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M. Sreenivasa Reddy

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Simplex method - 2, Duality Theory

Learning objectives

After Studying this chapter, you should be able to

- appreciate the significance of duality concept
- understand the relationship between the primal and dual LP problems
- convert a primal to dual problem and vice-versa
- interpretation of the duality
- use dual simplex method to solve LP problems

3.1 Introduction

In the preceding chapters we have seen how complex and cumbersome the iterative procedure in the simplex method can be. Every time a variable departs and new variable enters, complicated calculations requiring large space (computer memory) have to be performed to get the new element values. Successive iterations are carried out by using number of row operations using simplex method and its variants (Big M method and Two Phase Method.

If a LPP with large number of variables and constraints (which is in real life problems) has to be solved by the simplex method, it will need a lot of time and computer space as all the tables and data will have to be stored and retrieved repeatedly. In this chapter we discuss about a new approach and variant of simplex method duality theory and dual simplex method.

3.2 The Essence of Duality Theory (Concept of Duality)

The term 'dual' in a general sense implies two or double; In LPP duality implies that each problem can be analyzed in two different ways but having equivalent solutions. Each LP problem stated in the original form is associated with another LPP called dual LPP or in short dual. The two problems are replica of each other. The dual of the primal problem is unique. The simplex procedure is such that if the primal is solved it is equivalent to solving the dual. Thus if the optimal solution to dual can be obtained, if we know the optimal solution to primal.

3.3 Economic Interpretation of Duality

In order to make the concept of duality clear, let us consider the following problem on diet.

Let the following table give the amounts of two vitamins V_1 and V_2 per unit present in two different foods f_1 and f_2 respectively.

	Food		
Vitamins	f_1	f_2	Daily requirement
V_1	6	8	100
V_2	7	12	120
Cost per unit	12	20	

The last column of the table represents the number of units of the minimum daily requirement for the two vitamins whereas the last row represents the cost per unit for the two foods. The problem is to determine the minimum quantities of the two foods f_1 and f_2 so that the minimum daily requirement of the two vitamins is met and that at the same time, the cost of purchasing these quantities of f_1 and f_2 is minimum.

To formulate the problem mathematically let x_1 be the number of units of food f_J (J = 1, 2) to be purchased, then the above problem is to determine two real numbers x_1 and x_2, so as to

Minimize $Z = 12x_1 + 20\,x_2$ subject to the constrains

$6x_1 + 8\,x_2 \geq 100$, $7x_1 + 12\,x_2 \geq 120$, $x_1, x_2 \geq 0$

Now let us consider a different problem, associated with the above problem. Suppose there is a wholesale dealer who sells the two vitamins V_1 and V_2 along with some other commodities. The local shopkeepers purchase the vitamins from him and from the two foods f_1 and f_2 (the details are same as in the given table). The dealer knows very well that the food f_1 and f_2 have their market values only because of their vitamins contents, the problem of the dealer is to fix the maximum per unit selling prices for the two vitamins V_1 and V_2, in such a way that the resulting prices for food f_1 and f_2 do not exceed their existing market prices.

The selling prices are generally known as dual prices or shadow prices.

To formulate the problem mathematically, let the dealer decide to fix up the two prices at W_1 and W_2 per unit respectively.

Then the dealer's problem can be stated mathematically as to determine two real numbers W_1 and W_2 so as to

Minimize $Z^* = 100W_1 + 120\ W_2$, subject to the constrains

$$6W_1 + 7\ W_2 \le 12,\ 8\ W_1 + 12W_2 \le 20,\ W_1, W_2 \ge 0.$$

3.4 Unrestricted variables

Usually in an LP problem, it is assumed that all the variables x_i ($i = 1, 2.........., n$) should have non – negative values. In many practical situations, one or more of the variables can have either positive, negative or zero value. Variables which can assume positive, negative or zero value are called unrestricted variables. Simplex method requires that all the decision variables must have non – negative value at each iteration, therefore, in order to convert an LP problem involving unrestricted variables into an equivalent problem having only restricted variables, we have to express each of unrestricted variable as two difference of the non – negative variables.

Let variable x_u be unrestricted in sign. We defined two new variables say x_u^{I} and x_u^{II} such that

$$x_u = x_u^{I} - x_u^{II}\ ;\ x_u^{I}, x_u^{II} \ge 0$$

3.5 Key relationships between Primal and Dual problems

Weak duality property: If 'x' is a feasible solution for the primal then and y is a feasible solution for the dual problems then, $Cx \le yb$, this inequality must hold good for any pair of feasible solutions of primal and dual.

Weak duality property describes the relationship between any pair of solutions for the primal and dual problems, where solutions are feasible for their problems.

Strong duality property : If x^* is an optimal solution for the primal and y^* is an optimal solution for the dual problem, then $Cx^* = y^*b$.

[The relationship imply that $Cx < yb$ for feasible solutions if one or both of them are not optimal for their respective problems, where as equality holds when both are optimal]

Complimentary solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution x for the primal problem and a complimentary solution y for the dual problem. In $Cx = yb$ if x is not optimal for primal problem, then y is not feasible for the dual problem.

Complimentary optimal solutions property: At the final iteration the simplex method simultaneously identifies an optimal solution x^* for the primal problem and a complementary optimal solution y^* for the dual problem. The relationship can be expressed as $Cx^* = y^*b$

3.6 Primal Dual Relationship

Let the primal problem be

Max $Z = C_1 x_1 + C_2 x_2 + ... + C_n x_n$

Subject to $a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \le b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \le b_2$

.

