with y(0) = 1 at x = 0.2 by taking h = 0.2 **(06 Marks)**

Module-2

3 a. Obtain the solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data: **(05 Marks)**

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y′	2	2.3178	2.6725	3.0657

b. Express $f(x) = 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials. **(05 Marks)**

c. Obtain the series solution of Bessel's differential equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ **(06 Marks)**

OR

4 a. By Runge-Kutta method solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ for x = 0.2. Correct to four decimal places using the initial conditions y = 1 and y′ = 0 at x = 0, h = 0.2. **(05 Marks)**

b. Prove that $J_{+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ **(05 Marks)**

c. Prove the Rodrigues formula,

$$\rho_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$ **(06 Marks)**

Module-3

5 a. State and prove Cauchy's-Riemann equation in polar form. **(05 Marks)**

b. Discuss the transformation $W = e^z$ **(05 Marks)**

c. Evaluate $\int_C \left\{ \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2 (z-2)} \right\} dz$ using Cauchy's residue theorem where 'C' is the circle $|z| = 3$ **(06 Marks)**

OR

6 a. Find the analytic function whose real part is, $\frac{\sin 2x}{\cosh 2y - \cos 2x}$. **(05 Marks)**

b. State and prove Cauchy's integral formula. **(05 Marks)**

c. Find the bilinear transformation which maps z = ∞, i, 0 into ω = -1, -i, 1. Also find the fixed points of the transformation. **(06 Marks)**

Module-4

7 a. Find the mean and standard deviation of Poisson distribution. **(05 Marks)**

b. In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and S.D of 60 hours. Estimate the number of bulbs likely to burn for,

(i) more than 2150 hours.
(ii) less than 1950 hours
(iii) more than 1920 hours and less than 2160 hours.

[A(1.833) = 0.4664, A(1.5) = 0.4332, A(2) = 0.4772] **(05 Marks)**

c. The joint probability distribution of two random variables x and y is as follows:

x/y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Determine:

(i) Marginal distribution of x and y.
(ii) Covariance of x and y
(iii) Correlaiton of x and y. **(06 Marks)**

OR

8 a. The probability that a pen manufactured by a factory be defective is $\frac{1}{10}$. If 12 such pens are manufactured what is the probability that, (i) Exactly 2 are defective (ii) at least 2 are defective (iii) none of them are defective. **(05 Marks)**

b. Derive the expressions for mean and variance of binomial distribution. **(05 Marks)**

c. A random variable X take the values -3, -2, -1, 0, 1, 2, 3 such that $P(x = 0) = P(x < 0)$ and $P(x = -3) = P(x = -2) = P(x = -1) = P(x = 1) = P(x = 2) = P(x = 3)$. Find the probability distribution. **(06 Marks)**

Module-5

9 a. In 324 throws of a six faced 'die' an odd number turned up 181 times. Is it reasonable to think that the 'die' is an unbiased one? **(05 Marks)**

b. Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results:

Horse A:	28	30	32	33	33	29	34
Horse B:	29	30	30	24	27	29	

Test whether you can discriminate between the two horses.($t_{0.05}$=2.2 and $t_{0.02}$=2.72 for 11 d.f) **(05 Marks)**

c. Find the unique fixed probability vector for the regular stochastic matrix, $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ **(06 Marks)**

OR

10 a. Define the terms: (i) Null hypothesis (ii) Type–I and Type–II error (iii) Confidence limits. **(05 Marks)**

b. Prove that the Markov chain whose t.p.m $P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ is irreducible. Find the corresponding stationary probability vector. **(05 Marks)**

c. Three boys A, B, C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball find the probabilities that after three throws (i) A has the ball (ii) B has the ball. (iii) C has the ball. **(06 Marks)**

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CBCS Scheme

Fourth Semester B.E. Degree Examination, June/July 2017

Engineering Mathematics - IV

Time: 3 hrs. 15MAT41

Module - 1

1) a) Find by Taylor's Series method the value of y at $x = 0.1$ from $\frac{dy}{dx} = x^2 y - 1$, $y(0) = 1$ (upto 4th degree term)

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

$$+ \frac{(x-x_0)^4}{4!}y^{iv}(x_0)$$

By data $x_0 = 0, \quad y_0 = 1$

$$\therefore\ y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) \quad \text{—— (1)}$$

Given $y' = x^2 y - 1 \qquad \therefore y'(0) = -1$

$y'' = 2xy + x^2 y' \qquad y''(0) = 0$

$y''' = 2[xy' + y] + x^2 y'' + 2xy'$

$y''' = 2xy' + 2y + x^2 y'' + 2xy'$

$y''' = 4xy' + 2y + x^2 y'' \qquad \therefore y'''(0) = 2$

$y^{iv} = 4[xy'' + y'] + 2y' + x^2 y''' + 2xy''$

$y^{iv} = 6y' + 6xy'' + x^2 y''' \qquad y^{iv}(0) = -6$

(1) $\Rightarrow \quad y(x) = 1 - x + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6)$

$$\boxed{y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}}$$

1) b) The following table gives the sol^n of $5xy'+y^2-2=0$. Find the value of y at $x=4.5$ using Milne's predictor and corrector formulae.

x	4	4.1	4.2	4.3	4.4
y	1	1.0049	1.0097	1.0143	1.0187

>> By data $5xy'+y^2-2=0$ (or) $y' = \frac{2-y^2}{5x}$

We prepare the table.

