FUTURE VISION BIE

One Stop for All Study Materials

& Lab Programs

Future Vision

By K B Hemanth Raj

Scan the QR Code to Visit the Web Page

Or

Visit : https://hemanthrajhemu.github.io

**Gain Access to All Study Materials according to VTU,
CSE – Computer Science Engineering,
ISE – Information Science Engineering,
ECE - Electronics and Communication Engineering
& MORE...**

4

Transportation and Assignment Problems

Learning objectives

After Studying this chapter, you should be able to

- formulate a transportation problem
- obtain initial basic feasible solutions and optimal solution
- handle various types of transportation/assignment problems
- formulate an assignment problem and solve it by Hungarian method
- find solution(optimal and feasible) to a travelling salesman problem

4.1 Introduction

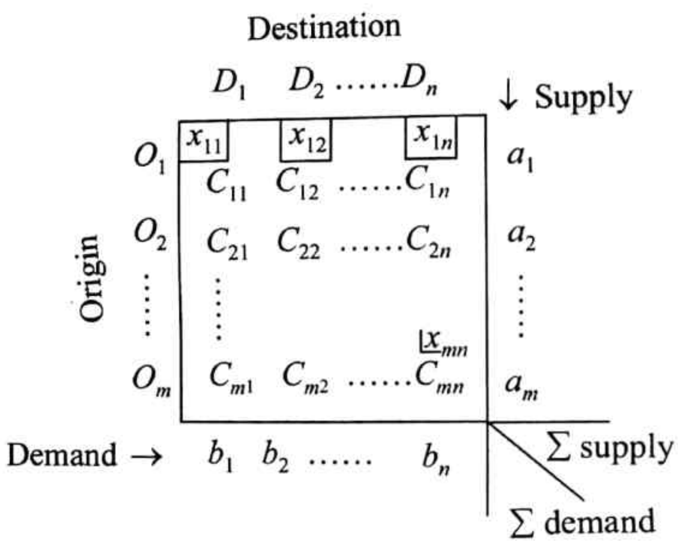
Some linear programming problems involving several product sources and destinations may be solved by a simplified version of simplex method. The transportation problem is one of the sub classes of LPP in where, the objective is to transport various quantities of a single homogeneous commodity, that are initially stored at various **origins** to different **destinations** in such a way that the total transportation cost is minimum. To achieve this objective it is required to know the demand and supply at destinations (market places) and sources/origins respectively. In addition, it is also required to known the unit transportation cost from source to destination. The origins/source is nothing but the place where a commodity is produced/stored. It may be a factory or a warehouse. The destination means the place where it is required, it may be a market place or a show room.

4.2 Formulation of a Transportation Problem

The transportation problem is generally stated in the form of cost matrix as shown in the table. There are 'm' origins (O_1, O_2... O_m) and 'n' destinations (D_1, D_2... D_n). The cost per unit is C_{ij}, transportation cost from i^{th} origin to j^{th} destination.

a_1, a_2... a_m and b_1,b_2... b_n are called rim conditions or supply and demand conditions.

x_{11},x_{12}... x_{mn} are the number of units that are to be transported from origins to destinations.



4.3 Initial Basic Feasible Solution (IBFS)

Basic feasible solution is one in which all the units produced at the sources, are transported to the destinations or market places. (The solution which satisfies the rim conditions).

There are several methods to obtain the basic feasible solution for a balanced transportation problem. They are,

i. *North–West corner method* *ii. Least–cost method*

iii. *Matrix minima method and*

iv. *Vogel's Approximation Method (VAM)*

To obtain an Initial Basic Feasible Solution by the above methods

i) ***Formulate the transportation table and***

ii) ***Check for $\sum$ Supply = $\sum$ Demand***

When $\sum$ Supply = $\sum$ Demand, the problem is stated as Balanced Transportation Problem, only for which Initial Basic Feasible Solution is obtained.

4.4 Applications of Transportation Problems

Following are the few applications of transportation problems in general.

i. *To minimize the shipping cost of the products from the origin to the destination.*

ii. *To determine the lowest cost location for a new firm / office / outlet.*

iii. *In Miltary distribution system.*

iv. *To maximize profit of an organization.*

v. *To minimize the delay cost of processing checks.*

Worked Examples

1. *Obtain an Initial basic feasible solution to the transportation table shown below, the elements of the matrix indicates cost in rupees.*

Destination

Origin		D_1	D_2	D_3	
	O_1	2	7	4	5
	O_2	3	3	1	8
	O_3	5	4	7	7
	O_4	1	6	2	14
		8	8	18	

Solution:

Before using / applying the above mentioned methods check for balance.