.

.

$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \le b_m$

$x_1, x_2 \ldots x_n \ge 0$

Dual of the problem is defined as (let w_1, w_2, w_3 be the dual variables)

Min $Z^* = b_1 w_1 + b_2 w_2 + \ldots\ldots\ldots + b_m w_m$

Subject to $a_{11} w_1 + a_{21} w_2 + \ldots\ldots\ldots + a_{m1} w_m \ge C_1$

$a_{12} w_1 + a_{22} w_2 + \ldots\ldots\ldots\ldots + a_{m2} w_m \ge C_2$

. .

. .

$a_{1n} w_1 + a_{2n} w_2 + \ldots\ldots\ldots\ldots + a_{mn} w_m \ge C_n$

$w_1, w_2, \ldots\ldots\ldots w_m \ge 0$

Where, $w_1, w_2 \ldots w_m \ge 0$ are called dual variables

Note: *In general Dual of the dual is primal.*

3.7 CHARACTERISTICS OF THE DUAL PROBLEM

Duality in linear programming has the following major characteristics

i Dual of the dual LP problem is primal.

ii. If either the primal or dual of the problem has a finite optimal solution, then the other one also will have the same.

iii. If any of the two problems has only an infeasible solution, then the value of the objective function of the other is unbounded.

iv. The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.

v. If either the primal or the dual has an unbounded objective function value then the solution to the other problem is infeasible.

vi. If the primal has a feasible solution but the dual does not have, then the primal will not have a finite optimal solution and vice-versa.

3.8 ADVANTAGES OF DUALITY

Following are the advantages of duality.

i. It yields a number of useful theorems.

ii. Solution of the dual checks the accuracy of the primal solution for computational errors.

iii. Indicates that fairly close relationship exists between linear programming and duality.

iv.. Economic interpretation of the dual helps the management in making future decisions.

v. Computational procedure can be considerably reduced.

vi. Duality can be used to solve an LP problem by the simplex method in which the initial solution is infeasible.

Worked Examples

1. *Write the dual of the following LPP*

Max $Z = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \geq -6$

$4x_1 + x_2 + x_3 \leq 6$ $x_1, x_2, x_3 \geq 0$

Solution:

Max $Z = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \leq 2$

Note: Put the second constraint in the standard form ($\leq$ form)

$2x_1 - x_2 + 5x_3 \leq 6$, $4x_1 + x_2 + x_3 \leq 6$ and $x_1, x_2, x_3 \geq 0$

Dual; Let w_1, w_2, w_3 be the dual variables

Then Min $Z^* = 2w_1 + 6w_2 + 6w_3$

Subject to $2w_1 + 2w_2 + 4w_3 \geq 1$, $w_1 - w_2 + w_3 \geq 2$, $-w_1 + 5w_2 + w_3 \geq 1$ and $w_1, w_2, w_3 \geq 0$

2. *Write dual of the following LPP*

Max $Z = 3x_1 - x_2 + x_3$

Subject to $4x_1 - x_2 \leq 8$, $8x_1 + x_2 + 3x_3 \geq 12$, $5x_1 - 6x_3 \leq 13$ *and* $x_1, x_2, x_3 \geq 0$

Solution:

Put the second constraint in the standard form ($\leq$ form) i.e., $-8x_1 - x_2 - 3x_3 \leq 12$

Dual; Min $Z^* = 8w_1 - 12w_2 + 13w_3$

Subject to $4w_1 - 8w_2 + 5w_3 \geq 3$, $-w_1 - w_2 + 0w_3 \geq -1$ and $0w_1 - 3w_2 + 6w_3 \geq 1$

3. *Write the dual of the following LPP*

Max $Z = 6x_1 + 10x_2$

Subject to $x_1 \le 14$, $x_2 \le 16$, $3x_1 + 2x_2 \le 18$ *and* $x_1, x_2 \ge 0$

Solution:

Let w_1, w_2 and w_3 be the dual variables

Min $Z^* = 14w_1 + 16w_2 + 18w_3$

Subject to $w_1 + 0w_2 + 3w_3 \ge 6$

$0w_1 + w_2 + 2w_3 \ge 10$ and $w_1, w_2, w_3 \ge 0$

4. *Write the dual of the following LPP*

Min $Z = 0.4x_1 + 0.5x_2$

Subject to $0.3x_1 + 0.1x_2 \le 2.7$, $0.5x_1 + 0.5x_2 = 6$

$0.6x_1 + 0.4x_2 \ge 6$ *and* $x_1, x_2 \ge 0$

Solution:

Max $Z^* = 2.7w_1 + 6w_2 + 6w_3$

Subject to $0.3w_1 + 0.5w_2 + 0.6w_3 \le 0.4$

$0.1w_1 + 0.5w_2 + 0.4w_3 \le 0.6$

$w_1 \le 0$, w_2 is unconstrained as it is equation, $w_3 \ge 0$

5. *Construct the dual problem for the following LPP*

Maximize $Z = 16x_1 + 14x_2 + 36x_3 + 6x_4$

Subject $14x_1 + 4x_2 + 14x_3 + 8x_4 = 21$; $13x_1 + 17x_2 + 80x_3 + 2x_4 \le 48$

$x_1, x_2 \ge 0$; x_3; x_4 *to unrestricted*

Solution:

Dual problem for the given primal problem

Minimize $Z^* = 21w_1 + 48w_2$

Subject to

$14w_1 + 13w_2 \ge 16, 4w_1 + 17w_2 \ge 14$

$14w_1 + 80w_2 = 36$, $8w_1 + 2w_2 = 6$

w_1 is un–restricted, $w_2 \ge 0$

The above dual can also be solved by graphical method where as primal can be solved using only simplex method due to more than 2 variables.