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \frac{2-1^2}{5\times4} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2-(1.0049)^2}{5\times4.1} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = 0.0452$
$x_4 = 4.4$	$y_4 = 1.0187$	$y_4' = 0.0437$
$x_5 = 4.5$	$y_5 = ?$	

$$y_5^{(P)} = y_1 + \frac{4h}{3}(2y_2' - y_3' + 2y_4')$$

$$y_5^{(P)} = 1.0049 + \frac{4(0.1)}{3}[2\times0.0467 - 0.0452 + 2\times0.0437]$$

$$\boxed{y_5^{(P)} = 1.023}$$

$$y_5' = \frac{2-y_5^2}{5x_5} = \frac{2-(1.023)^2}{5\times4.5} = 0.0424$$

$$y_5^{(C)} = y_3 + \frac{h}{3}(y_3' + 4y_4' + y_5')$$

$$y_5^{(C)} = 1.0143 + \frac{0.1}{3}[0.0452 + 4(0.0437) + 0.0424]$$

$$\boxed{y_5^{(C)} = 1.023}$$

Thus $\boxed{y(4.5) = 1.023}$

1) c) Using Euler's modified method. obtain a sol^n of the Eq^n $\frac{dy}{dx} = x + |\sqrt{y}|$, with initial conditions $y=1$ at $x=0$, for the range $0 \le x \le 0.4$ in steps of 0.2.

>> we need to compute $y(0.2)$ and $y(0.4)$ with $h=0.2$

where modulus sign indicates that we have to take only the positive value of $\sqrt{y}$.

I Stage. By data $x_0 = 0,\ y_0 = 1,\ f(x, y) = x + \sqrt{y},\ h = 0.2$

$$y(x_1) = y_1 \Rightarrow \boxed{y(0.2) = ?}$$

Euler's formula $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$$y_1^{(0)} = 1 + 0.2(1) = 1.2 \quad \therefore \boxed{y_1^{(0)} = 1.2}$$

modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + x_1 + \sqrt{y_1^{(0)}}\right]$$

$$y_1^{(1)} = 1.2295$$

$$y_1^{(2)} = 1 + 0.1\left[1.2 + \sqrt{1.2295}\right] = 1.2309$$

$$y_1^{(3)} = 1 + 0.1\left[1.2 + \sqrt{1.2309}\right] = 1.2309$$

$$\boxed{y(0.2) = 1.2309}$$

II Stage. Let $x_0 = 0.2,\ y_0 = 1.2309 \quad f(x_0, y_0) = 1.3095$

$$x_1 = x_0 + h = 0.4 \quad y(0.4) = ?$$

$$y_1^{(0)} = 1.4928$$

modified formula $y_1^{(1)} = 1.2309 + \frac{0.2}{2}\left[1.3095 + 0.4 + \sqrt{1.4928}\right]$

$$y_1^{(1)} = 1.524$$

$$y_1^{(2)} = 1.2309 + 0.1\left[1.7095 + \sqrt{1.524}\right] = 1.5253$$

$$y_1^{(3)} = 1.2309 + 0.1\left[1.7095 + \sqrt{1.5253}\right] = 1.5254$$

2) a) Using modified Euler's method find $y(20.2)$ and $y(20.4)$ given that $dy/dx = \log_{10}(x/y)$ with $y(20) = 5$ taking $h = 0.2$

I Stage Let $x_0 = 20$, $y_0 = 5$, $h = 0.2$ and $f(x,y) = \log_{10}(x/y)$

$$f(x_0, y_0) = \log_{10}(4) = 0.6021$$

$$x_1 = x_0 + h = 20.2 \qquad y(x_1) = y_1 \Rightarrow \boxed{y(20.2) = ?}$$

Euler's formulae $y_1^{(0)} = y_0 + h f(x_0, y_0) = 5.1204$

modified $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$$y_1^{(1)} = 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right] = 5.1198$$

$$y_1^{(2)} = 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1198}\right)\right] = 5.1198$$

Thus $\boxed{y(20.2) = 5.1198}$

II Stage Now let $x_0 = 20.2$, $y_0 = 5.1198$, $h = 0.2$

$$f(x_0, y_0) = 0.5961 \qquad x_1 = x_0 + h = 20.4$$

$$y(20.4) = ?$$

Euler's formulae

$$y_1^{(0)} = 5.1198 + .2(0.5961) = 5.239$$

modified euler's formulae

$$y_1^{(1)} = 5.1198 + \frac{0.2}{2}\left[0.5961 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$= 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.239}\right)\right] = 5.2384$$

$$y_1^{(2)} = 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.2384}\right)\right] = 5.2385$$

Thus $\boxed{y(20.4) = 5.2385}$

2) b) Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$ Evaluate $y(1.4)$ by Adams - Baghforth method.

>> prepare the following table : $(h = 0.1)$

x	y	$y' = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$y_0' = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$y_1' = 2.702$
$x_2 = 1.2$	$y_2 = 1.548$	$y_2' = 3.669$
$x_3 = 1.3$	$y_3 = 1.979$	$y_3' = 5.035$
$x_4 = 1.4$	$y_4 = ?$	

using $y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$

$$= 1.979 + \frac{0.1}{24}\left[55(5.035) - 59(3.669) + 37(2.702) - 9(2)\right]$$

$$\boxed{y_4^{(P)} = 2.573}$$

$$y_4' = f(x_4, y_4) = x_4^2(1+y_4) = 7.004$$

$$y_4^{(C)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[9(7.004) + 19(5.035) - 5(3.669) + 2.702\right]$$

$$\boxed{y_4^{(C)} = 2.575}$$

Thus $y(1.4) = 2.575$

2) c) Using Runge-kutta method of 4th order, Solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$ by taking $h=0.2$

→ Given $x_0=0$, $y_0=1$, $h=0.2$

$f(x,y) = \frac{y^2-x^2}{y^2+x^2}$ we shall find k_1, k_2, k_3, k_4

$k_1 = h f(x_0, y_0) = 0.2\, f(0,1) = 0.2(1) = 0.2$

$k_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.2\, f(0.1, 1.1) = 0.1967$

$k_3 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = 0.2\, f(0.1, 1.0984) = 0.1967$

$k_4 = h f(x_0+h, y_0+k_3) = 0.2\, f(0.2, 1.1967) = 0.1891$

we have $y(x_0+h) = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$$\boxed{y(0.2) = 1.196}$$

Module-02

3) a) Obtain the solⁿ of the Eqⁿ $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ by Computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the foll^ng data.