That is, $\sum$ Supply = $\sum$ Demand

$\sum$ Supply = 5 + 8 + 7 +14= 34 (sum of column rim)

$\sum$ Demand = 8+8+18 = 34 (sum of row rim)

Method 1 : North West Corner rule

Step 1

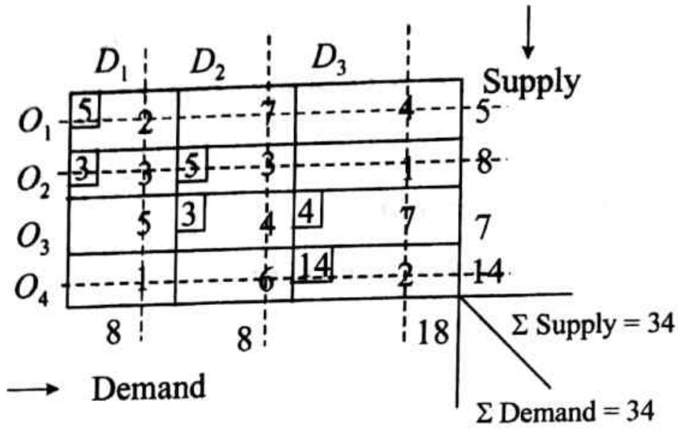
Identify the North West element in the given cost matrix and allocate as many units as possible (subject to / satisfying the row rim and column rim of that particular location).

Step 2

5 Units can be allocated to the cost cell 2, by this allocation first origin (O_1) is satisfied. Hence cancel the first row.

Step 3

Now the north west element is 3 for which maximum of 3 units can be allocated and the first demand (D_1) is satisfied, hence cancel the first column.



Step 4

Repeat the above steps until all the rim conditions are satisfied. Then, the basic feasible solution is,

Origin	Destination	Number of units	Cost	
O_1	D_1	5	10	(that is 5 × 2, 3 × 3 and so on)
O_2	D_1	3	9	
O_2	D_2	5	15	
O_3	D_2	3	12	
O_3	D_3	4	28	
O_4	D_3	14	28	
			₹102/–	

That is, $5 \times 2 + 3 \times 3 + 5 \times 3 + 3 \times 4 + 4 \times 7 + 14 \times 2 =$ ₹102/–.

Method 2 : Least Cost Method

Step 1

Identify the least element (least cost) in the entire matrix and allocate to it as many units as possible.

In the given matrix, the least element is 1 (as it occurs twice, choose in first come first serve order). Thus, after allocation of 8 units to the least cost cell second origin and first destination are satisfied and hence canceling it we get,

In the given matrix, the least element is 1 (as it is repeated, the choice is arbitrary twice, any one may be considered).

Step 2

The next least element is 2, allocating 6 units to it, 4th row is satisfied. Canceling the satisfied row rim condition we get,

Step 3

After each cancellation, find the least element in the remaining matrix and continue the procedure.

	D_1	D_2	D_3	
O_1	2	$\boxed{1}$ 7	$\boxed{4}$ 4	5
O_2	3	3	$\boxed{8}$ 1	8
O_3	5	$\boxed{7}$ 4	7	7
O_4	$\boxed{8}$ 1	6	$\boxed{6}$ 2	14
	8	8	18	

Then we get,

Origin	Destination	Units	Cost
O_1	D_2	1	7
O_1	D_3	4	16
O_2	D_3	8	8
O_3	D_2	7	28
O_4	D_1	8	8
O_4	D_3	6	12
			Rs 79

That is, $1 \times 7 + 4 \times 4 + 8 \times 1 + 7 \times 4 + 8 \times 1 + 6 \times 2 =$ Rs.79/–

Method 3 : Matrix Minima Method

This method consists of two sub methods listed and discussed hereunder.

i. *Row Minima Method*

Step 1

Select the minimum element in each row

Step 2

Allocate as many units as possible to subject to the rim conditions.

Step 3

Continue the procedure until all the rim conditions are satisfied.

	D_1	D_2	D_3	
O_1	$\boxed{5}$ 2	7	4	5
O_2	3	3	$\boxed{8}$ 1	8
O_3	5	$\boxed{7}$ 4	7	7
O_4	$\boxed{3}$ 1	$\boxed{1}$ 6	$\boxed{10}$ 2	14
	8	8	18	

Now, the cost is $5 \times 2 + 8 \times 1 + 7 \times 4 + 3 \times 1 + 1 \times 6 + 10 \times 2$

$= 10 + 8 + 28 + 3 + 6 + 20 =$ ₹75/–

ii. *Column Minima Method*

Step 1

Select the minimum element in each column.

Step 2

Allocate as many units as possible to satisfy the rim condition.

Step 3

Continue the procedure until all the rim conditions are satisfied.