6. ***Obtain the dual of the following LPP***

$Z_{max} = 3x_1 + 4x_2$

Subject to $2x_1 + 6x_2 \leq 16,\ 5x_1 + 2x_2 \geq 20;\ x_1, x_2 \geq 0$

Solution:

Since the objective function of the given LP problem is of maximization, the direction of each of '$\geq$' has to be changed.

Multiplying the second constraint on both sides by -1, we get

$(-1)\ (5x_1 + 2x_2) \geq (-1)\ (20)$ or

$-5x_1 - 2x_2 \leq -20$

The standard primal LP problem so obtained is:

$Z_{max} = 3x_1 + 4x_2$

Subject to the constraints

$2x_1 + 3x_2 \leq 16,\ 5x_1 - 2x_2 \leq -20$ and $x_1 \geq 0,\ x_2 \geq 0.$

Let w_1, w_2 are the duals then, the dual is

$Z_{max} = 16w_1 - 20w_2$

Subject to the constraints

$2w_1 - 5w_2 \geq 3,\ 3w_1 - 2w_2 \geq 4$ and $w_1 \geq 0,\ w_2 \geq 0.$

7. ***Write the dual for the following LPP***

$Z_{max} = 3x_1 + 5x_2 + 7x_3$

Subject to the constraints

$x_1 + x_2 + 3x_3 \leq 10,\ 4x_1 - x_2 + 2x_3 \geq 15$

$x_1, x_2 \geq 0;\ x_3$ *is unrestricted variable.*

Solution:

$Z_{min} = 10w_1 - 15w_2$

Subject to the constraints

$w_1 - 4w_2 \geq 5,\ w_1 + w_2 \geq 5$

$3w_1 - 2w_2 \geq 7,\ -3w_1 + 2w_2 \geq -7$ or

$3w_1 - 2w_2 \leq 7,\ w_1, w_2 \geq 0.$

8. ***Obtain the dual problem of the following primal LP problem.***

Maximize $Z = 40x_1 + 120x_2$ *subject to the constraints,* $x_1 - 2x_2 \leq 8,\ 3x_1 + 5x_2 = 90,\ 15x_1 + 44x_2 \leq 660,$

$x_1 \geq 0,\ x_2 \geq 0$

Solution:

Transform the '≤' type constraint to a '≥' type constraint by multiplying the constraints by −1. Also write '=' type constraint equivalent to the constraint of the type '≥' and '≤', then the given primal problem can be written as,

$Z_{min} = 40x_1 + 120x_2$

Subject to the constraints

$-x_1 + 2x_2 \geq -8,\ 3x_1 + 5x_2 \geq 90$

$3x_1 + 5x_2 \geq 90$ or $-3x_1 - 5x_2 \geq -90$

$-15x_1 - 44x_2 \geq -660$ and $x_1 \geq 0,\ x_2 \geq 0$

Let w_1, w_2, w_3 and w_4 be the dual variables corresponding to the four constraints in given order, then the dual of the given problem can be formulated as:

$Z_{max} = 8w_1 + 90w_2 - 90w_3 - 660w_4$

Subject to the constraints

$-w_1 + 3w_2 - 3w_3 - 15w_4 \leq 40$

$2w_1 + 5w_2 - 5w_3 - 44w_4 \leq 120$

$w_1 \geq 0,\ w_2 \geq 0,\ w_3 \geq 0,\ w_4 \geq 0$

Let $w = w_2 - w_3$, the above dual problem reduces to the form.

$Z_{max} = 8w_1 + 90w - 660w_4$

Subject to the constraints

$-w_1 + 3w - 15w_4 \leq 40$

$2w_1 + 5w - 44w_4 \leq 120$

$w_1 \geq 0,\ w_4 \geq 0$; w is unrestricted in sign.

Here, it may be noted that the second constraint in the primal is equality. Therefore the corresponding second dual variable w_2 should be unrestricted in sign.

3.9 Dual Simplex Method

In simplex method, if one or more solution values (that is x_B) are negative and optimality condition is satisfied, then the solution may be optimum but not feasible [as it must satisfy the non negative constraint]. In such cases, a variant of the simplex method called the dual – simplex method would be used. In the dual simplex method we always attempt to retain optimality while bringing the primal back to feasibility (that is $x_{bi} \geq 0$ for all i).

3.9.1 Procedure of Dual Simplex Method

Step 1

Rewrite the linear programming problem by expressing all the constraints in ≤ form and transform them into equations through slack variables.

Step 2

Express the above problem in the form of a simplex tableau. If the optimality condition is satisfied and one or more basic variables have negative values, the dual simplex method is applicable.

Step 3

Feasibility condition: The basic variable with the most negative value becomes the departing / leaving variable (LV). Call the row in which this value appears as pivot row. If more than one element for LV exists, choose one.