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

>> Dividing the given eq^n by 2 we have

$$\frac{d^2y}{dx^2} = 2x + \frac{dy}{2dx} \quad (or) \quad y'' = 2x + \frac{y'}{2}$$

put $y' = z$ we get $y'' = z'$

$$\therefore \; z' = 2x + \frac{z}{2}$$

$$z_0' = 2(1) + \frac{2}{2} = 3, \quad z_1' = 3.3589 \quad z_2' = 3.73625$$

$$z_3' = 4.13285$$

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z_0' = 3$	$z_1' = 3.3589$	$z_2' = 3.73625$	$z_3' = 4.13288$

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3) = 3.0793$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3') = 3.4996$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$z_4' = 2x_4 + \frac{z_4^{(P)}}{2}$$

$$= 2(1.4) + \frac{3.4996}{2} = 4.5498$$

$$z_4' = 4.5498$$

$$y_4^{(C)} = 3.0794 \quad \text{and} \quad z_4^{(C)} = 3.4997$$

Thus the required value of y is 3.0794

3) b) Express $f(x) = 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.

Let $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$\therefore x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$P_3(x) = \frac{1}{2}\left[5x^3 - 3x\right]$$

$$\therefore x^3 = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right]$$

$$f(x) = 3x^3 - x^2 + 5x - 2$$

$$f(x) = 3\left[\frac{2}{5}P_3 + \frac{3}{5}P_1\right] - \left[\frac{2}{3}P_2 + \frac{1}{3}P_0\right] + 5P_1 - 2P_0$$

$$= \frac{6}{5}P_3 + \frac{9}{5}P_1 - \frac{2}{3}P_2 - \frac{1}{3}P_0 + 5P_1 - 2P_0$$

$$= \frac{6}{5}P_3 + \left(\frac{9}{5} + 5\right)P_1 - \frac{2}{3}P_2 + \left(-\frac{1}{3} - 2\right)P_0$$

$$f(x) = \frac{6}{5}P_3 - \frac{2}{3}P_2 + \frac{34}{5}P_1 - \frac{7}{3}P_0$$

---

3) c) Obtain the series sol^n of Bessel's differential eq^n

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

>> Frobenius method

$y'' = x^2 = P_0(x)$ and $P_0(x) = 0$ at $x = 0$

We assume

$$y = \sum_{r=0}^{\infty} a_r x^{k+r}, \quad y' = \sum_0^{\infty} a_r (k+r) x^{k+r-1}$$

$$y'' = \sum_0^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

Consider Bessel eq^n $x^2y'' + xy' + (x^2 - n^2)y = 0$ —— (1)

using y, y', y'' (1) $\Rightarrow$

$$\sum_0^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_0^{\infty} a_r (k+r) x^{k+r} + \sum_0^{\infty} a_r x^{k+r+2} -$$

$$\sum_0^\infty a_r x^{k+r}\left[(k+r)^2-n^2\right]+\sum_0^\infty a_r x^{k+r+2}=0$$

equate the coefficient of the lowest degree term in x,

ie x^k to zero. $a_0(k^2-n^2)=0$. $a_0\neq 0$, $k=\pm n$.

also x^{k+1} to zero. $a_1\left[(k+1)^2-n^2\right]=0$

$a_1=0$, $(k+1)^2=n^2$ (or) $(k+1)=\pm n$

which can't be accepted as we have already $k=\pm n$.

x^{k+r} $(r\geq 2)$. to zero

$$a_r\left[(k+r)^2-n^2\right]+a_{r-2}=0$$

$$a_r=\frac{-a_{r-2}}{\left[(k+r)^2-n^2\right]}\quad (r\geq 2) \text{ --- (2)}$$

when $k=+n$, (2) $\Rightarrow$

$$a_r=\frac{-a_{r-2}}{(n+r)^2-n^2}=\frac{-a_{r-2}}{2nr+r^2}$$

putting $r=2,3,4$ -------

$a_2=\dfrac{-a_0}{4(n+1)}$, $a_3=0$, $a_5=a_7=0\ldots$

$$a_4=\frac{a_0}{32(n+1)(n+2)}$$

wkt $y=x^k a^r$

$y=x^k(a_0+a_1x+a_2x^2+\text{-----})$ also $k=+n$, denoted by y_1

$$\therefore y_1=x^n\left[a_0-\frac{a_0}{4(n+1)}x^2+\frac{a_0}{32(n+1)(n+2)}x^4-\text{-----}\right]$$

$$y_1=a_0x^n\left[1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2^5(n+1)(n+2)}-\ldots\right] \text{ --- (3)}$$

Since we also have $k=-n$, denoted by y_2 (3) $n\to -n$

$$-n\left[1-\frac{x^2}{\quad}+\frac{x^4}{\quad}-\ldots\right] \text{ --- (4)}$$

Complete solⁿ is

$y = AY_1 + BY_2$, where A, B are arbitary constants.

we shall standardize (3) $\Rightarrow$ $a_0 = \dfrac{1}{2^n\Gamma(n+1)}$

$$Y_1 = \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2\frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4\frac{1}{(n+1)(n+2)\Gamma(n+1)2} \dots\right]$$

$$Y_1 = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1).r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{0}^{\infty} (-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1).r!}$$

This funⁿ is called the Bessel funⁿ of the first kind of order n denoted by $J_n(x)$.

Thus $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1).r!}$$

further, the solⁿ for $k=-n$, denoted by $J_{-n}(x)$

hence the general solⁿ of the Bessel's eqⁿ is

$$\underline{y = aJ_n(x) + bJ_{-n}(x)}$$ a, b are arbitary constants, n not an integer.

4) a) By Runge-kutta method solve $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$ for $x = 0.2$ correct to four decimal places using the initial conditions $y=1$ and $y'=0$ at $x=0$, $h=0.2$

put $\dfrac{dy}{dx} = z$, $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

$\therefore$ The given eqⁿ becomes.

$z=0$ when $x=0$

we have a system of equations $\frac{dy}{dx} = z$, $\frac{dz}{dx} = xz^2 - y^2$

$f(x, y, z) = z$, $g(x, y, z) = xz^2 - y^2$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and $h = 0.2$

we shall first compute

$k_1 = h f(x_0, y_0, z_0)$

$= 0.2 \, f(0, 1, 0)$

$= 0.2(0)$

$\boxed{k_1 = 0}$

$l_1 = h g(x_0, y_0, z_0)$

$l_1 = 0.2 \, g(0, 1, 0)$

$l_1 = 0.2 [0(0)^2 - 1^2]$

$\boxed{l_1 = -0.2}$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_2 = 0.2 \, f(0.1, 1, -0.1)$

$k_2 = 0.2(-0.1) = -0.02$

$\boxed{k_2 = -0.02}$

$l_2 = h g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$l_2 = 0.2 [0.1(-0.1)^2 - 1^2]$

$\boxed{l_2 = -0.1998}$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$k_3 = 0.2 \, f(0.1, 0.99, -0.0999)$

$\boxed{k_3 = -0.01998}$

$l_3 = h g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$\cancel{l_3 = 0.2 [0.1 \, (-0.0999)^2]}$

$l_3 = 0.2 [0.1(-0.0999)^2 - (0.99)^2]$

$\boxed{l_3 = -0.1958}$

$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$

$k_4 = 0.2 \, f(0.2, 0.98002, -0.1958)$

$\boxed{k_4 = -0.03916}$

$l_4 = h g(x_0 + h, y_0 + k_3, z_0 + l_3)$

$\boxed{l_4 = -0.19055}$

$y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\boxed{y(0.2) = 0.9801}$

4) b) Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

By definition

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!} \quad \text{--- (1)}$$

putting $n = 1/2$ in (1)

$$J_{1/2}(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{1/2+2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

$$= \sqrt{\frac{x}{2}} \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

on expanding summation we've

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)\, 1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)\, 2!} - \cdots \right] \quad \text{--- (2)}$$

wkt $\Gamma(1/2) = \sqrt{\pi}$. $\Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$, $\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$, $\Gamma(7/2) = \frac{15\sqrt{\pi}}{8}$

(2) ⇒

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4} \cdot \frac{4}{3\sqrt{\pi}} + \frac{x^4}{16} \cdot \frac{8}{15\sqrt{\pi} \cdot 2} - \cdots \right]$$

$$= \sqrt{\frac{x}{2\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots \right]$$

$$= \sqrt{\frac{x}{2\pi}} \, \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \right]$$

$2/x$ as a common factor keeping in view of the dised result

$$\text{ie } J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

Thus $\boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$

$$\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

4) c) Prove the Rodrigue formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Let $u = (x^2-1)^n$

Legendre's differential eq^n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{------ (1)}$$

Diff u w.r.t x

$$u_1 = n(x^2-1)^{n-1} \cdot 2x$$

(or) $(x^2-1)u_1 = 2nx(x^2-1)^n$

$$(x^2-1)u_1 = 2nxu$$

again Diff w.r.t x

$$(x^2-1)u_2 + 2xu_1 = 2n(xu_1 + u)$$

Apply Leibnitz rule we get

$$(uv)_n = uv_n + nu_1v_{n-1} + \frac{n(n-1)}{2!}u_2v_{n-2} + \cdots + u_nv$$

$$\therefore [(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$$

$$\left[(x^2-1)u_{n+2} + n.2xu_{n+1} + \frac{n(n-1)}{2} \cdot 2.u_n\right] + 2\left[xu_{n+1} + n.u_n\right]$$

$$= 2n[xu_{n+1} + nu_n] + 2nu_n$$

$$(x^2-1)u_{n+2} + 2nxu_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n =$$
$$2nxu_{n+1} + 2n^2u_n + 2nu_n$$

$$(x^2-1)u_{n+2} + 2xu_{n+1} - n^2u_n - nu_n = 0$$

$$(x^2-1)u_{n+2} + 2xu_{n+1} - nu_n(n+1) = 0$$

(or) $(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$

Thus can be put in the form

$$(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \quad \text{------ (2)}$$

Comparing (2) with (1) we conclude that u_n is a sol^n of

[illegible] ... It may be observed that u is a polynomi

Also $P_n(x)$ which satisfies the Legendre's d. eq^n is also a polynomial of degree n. must be the same as $P_n(x)$ but for some constant factor k.

ie $\quad P_n(x) = k u_n = k\left[(x^2-1)^n\right]_n$

$$P_n(x) = k\left[(x-1)^n (x+1)^n\right]_n$$

Apply Leibnitz rule for the RHS

$$P_n(x) = k\Big[(x-1)^n \{(x+1)^n\}_n + n \cdot n(x-1)^{n-1}\{(x+1)^n\}_{n-1}$$

$$+ \frac{n(n-1)}{2}\, n(n-1)(x-1)^{n-2}\{(x+1)^n\}_{n-2}$$

$$+ \text{---} \; \{(x-1)^n\}_n (x+1)^n\Big] \text{ —— } (3)$$

Now $\; z = (x-1)^n$ then

$z_1 = n(x-1)^{n-1}$

$z_2 = n(n-1)(x-1)^{n-2}$

$\vdots$

$z_n = n(n-1)(n-2) \dots 2 \cdot 1 \cdot (x-1)^{n-n}$

$z_n = n!\,(x-1)^0 = n!$

$\therefore \{(x-1)^n\}_n = n!$

putting $x=1$ in ③ all the terms in RHS become zero except the last term which becomes $n!\,(x+1)^n = n!\,2^n$

ie $\; P_n(1) = k \cdot n!\, 2^n$

and $P_n(1) = 1$ by def^n of $P_n(x)$

$\therefore \quad 1 = k \cdot n!\, 2^n$

(or) $\quad k = \dfrac{1}{n!\,2^n}$

since $\; P_n(x) = k u_n$ we have

$$P_n(x) = \frac{1}{n!\,2^n}\{(x^2-1)^n\}_n$$

Module - 03

5) a) State and prove Cauchy's - Riemann eqn in polar form.