Step 4

Optimality condition: Form ratios by dividing all $C_j - Z_j$ values by the corresponding pivotal row, the entering variable is the one having mminimum ratio. If no element in the pivot row is negative, the problem has no feasible solution. that col' $\theta' = \frac{C_j - Z_j}{PR}$ = smallest ~~value~~ value in the SV

P.E in the one,

Step 5

Use elementary row operations to convert the pivot element to 1 and then reduce all the other elements in the key/pivot column to zero.

Step 6

Repeat steps 3 through 5 until there are no negative values for the basic variables.

Note: *The advantage of dual simplex method is, it can avoid the artificial variables introduced in the constraints along with the surplus variables as all '≥' constraints are converted into '≤' type.*

9. *Use dual simplex method to solve the following problem.*
   Maximize $Z = -2x_1 - 3x_2$
   Subject to $x_1 + x_2 \geq 2$, $2x_1 + x_2 \leq 10$ *and* $x_1 + x_2 \leq 8$
   with x_1 *and* x_2 *non negative*

Solution:

The given problem is of maximization type hence, it is in standard form. Converting '≥' as '≤' of first constraint.

$$-x_1 - x_2 \leq -2$$

Converting the constraints as equations,

$-x_1 - x_2 + u_1 = -2$ where u_1, u_2, u_3 are slack variables

$2x_1 + x_2 + u_2 = 10$

$x_1 + x_2 + u_3 = 8$

Modified objective function is,

$Z_{max} = -2x_1 - 3x_2 + 0u_1 + 0u_2 + 0u_3$

First Simplex Table

Basis	C_B	x_1 (EV ↓)	x_2	u_1	u_2	u_3	B
LV ← u_1	0	$\boxed{-1}$ PE	−1	1	0	0	−2 ← PR
u_2	0	2	1	0	1	0	10
u_3	0	1	1	0	0	1	8
C_j		−2	−3	0	0	0	–
Z_j		0	0	0	0	0	0
$C_j - Z_j$		−2 (↑ PC)	−3	0	0	0	–

Since all the $(C_j - Z_j)$ values are ≤ 0, the above solution is optimal. However, it is infeasible because it has a non positive value for the basic variable u_1 $(= -2)$. Since u_1 is the only non positive variable, it becomes the leaving variable (LV).

Table to Obtain the Ratio

	x_1	x_2	u_1	u_2	u_3
$C_j - Z_j$ row	−2	−3	0	0	0
Pivot row	−1	−1	1	0	0
Ratio	2	3	–	–	–

As x_1 has the smallest ratio, it becomes the entering variable (EV)

Thus, the element −1, is the pivot element. Using the elementary row operations, we obtain second simplex table.

Second Simplex Table

Basis	C_B	x_1	x_2	u_1	u_2	u_3	B
x_1	– 2	1	1	– 1	0	0	2
u_2	0	$\boxed{0}$ PE	– 1	2	1	0	6 ← PR
u_3	0	0	0	1	0	1	6
C_j		– 2	– 3	0	0	0	–
Z_j		– 2	– 2	2	0	0	– 4
$C_j - Z_j$		0	– 1	– 2	0	0	–

Since all the $(C_j - Z_j)$ values are ≤ 0, the above optimal solution is feasible. The optimal and feasible solution is $x_1 = 2$, $x_2 = 0$ with $Z_{max} = -4$.

10. *Use dual simplex method to solve the LPP*

$Z_{max} = -2x_1 - x_3$ *subject to the constraints*

$x_1 + x_2 - x_3 \geq 5$

$x_1 - 2x_2 + 4x_3 \geq 8$

Solution:

Since the objective function of LP problem is of maximisation, therefore all the constraints should be of $\leq$ type. Thus convert the constraints of the $\leq$ type by multiplying both sides by -1 and re-writting the LP problem.

$Z_{max} = -2x_1 + 0.x_2 - x_3$

subject to the constraints:

$$-x_1 - x_2 + x_3 \leq -5$$
$$-x_1 + 2x_2 - 4x_3 \leq -8$$
$$x_1, x_2, x_3 \geq 0$$

Converting the constraints into equations by adding slack variables we get,

$Z_{max} = -2x_1 + 0x_2 - x_3 + 0u_1 + 0u_2$

subject to the constraints

$$-x_1 - x_2 + x_3 + u_1 = -5$$
$$-x_1 + 2x_2 - 4x_3 + u_2 = -8$$
$$x_1, x_2, x_3, u_1, u_2 \geq 0$$

Simplex Table - 1

Basis	c_B	x_1	x_2	x_3	u_1	u_2	B
u_1	0	–1	–1	1	1	0	–5
PR ← u_2	0	–1	2	–4	0	1	–8 ← PR
C_j		–3	0	–1	0	0	
Z_j		0	0	0	0	0	0
$C_j - Z_j$		–2	0	–1	0	0	

The solution is infeasible as $u_1 = -5$ and $u_2 = -8$, but it is optimal as all $C_j - Z_j \leq 0$. Thus we need to apply the dual simplex method to get both feasible as well as on optimal solution.

Iteration - 1

$u_2 = -8$ is the most negative valueand hence u_2 leaves the basis.