Stmt :- If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

These are known as Cauchy - Riemann eqns in the polar form.

Proof :- Let $f(z)$ be analytic at a point $z = re^{i\theta}$

$$\therefore f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

In the polar form $f(z) = u(r,\theta) + iv(r,\theta)$ and let δz be increment in z corresponding to the increments $\delta r, \delta\theta$, in r, θ

$$f'(z) = \lim_{\delta z \to 0} \frac{u[(r+\delta r, \theta+\delta\theta)] + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i\lim_{\delta z \to 0} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \quad \text{--- (1)}$$

$$\delta z = e^{i\theta}\delta r + ire^{i\theta}\delta\theta$$

case i) Let $\delta\theta = 0$ so that $\delta z = e^{i\theta}\delta r$

and $\delta z \to 0$ imply $\delta r \to 0$

(1) =>

$$f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}\delta r} + i\lim_{\delta r \to 0} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta}\delta r}$$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{--- (2)}$$

Case ii) Let $\delta r = 0$ so that $\delta z = ire^{i\theta}\delta\theta$ and $\delta z \to 0$ imply $\delta\theta \to 0$. Now ① $\Rightarrow$

$$f'(z) = \lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{ire^{i\theta}\delta\theta} + i\lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{ire^{i\theta}\delta\theta}$$

$$f'(z) = \frac{1}{ire^{i\theta}}\left[\lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{\delta\theta} + i\lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{\delta\theta}\right]$$

$$f'(z) = \frac{1}{ire^{i\theta}}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right]$$

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right] \quad \text{—③} \qquad \frac{1}{i} = -i$$

Equating the RHS of ② and ③ we have

$$e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right]$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

These are C-R eqns in the polar form.

5) b) Discuss the transformation $w = e^z$.

Consider $w = e^z$

ie $u + iv = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$\therefore u = e^x \cos y, \quad v = e^x \sin y$

We shall find the image in the w-plane corresponding to the straight lines ~~passes~~ parallel to the co-ordinate axes in the z-plane. That is x = constant = y

Let us eliminate x and y separately from (1)

Squaring and adding we get

$$u^2+v^2=e^{2x}$$

Also dividing we get $\frac{v}{u}=\frac{e^x \sin y}{e^x \cos y}$

$$\frac{v}{u}=\tan y$$

case i) Let $x=c_1$ where c_1 is a constant.

Eq^n (2) $\Rightarrow u^2+v^2=e^{2c_1}=$ constant $=r^2$

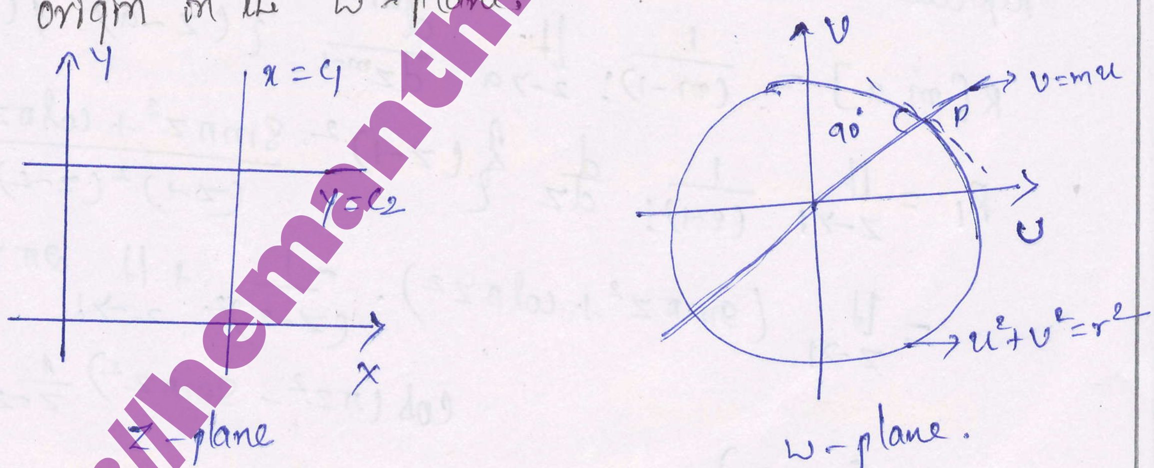
This represents a circle with centre origin and radius in the w-plane.

case ii) Let $y=c_2$ where c_2 is a constant

Eq^n (3) $\Rightarrow \frac{v}{u}=\tan c_2=m$

$$\therefore v=mu$$

represents a straight line passing through the origin in the w-plane.



Conclusion:

The straight line parallel to the x-axis ($y=c_2$) in the z-plane maps onto a straight line passing through the origin in the w-plane. The St line parallel to the y-axis ($x=c_1$) in the z-plane maps onto a circle with centre origin and radius

Suppose we draw a tangent at the point of intersection of these two curves on the w-plane (At pos on fig) the angle subtended is equal to 90°, hence these two curves can be regarded as orthogonal trajectories of each other.

5) c) Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} dz$ using Cauchy's residue theorem where 'C' is the circle $|z| = 3$.

Let $f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}$; $C : |z| = 3$

$z=1$ is a pole of order 2 and $z=2$ is a pole of order 1. Both of them lies within the circle $|z|=3$.