Table to obtain the ratio.

	x_1	x_2	x_3	u_1	u_2
$C_j - Z_j$ row	–2	0	–1	0	0
Pivot row	–1	2	–4	0	1
Ratio	2	–	1/4	–	–

As x_3 has the smallest ratio, it becomes the entering variable (EV). The intersection element of PR and column corresponding to EV is PE = 2

Making the PE as unity and other element of PC as 1 by suitable row operations we get

Simplex Table - 2

Basis	c_B	x_1	x_2	x_3	u_1	u_2	B
u_1	0	–5/4	–1/2	0	1	1/4	–7
LV ← x_3	–1	1/4	–1/2	1	0	–1/4	2 ← PR
C_j		–2	0	–1	0	0	
Z_j		–1/4	1/2	–1	0	1/4	
$C_j - Z_j$		–7/4	–1/2	0	0	–1/4	

Still the solution is infeasible as $u_1 = -7$, but optimal as all $C_j - Z_j \leq 0$.

So we proceed to second iteration.

Iteration - 2

Since, the variable $u_1 = -7$ is the only variable having negative value, we should select the variable u_1 to leave the basis.

Table to obtain the ratio

	x_1	x_2	x_3	u_1	u_2
$C_j - Z_j$ row	$-7/4$	$-1/2$	0	0	$-1/4$
Pivot row	$-5/4$	$-1/2$	0	1	$1/4$
Ratio	$35/4$	1	–	–	–

The least ratio is 1 which corresponds to x_2 and hence x_2 enters into the basis. Making the corresponding PE as 1 in the Simplex Table - 2 we get,

Simplex Table - 3

Basis	c_B	x_1	x_2	x_3	u_1	u_2	B
x_2	0	$5/2$	1	0	-2	$-1/2$	14
x_3	-1	$3/2$	0	1	-1	$-1/2$	9
C_j		-2	0	-1	0	0	–
Z_j		$-3/2$	0	-1	1	$1/2$	
$C_j - Z_j$		$-1/2$	0	0	-1	$-1/2$	

From the above table, all $C_j - Z_j \leq 0$ and also all the basic variables are positive. Hence, the solution is feasible and optimal.

Thus, $x_1 = 0, x_2 = 14, x_3 = 9$

$$Z_{max} = -2\ (0) - 9 = -9$$

11. *Solve the following LPP using dual simplex method.*

$Z_{min} = 3x_1 + x_2$ subject to:

$x_1 + x_2 \geq 1, \qquad 2x_1 + 3x_2 \geq 2, x_1$ and $x_2 \geq 0$

Solution:

Converting the minimisation problem to maximization problem and the constraints of $\geq$ type to $\leq$ type we get,

$$Z_{max} = -3x_1 - x_2$$

subject to $-x_1 - x_2 \leq -1, \ -2x_1 - 3x_2 \leq -2$

x_1 and $x_2 \geq 0$

Adding the slack variables to the constraints,

$Zmax = -3x_1 - x_2 + ou_1 + 0u_2$

Such that $-x_1 - x_2 + u_1 = -1$

$-2x_1 - 3x_2 + u_2 = -2$

$x_1, x_2, u_1, u_2 \geq 0$

Simple Table - 1

EV ↓

	Basis	c_B	x_1	x_2	u_1	u_2	B	
	u_1	0	−1	−1	1	0	−1	
LV ←	u_2	0	−2	−3	0	1	−2	← PR
	C_j		−3	−1	0	0	−	
	Z_j		0	0	0	0	0	
	$C_j - Z_j$		−3	−1	0	0	−	

From the table, we can notice that the solution is infeasible as $u_1 = -1$, $u_2 = -2$ but it is optimal as all $C_j - Z_j \leq 0$. Thus, we need to apply dual simplex method to get both feasible as well as optimal solution.

u_2 is the leaving variable as it has most negative value. To determine the entering variable it is required to obtain the ratio.

	x_1	x_2	u_1	u_2
$C_j - Z_j$ row	−3	−1	0	0
Pivot row	−2	−3	0	1
Ratio	3/2	1/3	−	−

The least positive ratio is 1/3 which corresponds to column 2 and hence x_2 enters into the basis.

Replacing u_2 with x_2 we get,

Simple Table - 2

	Basis	c_B	x_1	x_2	u_1	u_2	B	
LV ←	u_1	0	−1/3	0	1	−1/3	−1/3	← PR
	x_2	−1	2/3	1	0	−1/3	2/3	
	C_j		−3	−1	0	0		
	Z_j		−2/3	−1	0	1/3		
	$C_j - Z_j$		−7/3	0	0	−1/3		

Solution is still infeasible as $u_1 = -1/3$ but optimal so proceeding to the next iteration.

As u_1 is non - positive it becomes the LV, to determine the EV determine the ratio as shown below.

	x_1	x_2	u_1	u_2
$C_j - Z_j$ row	−7/3	0	0	−1/3
Pivot row	−1/3	0	1	−1/3
Ratio	8/3	−	−	1

The min. ratio is 1 which corresponds to Column IV hence, u_2 becomes the entry variable. Replacing u_1 with u_2 making PE as 1 in the Simplex Table - 2 we get,

Basis	c_B	x_1	x_2	u_1	u_2	*B*
u_2	0	1	0	–3	1	1
x_2	–1	1	1	–1	0	1
C_j		–3	–1	0	0	–
Z_j		–1	–1	1	0	
$C_j - Z_j$		–2	0	–1	0	

As all $C_j - Z_j \le 0$ and the variables (of basis) are positive the solution is feasible and optimal.

Hence, the optimal solution is

$$x_1 = 0, \quad x_2 = 1$$

$$Z_{min} = 1$$

Note:

i. The least ratio may corresponds to the slack variable also, in such cases the slack variables becomes the entering variable replacing the existing variable in the basis.

ii. In dual simplex method, the solution will be declared as final only when it is optimal and feasible and i.e., only when all $C_j - Z_j \le 0$ and all the variables of the basis are positive.

12. *Solve the following LPP using dual simplex method.*

$$Z_{min} = 2x_1 + x_2$$

Subject to $3x_1 + x_2 \ge 3$

$$4x_1 + 3x_2 \ge 6, \quad x_1 + 2x_2 \ge 3 \quad \text{and} \quad x_1, x_2 \ge 0.$$

Solution:

Converting the given objective function to maximization type

$$Z_{max} = -2x_1 - x_2$$

Converting all the constraints into $\le$ type.

$$-3x_1 - x_2 \le -3$$

$$-4x_1 - 3x_2 \le -6.$$

$$-x_1 - 2x_2 \le -3 \quad \text{and} \quad x_1, x_2 \ge 0.$$

Introducing the slack variables we get,

$$Z\text{max} = -2x_1 - x_2 + 0u_1 + 0u_2 + 0u_3$$

Subject to = $-3x_1 - x_2 + u_1 = -3$

$-4x_1 - 3x_2 + u_2 = -6$

$-x_1 - 2x_2 + u_3 = -3$ and $x_1, x_2, u_1, u_2, u_3 \geq 0$

Simplex Table - 1

	Basis	c_B	x_1	x_2 (EV)	u_1	u_2	u_3	B	
	u_1	0	–3	–1	1	0	0	–3	
LV ←	u_2	0	–4	PE $\boxed{-3}$	0	1	0	–6	← PR
	u_3	0	–1	–2	0	0	1	–3	
	C_j		–2	–1	0	0	0		
	Z_j		0	0	0	0	0	0	
	$C_j - Z_j$		–2	–1	0	0	0	–	

As all $C_j - Z_j \leq 0$, the solution is optimal. However, it is not feasible as it is having non - positive basic variables.

The most negative basic variable is u_2 and hence it will be leaving variable.

Table to obtain the ratio.

	x_1	x_2	u_1	u_2	u_3
$C_j - Z_j$ row	–2	–1	0	0	0
Pivot row	–4	–3	0	1	0
Ratio	1/2	1/3	–	–	–

As x_2 has the smallest ratio, it becomes the entering variable.

The element – 3 is the pivot element, using the elementary row operations replacing u_2 with x_2 in simplex table - 1, we get,

Simplex Table - 2

	Basis	c_B	x_1 (EV)	x_2	u_1	u_2	u_3	B	
LV ←	u_1	0	PE $\boxed{-5/3}$	0	1	–1/3	0	–1	← PR
	x_2	–1	4/3	1	0	–1/3	0	2	
	u_3	0	5/3	0	0	–2/3	1	1	
	C_j		–2	–1	0	0	0		
	Z_j		–4/3	–1	0	1/3	0		
	$C_j - Z_j$		–2/3	0	0	–1/3	0		

Since all $C_j - Z_j \leq 0$ the solution is optimal but $u_1 = -1$ is the only negative value hence u_1 becomes the leaving variable.

Table to obtain the ratio to decide the entering variable

	x_1	x_2	u_1	u_2	u_3
$C_j - Z_j$ *row*	$-2/3$	0	0	$-1/3$	0
Pivot row	$-5/3$	0	1	$-1/3$	0
Ratio	$2/5$	–	0	1	–

x_1 is the entering variable as it's ratio is least/minimum.

Simple Table - 3

Replacing u_1 with x_1 and making PE as 1 by row operations

Basis	c_B	x_1	x_2	u_1	u_2	u_3	B
x_1	-2	1	0	$-3/5$	$1/5$	0	$3/5$
x_2	-1	0	1	$4/5$	$-3/5$	0	$6/5$
u_3	0	0	0	1	-1	1	0
C_j		-2	-1	0	0	0	
Z_j		-2	-1	$2/5$	$1/5$	0	
$C_j - Z_j$		0	0	$-2/5$	$-1/5$	0	

Since all $C_j - Z_j \leq 0$ and the variables of the basis ≥ 0 (non-negative) the solution is optimal and feasible.

Hence, $x_1 = 3/5$, $x_2 = 6/5$

$Z_{max} = -12/5$ or $Z_{min} = 12/5$ is the solution.

13. ***Use dual simplex method to solve the following problem.***

Minimize $Z = 2x_1 + x_2 + 3x_3$

Subject to $x_1 - 2x_2 + x_3 \geq 4,\ 2x_1 + x_2 + x_3 \leq 8,\ x_1 - x_3 \geq 0$

and with all variables non negative.

Solution:

As the given problem is minimization, convert into maximization (standard form)

$Z_{max} = -2x_1 - x_2 - x_3$

Expressing the constraints in $\leq$ form and adding the slack variables; the problem becomes

$Z_{max} = -2x_1 - x_2 - 3x_3 + 0u_1 + 0u_2 + 0u_3$

Subject to $-x_1 + 2x_2 - x_3 + u_1 = -4$

$$2x_1 + x_2 + x_3 + u_2 = 8$$
$$-x_1 + x_3 + u_3 = 0$$

With all variables non negative.