Residue at $z=1$ be denoted by R_1 and we've

$$R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

$$R_1 = \lim_{z\to 1} \frac{1}{(2-1)!} \frac{d}{dz} \left\{(z-1)^2 \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\right\}$$

$$= \lim_{z\to 1} (\sin\pi z^2 + \cos\pi z^2) \cdot \frac{-1}{(z-2)^2} + \lim_{z\to 1} 2\pi z \cos(\pi z^2 - \sin\pi z^2) \frac{1}{z-2}$$

$R_1 = (1+2\pi)$

$$R_2 = \lim_{z\to 2} (z-2) \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} = 1$$

$R_2 = 1$

$$\int_C f(z)dz = 2\pi i[R_1 + R_2] = 4\pi i(1+\pi)$$

Thus $\int \frac{\sin\pi z^2 + \cos\pi z^2}{\ldots} dz = 4\pi i(1+\pi)$

6) a) Find the analytic f^n whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

>> Let $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\therefore u_x = \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{-\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

Consider $f'(z) = u_x + i v_x = u_x - i u_y$ by CR eq^n

Putting $x = z,\ y = 0$

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

$$f'(z) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2} - i(0)$$

$$f'(z) = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = \frac{-2}{(1-\cos 2z)} = \frac{-2}{2\sin^2 z}$$

$$f'(z) = -\operatorname{cosec}^2 z$$

Thus $\boxed{f(z) = \cot z + C}$

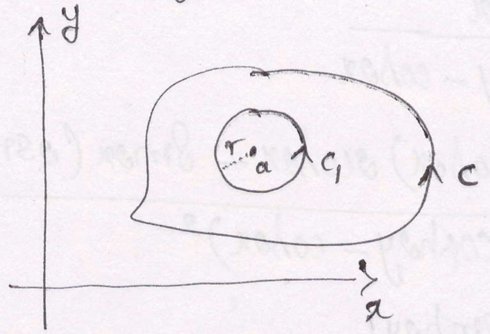
6) b) State and Prove Cauchy's integral formula.

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)}\, dz$$

Proof :- Since 'a' is a point within C, we shall enclose it by a circle C_1 with $z = a$ as centre

The fu^n $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region b/w C and C_1



Now, as a consequence of Cauchy's theorem

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{--- (1)}$$

The eq^n of C_1 can be written in the form $|z-a| = r$

$$\Rightarrow z-a = re^{i\theta} \quad (\text{or}) \quad z = a + re^{i\theta}$$

$$0 \le \theta \le 2\pi, \quad dz = ire^{i\theta}\,d\theta$$

RHS of (1)

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

$r \to 0$ hence $r \to 0$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta = i f(a)\,\theta\Big|_0^{2\pi}$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Thus $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

6) c) Find the bilinear transformation which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find the fixed points of the tranformation.

The required tranlformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

By data $z = \infty, i, 0 \Rightarrow z_1 = \infty, z_2 = i, z_3 = 0$

$w = -1, -i, 1 \Rightarrow w_1 = -1, w_2 = -i, w_3 = 1$

$$\therefore \frac{(w+1)(-1-i)}{(w-1)(1-i)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{z_1\left(z-z_3\right)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w+1)(-1-i)}{(w-1)(1-i)} = \frac{(0-1)(i-0)}{(z-0)(0-1)} = \frac{-i}{-z}$$

$$\frac{(w+1)(-1-i)}{(w-1)(1-i)} = \frac{i}{z}$$

$$\frac{(w+1)}{(w-1)} = \frac{i}{z}\,\frac{(1-i)}{(-1-i)} = \frac{i-i^2}{z(-i-1)} = \frac{i+1}{z(-i-1)}$$

$$\frac{(w+1)}{(w-1)} = \frac{\cancel{(i+1)}}{-z\cancel{(i+1)}} = \frac{-1}{z}$$

$$wz + z = -w + 1$$

$$wz + w = 1 - z$$

$$w(z+1) = 1 - z$$

$$\boxed{w = \frac{1-z}{1+z}}$$

$$z = \frac{1-z}{1+z}$$

$$z^2 + 2z - 1 = 0$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$z = \frac{-2 \pm \sqrt{4+4}}{2}$$

$$z = -1 \pm \sqrt{2}$$

Thus, $-1+\sqrt{2}$ and $-1-\sqrt{2}$ are the fixed points.

Module-04

7) a) Find the mean and standard deviation of poission distribution.

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}, \text{ where } \alpha > 0 \text{ is known}$$

as the exponential distribution.

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x\,\alpha e^{-\alpha x}\,dx = \alpha\int_{0}^{\infty} x e^{-\alpha x}\,dx$$

Apply Bernoulli's rule

$$\mu = \alpha\left[x\frac{e^{-\alpha x}}{-\alpha} - 1\cdot\frac{e^{-\alpha x}}{(-\alpha)^2}\right]_0^{\infty}$$

$$= \alpha\left[0 - \frac{1}{\alpha^2}(0-1)\right] = \frac{1}{\alpha}$$

$$\therefore \boxed{\mu = \frac{1}{\alpha}}$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\sigma^2 = \alpha\int_{-\infty}^{\infty} (x-\mu)^2 e^{-\alpha x}\,dx$$

$$= \alpha\left[(x-\mu)^2\left(\frac{e^{-\alpha x}}{-\alpha}\right) - 2(x-\mu)\left(\frac{e^{-\alpha x}}{\alpha^2}\right) + 2\frac{e^{-\alpha x}}{-\alpha^3}\right]_0^{\infty}$$

$$= \alpha\left\{\frac{\mu^2}{\alpha} - \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3}\right\} \qquad \mu = \frac{1}{\alpha}$$

$$\sigma^2 = \alpha\left(\frac{1}{\alpha^3} - \frac{2}{\alpha^3} + \frac{2}{\alpha^3}\right)$$

$$\boxed{\sigma^2 = \frac{1}{\alpha^2}}$$

$$\boxed{SD\,(\sigma) = \frac{1}{\alpha}}$$

7) b) In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and S.D of 60 hours. Estimate the no of bulbs likely to burn for. i) more than 2150 hours

ii) less than 1950 hours.

iii) more than 1920 hours and less than 2160 hrs

[$A(1.833) = 0.4664$, $A(1.5) = 0.4332$, $A(2) = 0.4772$]

By data Mean $= \mu = 2040$ $\quad SD = (\sigma) = 60$

SNV $\quad z = \dfrac{x-\mu}{\sigma} = \dfrac{x-2000}{60}$

i) $p(x > 2150)$

If $x = 2150$, $z = 1.83$

$p(z > 1.83) = 0.5 - \phi(1.83) = 0.0336$

Now $2000 \times 0.0336 = 67.2 \simeq 67$

ii) $p(x < 1950)$

If $x = 1950$, $z = -1.5$

$\therefore p(z < -1.5) = p(z > 1.5) = 0.5 - \phi(1.5) = 0.0668$

Now $2000 \times 0.0668 = 133.6 \simeq 133$

iii) $p(1920 < x < 2160)$

If $x = 1920$, $z = -2$, If $x = 2160$, $z = 2$

$\therefore p(-2 < z < 2) = 2\, p(0 < z < 2)$

$= 2(0.4772)$

$= 0.9544$

Now $2000 \times 0.9544 = 1908.8 = \underline{\underline{1908}}$

7) c) The joint Prob distribution of two random variables x and y is as follows:

x \ y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Determine:

i) marginal distribution of x and y

ii) cov (of x and y) iii) Correlation of x and y.

Marginal distribution of x and y

x	1	5
f(x)	1/2	1/2

y	-4	2	7
g(y)	3/8	3/8	1/4

$E(x) = \mu_x = \sum x f(x) = 1 \times \frac{1}{2} + 5 \times \frac{1}{2} = \frac{6}{2} = 3$

$E(y) = \mu_y = \sum y g(y) = -4 \times \frac{3}{8} + 2 \times \frac{3}{8} + 7 \times \frac{1}{4} = 1$

$E(xy) = \sum x y \, p_{ij} = 1.5$

$cov(x, y) = E(xy) - E(x) E(y) = -1.5$

$\sigma_x^2 = E(x^2) - (E(x))^2$

But $E(x^2) = \sum x^2 f(x) = 1 \times \frac{1}{2} + 25 \times \frac{1}{2} = 13$

$\therefore \sigma_x^2 = 13 - 9 = 4$

$\therefore \boxed{\sigma_x = 2}$

$\sigma_y^2 = E(y^2) - (E(y))^2$

$E(y^2) = \sum y^2 g(y_j) = 19.75$

$\sigma_y^2 = 19.75 - 1 = 18.75$

$\sigma_y = 4.330$

$\therefore \rho(x, y) = \frac{cov(x, y)}{\sigma_x \sigma_y} = -0.1732$

8) a) The prob that a pen manufactured by a factory be defective is $\frac{1}{10}$. If 12 such pens are manufactured what is the prob that i) exactly 2 are defective ii) atleast 2 are defective iii) none of them are defective.

>> Prob of a defective pen is $p = 1/10 = 0.1$

prob of a non-defective pen $q = 1-p = 1-0.1 = 0.9$

we have $p(x) = {}^nC_x\, p^x q^{n-x}$ and $n = 12$

i) prob (exactly two defective) is $p(x=2)$

$$= {}^{12}C_2 (0.1)^2 (0.9)^{10} = \underline{\underline{0.2301}}$$

ii) prob (atleast 2 defective) is

$$= 1 - \{p[x=0] + p(x=1)\}$$

$$= 1 - [\,{}^{12}C_0 (0.1)^0 (0.9)^{12} + {}^{12}C_1 (0.1)^1 (0.9)^{11}]$$

$$= \underline{0.341}$$

iii) prob (no defective) is $p(x=0)$

$$= {}^{12}C_0 (0.1)^0 (0.9)^{12} = \underline{\underline{0.2824}}$$

8) b) Derive the exp^n for mean and variance of binomial distribution.

$$\text{Mean}(\mu) = \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{0}^{n} x \cdot {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

$$= \sum_{0}^{n} \frac{n\,(n-1)!}{(x-1)!\,(n-x)!}\, p \cdot p^{x-1} q^{n-x}$$

$$= \sum^{n} \cdots\, p^{x-1} q^{(n-1)-(x-1)}$$

$$\mu = np \sum_{1}^{n} {}^{(n-1)}C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$\mu = np(q+p)^{n-1} = np$$

$$\boxed{\text{Mean } (\mu) = np}$$

$$\text{Variance } (V) = \sum_{x=0}^{n} x^2 p(x) - \mu^2 \quad \text{—— (1)}$$

$$\text{Now } \sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} [x(x-1) + x]\, p(x)$$

$$= \sum_{0}^{n} x(x-1)p(x) + \sum_{0}^{n} x\,p(x)$$

$$= \sum_{0}^{n} x(x-1)\, {}^{n}C_{x}\, p^x q^{n-x} + np$$

$$= \sum_{0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{0}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!} p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{2}^{n} {}^{(n-2)}C_{(x-2)}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

$$(1) \Rightarrow \quad V = n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= -np^2 + np$$

$$= np - np^2 = np[1-p]$$

$$= npq$$

8) c) A random variable X take the values $-3, -2, -1, 0, 1, 2, 3$ such that $p(x=0) = p(x<0)$ and $p(x=-3) = p(x=-2) = p(x=-1) = p(x=1) = p(x=2) = p(x=3)$. Find the probability distribution.

>> Let the distribution $[x, p(x)]$ be as follows.

x	-3	-2	-1	0	1	2	3
$p(x)$	p_1	p_2	p_3	p_4	p_5	p_6	p_7

By data $p(x=0) = p(x<0)$

$$\Rightarrow p(x=0) = p(x=-1) + p(x=-2) + p(x=-3)$$

ie $p_4 = p_3 + p_2 + p_1$ —— (1)

Also we have by data —— (2)

$$p_1 = p_2 = p_3 = p_5 = p_6 = p_7$$

Further we must have

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1 \quad \text{—— (3)}$$

using (2) in (1) we get $3p_1 = p_4$

using (2) in (3) we get $6p_1 + p_4 = 1$ But $p_4 = 3p_1$

$\therefore 9p_1 = 1$ (or) $p_1 = 1/9$

hence $p_4 = 3 \cdot \left(\frac{1}{9}\right) = \frac{1}{3}$

Thus the prob distribution is as follows

x	-3	-2	-1	0	1	2	3
$p(x)$	$1/9$	$1/9$	$1/9$	$1/3$	$1/9$	$1/9$	$1/9$

9) a) In 324 throws of a six faced 'die' an odd number turned up 181 times. Is it reasonable to think that the 'die' is an unbiased one?