First Simplex Table

↓ EV

Basis	C_B	x_1	x_2	x_3	u_1	u_2	u_3	B
LV ← u_1	0	$\boxed{-1}$ PE	2	−1	1	0	0	−4 ← PR
u_2	0	2	1	1	0	1	0	8
u_3	0	−1	0	1	0	0	1	0
C_j		−2	−1	−3	0	0	0	–
Z_j		0	0	0	0	0	0	0
$C_j - Z_j$		−2	−1	−3	0	0	0	–

↑ PC

Since all the $(C_j - Z_j)$ values are ≤ 0, the above solution is optimal. However, it is infeasible because it has a non positive value for the basic variable u_1 (= −4). Since u_1 is the only negative variable, it becomes the leaving variable (LV).

Table to obtain the ratio

	x_1	x_2	x_3	u_1	u_2	u_3
$C_j - Z_j$ row	−2	−1	−3	0	0	0
Pivot row	−1	2	−1	1	0	0
Ratio	2	–	3	–	–	–

As x_1 has the smallest ratio, it becomes the entering variable (EV).

Thus, the element −1 becomes the pivot element. Using the elementary row operations, we get,

Basis	C_B	x_1	x_2	x_3	u_1	u_2	u_3	B
x_1	−2	$\boxed{1}$ PE	−2	1	−1	0	0	← PR 4
u_2	0	0	5	−1	2	1	0	0
u_3	0	0	−2	2	−1	0	1	4
C_j		−2	−1	−3	0	0	0	–
Z_j		−2	4	−2	2	0	0	−8
$C_j - Z_j$		0	−5	−1	−2	0	0	–

Since all the variables have non negative values ($x_1 = 4$, $u_2 = 0$, $u_3 = 4$), the above solution is optimal and feasible. The optimal and feasible solution is, $x_1 = 4$, $x_2 = 0$, $x_3 = 0$ with Z min. = 8

[Substituting the values of x_1, x_2 and x_3, in the original objective function]

14. *Use dual simplex method to solve the LPP*

Max. $Z = -3x_1 - 2x_2$

Subject to $x_1 + x_2 \geq 1$, $x_1 + x_2 \leq 7$, $x_1 + 2x_2 \geq 10$, $x_2 \leq 3$ *and* $x_1, x_2 \geq 0$

Solution:

Interchanging the $\geq$ inequality of the constraints into $\leq$, the given LPP becomes.

Max. $Z = -3x_1 - 2x_2$

Subject to $-x_1 - x_2 \leq -1$

$x_1 + x_2 \leq 7$

$-x_1 - 2x_2 \leq -10$

$x_2 \leq 3$

By introducing the slack variables u_1, u_2, u_3, u_4, the standard form of the LPP becomes

Max $Z = -3x_1 - 2x_2 + 0u_1 + 0u_2 + 0u_3 + 0u_4$

Subject to $-x_1 - x_2 + u_1 = -1$

$x_1 + x_2 + u_1 = 7$

$-x_1 - 2x_2 + u_3 = -10$

$0x_1 + x_2 + u_4 = 3$

First Simplex Table

↓ EV

Basis	C_B	x_1	x_2	u_1	u_2	u_3	u_4	B
u_1	0	−1	−1	1	0	0	0	−1
u_2	0	1	1	0	1	0	0	7
LV ← u_3	0	−1	$\boxed{-2}$ PE	0	0	1	0	−10 ← PR
u_4	0	0	1	0	0	0	1	3
C_j		−3	−2	0	0	0	0	–
Z_j		0	0	0	0	0	0	0
$C_j - Z_j$		−3	−2	0	0	0	0	–

↑PC

Since all $(C_j - Z_j) \leq 0$ the solution is optimal but as $x_{Bi} \leq 0$ it is not feasible. $u_3 = -10$ is the most negative, the basic variable u_3 leaves the basis.

Table to obtain the ratio

	x_1	x_2	u_1	u_2	u_3	u_4
$C_j - Z_j$ row	– 3	– 2	0	0	0	0
Pivot row	– 1	– 2	0	0	1	0
Ratio	3	1	–	–	–	–

As x_2 has the smallest ratio, it becomes the entering variable (EV). The element – 2 is the pivot element, using elementary row operations, we get,

Second Simplex Table

↓ EV

Basis	C_B	x_1	x_2	u_1	u_2	u_3	u_4	B
u_1	0	– 1 / 2	0	1	0	– 1 / 2	0	4
u_2	0	1 / 2	0	0	1	1 / 2	0	2
x_2	– 2	1 / 2	1	0	0	– 1 / 2	0	5
LV ← u_4	0	$\boxed{-1/2}$ PE	0	0	0	1 / 2	1	– 2 ← PR
C_j		– 3	– 2	0	0	0	0	–
Z_j		– 1	– 2	0	0	0	0	–
$C_j - Z_j$		– 2	0	0	0	– 1	0	–

↑PC

$u_4 = -2$ is only the negative value. Hence, u_4 becomes leaving variable x_1 enters or is entering variable.

Table to obtain the ratio

	x_1	x_2	u_1	u_2	u_3	u_4
$C_j - Z_j$ row	– 2	0	0	0	– 1	0
Pivot row	– 1 / 2	0	0	0	1/2	0
Ratio	4	0	–	–	–	–

Third Simplex Table

Basis	C_B	x_1	x_2	u_1	u_2	u_3	u_4	B
u_1	0	0	0	1	0	−1	−1	2
u_2	0	0	0	0	1	1	1	0
x_2	−2	0	1	0	0	0	1	3
x_1	−3	$\boxed{1}$ PE	0	0	0	−1	−2	4 ← PR
C_j		−3	−2	0	0	0	0	–
Z_j		−3	−2	0	0	3	4	−18
$C_j - Z_j$		0	0	0	0	−3	−4	–

As all $C_j - Z_j \leq 0$ and all $x_{Bi} \geq 0$, the solution is optimal and feasible.