>> Prob of the turn up of an odd no is $p = 3/6 = 1/2$

hence $q = 1 - p = 1/2$

Expected no of successes $= 1/2 \times 324 = 162$

$\therefore$ observed no of successes $= 181$

$\therefore$ difference $= 181 - 162 = 19$

Consider $z = \dfrac{x - np}{\sqrt{npq}} = \dfrac{19}{\sqrt{324 \times 1/2 \times 1/2}} = 2.11$

Thus $z = 2.11 < 2.58$ (1% level of significance two tailed test)

Thus we conclude that the die is unbiassed.

9) b) Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results:

Horse A:	28	30	32	33	33	29	34
Horse B:	29	30	30	24	27	29	

Test whether you can discriminate b/w the two horses ($t_{0.05} = 2.2$ and $t_{0.02} = 2.72$ for 11 d.f)

>> Let the variables x and y respectively correspond to horse A and horse B.

x: 28, 30, 32, 33, 33, 29, 34

y: 29, 30, 30, 24, 27, 29

$n_1 = 7, \; n_2 = 6$

$\bar{x} = \dfrac{\Sigma x}{n_1} = \dfrac{219}{7} = 31.3$

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$$\sum_1^{n_1} (x-\bar{x})^2 = 31.43 \quad , \quad \sum_1^{n_2} (y-\bar{y})^2 = 26.84$$

$$S^2 = \frac{1}{n_1+n_2-2} \left\{ \sum_1^{n_1} (x-\bar{x})^2 + \sum_1^{n_2} (y-\bar{y})^2 \right\}$$

$$S^2 = \frac{1}{11}(31.43 + 26.84) = 5.2973$$

$$\therefore S = 2.3016$$

Consider $t = \dfrac{\bar{x} - \bar{y}}{S\sqrt{1/n_1 + 1/n_2}}$

$$t = \frac{(31.3 - 28.2)}{2.3016\sqrt{1/7 + 1/6}} = 2.42$$

But $t_{0.05} = 2.2$ and $t_{0.02} = 2.72$ for 11 d.f

$$t = 2.42 \begin{cases} > t_{0.05} = 2.2 \\ < t_{0.02} = 2.72 \end{cases}$$

The discrimination b/w the horses is significant at 5% level but not at 2% level of significance.

9) c) Find the unique fixed prob vector for the regular stochastic matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

$\Rightarrow$ we have to find $v = (x, y, z)$ where $x + y + z = 1$ such that $vA = v$

$$\Rightarrow [x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

ie $\left[\dfrac{y}{6}, x + \dfrac{y}{2} + \dfrac{2z}{3}, \dfrac{y}{3} + \dfrac{z}{3} \right] = [x, y, z]$

$$\Rightarrow \frac{y}{6} = x, \quad x + \frac{y}{2} + \frac{2z}{3} = y, \quad y + z = 3z$$

$$y = 6x, \quad 6x + 3y + 4z = 6y, \quad y + z = 3z$$

$$y = 6x, \quad z = 1 - x - y = 1 - x - 6x = 1 - 7x$$

in $6x - 3y + 4z = 0$ we have

$$6x - 18x + 4 - 28x = 0$$

$$\therefore \boxed{x = 1/10} \quad \boxed{y = 6/10} \quad \boxed{z = 3/10}$$

Thus the required unique fixed probability vector v is given by

$$v = (1/10, 6/10, 3/10)$$

10) a) Define terms:

i) <u>Null hypothesis</u>:

In order to arrive at a decision regarding the population through a sample of the population we've to make certain assumption referred to as hypothesis which may (or) may not be true. Much depends on the framing of hypothesis.

The hypothesis formulated for the purpose of its rejection under the assumption that it is true is called the <u>Null hypothesis</u>, denoted by H_0.

ii) Type-I and Type-II error

* If a hypothesis is rejected while it should have been accepted is known as Type-I error.

* If a hypothesis is accepted, while it should have been rejected is known as Type-II error.

iii) Confidence limits.

$S \pm 1.96\sigma$ and $S \pm 2.58\sigma$ are called the confident limits for S, 95% and 99%.

10) b) Prove that the markov chain whose t.p.m $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible. Find the corresponding stationary probability vector.

>> we shall show that P is a regular stochastic matrix, For convenience we shall write the given matrix in the form.

$$P = \frac{1}{6}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$P^2 = \frac{1}{36}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{bmatrix}$$

Since all the entries in P^2 are positive we conclude that the t.p.m P is regular.

Hence the markov chain having t.p.m P is irreducible.

Now find fixed probability vector of P.

If $v = (x, y, z)$, $vP = v$ where $x + y + z = 1$

ie. $[x, y, z]\ \frac{1}{6}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} = [x, y, z]$

$$\Rightarrow \frac{1}{6}[3y + 3z,\ 4x + 3z,\ 2x + 3y] = [x, y, z]$$

$$3y + 3z = 6x, \quad 4x + 3z = 6y, \quad 2x + 3y = 6z$$

solving $x = 1/3$, $y = 10/27$, $z = 8/27$

Thus $v = \left(1/3,\ 10/27,\ 8/27\right)$ is the required stationary probability vector.

10) c) Three boys A, B, C are throwing ball to each other, A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball find the probability that after three throws i) A has the ball

ii) B has the ball

iii) C has the ball.

>> State Space = {A, B, C} and the associated t.p.m is as follows.

$$P = \begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Initially if C has the ball, the associated initial probability vector is given by

$$p^{(0)} = (0, 0, 1)$$

Since the prob are desired after three throws we've to find $p^{(3)} = p^{(0)} P^3$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\therefore p^{(3)} = p^{(0)} P^3 = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right] = \left[P_A^3, P_B^3, P_C^3\right]$$

Thus after three throws the prob that the ball is with A is $\frac{1}{4}$, with B is $1/4$ and with C is $\frac{1}{2}$.