The optimal feasible solution is,

Max $Z = -18$ with $x_1 = 4$, $x_2 = 3$

Review Questions

1. *Give the characteristics of a dual problem.*
2. *Mention the advantages of duality.*
3. *Explain the economic interpretation of duality with an example.*
4. *Explain the relation between the solution of primal and dual.*
5. *Briefly explain economic interpretation of duality.*
6. *Show that dual of dual is primal.*

   Hint Consider an LPP, write its dual and again write the dual of the obtain dual. This dual will be same as that of the original LPP considered in the beginning.
7. *Define duality in LPP and write important characteristics of it.*
8. *Explain the following:*

   i. The essence of duality theory *ii. Primal–Dual relationship*
9. *Explain upper bound technique used to solve a LPP.*
10. *Write a note on*

    i) Weak duality property *ii) Strong duality property*

    iii) Complementary solutions property *iv) Complementary optimal solutions property*

PROBLEMS

1. *Write the dual of the following LPP*

$Z_{max} = 3x_1 - x_2 + x_3$

Subject to the constraints $4x_1 - x_2 \le 8$, $8x_1 + x_2 + 3x_3 \ge 12$ and $5x_1 - 6x_2 \le 13$

Ans: $Z^*_{min} = 8w_1 - 12w_2 + 13w_3$

Subject to the constraints

$4w_1 - 8w_2 + 5w_3 \ge 3$, $-w_1 - w_2 + 0w_3 \ge -1$, $w_1 - 3w_2 + 6w_3 \ge 1$ and $w_1, w_2, w_3 \ge 0$

2. *Construct the dual of the LPP*

$Z_{max} = 4x_1 + 6x_2 + 18x_3$

Subject to the constraints $x_1 + 3x_2 \ge 3$ and $x_1 + 2x_3 \ge 5$

Ans: $Z^*_{min} = 3w_1 + 5w_2$

Subject to the constraints

$w_1 \le 4$, $3w_1 + w_2 \le 6$, $2w_2 \le 18$ and $w_1, w_2 \ge 0$

3. $Z_{max} = 3x_1 + 17x_2 + 9x_3$

Subject to the constraints $x_1 - x_2 + x_3 \ge 3$ and $-3x_1 + 2x_3 \le 1$

Write the dual for the above LPP

Ans: $Z^*_{min} = -3w_1 + w_2$

Subject to the constraints $w_1 - 3w_2 \ge 3$, $-w_1 \ge 17$, $w_1 + 2w_2 \ge 9$ and $w_1, w_2 \ge 0$

4. *Write dual of the following LPP*

$Z_{max} = 6x_1 + 10x_2$

Subject to $3x_1 + 2x_2 \le 18$, $x_1 \le 14$, $x_2 \le 16$ and $x_1, x_2 \ge 0$

Ans: $Z^*min = 14w_1 + 16w_2 + 18w_3$

Subject to $w_1 + 3w_3 \ge 6$, $w_2 + 2w_3 \ge 10$ and $w_1, w_2, w_3 \ge 0$

5. ***Use dual simplex method to solve Max. $Z = -3x_1 - x_2$ subject to $x_1 + x_2 \ge 1$, $x_1 + 3x_2 \ge 2$, $x_1, x_2 \ge 0$.***

Ans: ***Max $Z = -1$, $x_1 = 0$, $x_2 = 1$***

6. ***Using dual simplex method solve the following LPP***

Min $Z = 10x_1 + 6x_2 + 2x_3$

Subject to $-x_1 + x_2 + x_3 \ge 1$, $3x_1 + x_2 - x_3 \ge 2$, $x_1, x_2, x_3 \ge 0$

Ans: $x_1 = 1/4$, $x_2 = 5/4$,

$x_3 = 0$, $Z_{min} = 10$

7. ***Use dual simplex method and solve***

   Max. $Z = -2x_1 - x_3$

   Subject to $x_1 + x_2 - x_3 \geq 5$, $x_1 - 2x_2 + 4x_3 \geq 8$, $x_1, x_2, x_3 \geq 0$

Ans: $x_1 = 0$, $x_2 = 14$, $x_3 = 9$ and $Z_{max} = -9$

8. ***Using dual simplex method solve the LPP***

   Maximize $Z = -3x_1 - 2x_2$

   Subject to $x_1 + x_2 \geq 1$, $x_1 + x_2 \geq 7$, $x_1 + 2x_2 \geq 10$, $x_2 \geq 3$ and $x_1, x_2 \geq 0$

Ans: $x_1 = 4$, $x_2 = 3$ and $Z_{max} = -18$

9. ***Using dual simplex method solve,***

   $Z_{min} = 2x_1 + x_2$

   Subject to $3x_1 + x_2 \geq 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \geq 3$ where $x_1, x_2 \geq 0$.

Ans: $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, and $Z_{min} = \frac{12}{5}$

10. ***Find the maximum of*** $Z = 6x_1 + 8x_2$

    Subject to $5x_1 + 2x_2 \leq 20$, $x_1 + 2x_2 \leq 10$, x_1 and $x_2 \geq 2$

    by using dual simplex method

Ans: $x_1 = 5/2$, $x_2 = 15/4$ and $Z_{max} = 45$