	D_1	D_2	D_3	
O_1	2	7	$\boxed{5}$ 4	5
O_2	3	$\boxed{8}$ 3	1	8
O_3	5	4	$\boxed{7}$ 7	7
O_4	$\boxed{8}$ 1	6	$\boxed{6}$ 2	14
	8	8	18	

The cost is, $5 \times 4 + 8 \times 3 + 7 \times 7 + 8 \times 1 + 6 \times 2$

$= 20 + 24 + 49 + 8 + 12$

$=$ ₹113/–

Method 4 : Vogel's Approximation Method (VAM) or Penaulty Method

Step 1

Get the difference between the least and next least element of each row and column and write it against that particular row/column.

Step 2

Identify the highest difference among these differences and corresponding to this highest difference select the least element in the respective row or column.

Step 3

Now, to this element allocate as many units as possible and cancel out the respective rim condition.

Step 4

Find the differences for the remaining matrix and continue the procedure until all the rim conditions are satisfied.

The highest difference in the remaining matrix is 2, and it corresponds to second row. In this row 1 is the least element; allocating 8 units that are possible to it, cancel the second row rim as shown.

Now the cost is, $5 \times 2 + 8 \times 1 + 7 \times 4 + 3 \times 1 + 1 \times 6 + 10 \times 2$

$$= 10 + 8 + 28 + 3 + 6 + 20$$

$$= ₹\ 75/–$$

	D_1	D_2	D_3					
O_1	$\boxed{5}$ 2	7	4	5	(2)			
O_2	3	3	$\boxed{8}$ 1	8	(2)	(2)		
O_3	5	$\boxed{7}$ 4	7	7	(1)	(1)	(1)	(1)
O_4	$\boxed{3}$ 1	$\boxed{1}$ 6	$\boxed{10}$ 2	14	(1)	(1)	(1)	(5)
	8	8	18					
	(1)	(1)	(1)					
	(2)	(1)	(1)					
	(4)	(2)	(5)					
	(4)	(2)	–					

Some special cases while finding difference (between least and next lest element) in VAM

i. ***Whenever least element is repeated, then difference against that particular column/row is zero.***

Example

a. *If the row to find the difference is*

2	2	7

Then the difference is '0'

b. *In the same way, for the column*

3
3
8

the difference is '0'

ii. ***Whenever there is tie with the highest difference obtained is same/repeated, then select a row/ column for allocation having least element in that particular row / column.***

Example

2	5	7	(3)
1	6	4	(3)

Prefer for allocation as least element (cost) exists though the difference is same.

Note: Vogel's Approximation Method (VAM) gives the best starting solution compared to any other methods discussed. Sometimes, Basic Feasible solution obtained by VAM itself will be optimum, or optimum solution can be reached in minimum steps. Hence, it is preferred to find the basic feasible solution while finding the optimum solution of a transportation problem unless otherwise specified.

4.5 Optimality Check

The solution obtained by the methods discussed is only a basic feasible solution which satisfies the supply/demand conditions. It may or may not be an optimum solution. To check the optimality we have the following methods.

i. ***Modified Distribution Method (MODI) / U–V Method***

ii. ***Stepping Stone Method***

2. *Larsen and Toubro Construction Company needs 3, 3, 4 and 5 million cubic feet of fill at four earthen dam sites I, II, III and IV in Karanataka. It can transfer the fill from three mounds A, B and C where 2, 6 and 7 million cubic feet of fill is available respectively. Costs of transporting one million cubic feet of fill from mounds to the four sites in lakhs are given in the following table.*

		To			
		I	*II*	*III*	*IV*
From	*A*	*15*	*10*	*17*	*18*
	B	*16*	*13*	*12*	*13*
	C	*12*	*17*	*20*	*11*

Determine the optimum transportation plan, which minimizes the total transportation cost to the company.

Solution:

The transportation table is, $\sum$ Supply = $\sum$ Demand

Hence, VAM can be applied to find initial basic feasible solution.

15	[2] 10	17	18	2	(5)			
16	[1] 13	[4] 12	[1] 13	6	(1)	(1)	(0)	(0)
[3] 12	17	20	[4] 11	7	(1)	(1)	(1)	(6)
3	3	4	5					
(3)	(3)	(5)	(2)					
(4)	(4)	(8)	(2)					
(4)	(4)		(2)					
	(4)	–	(2)					

The initial basic feasible solution is

$$2 \times 10 + 1 \times 13 + 4 \times 12 + 1 \times 13 + 3 \times 12 + 4 \times 11 = ₹.174 \text{ lakhs}$$

Optimality Check (Using U–V / MODI Method)

To perform optimality test, the prime condition is m + n – 1 = number of allocations

where, m = number of rows, n = number of columns in this case, m = 3, n = 4 Hence, m + n – 1 = 3 + 4 –1 = 6 Number of allocations = 6

Thus, optimality check can be performed for the given problem.

	$v_1 = 14$	$v_2 = 13$	$v_3 = 12$	$v_4 = 13$	
$u_1 = -3$	15	$\boxed{2}$ 10	17	18	2
$u_2 = 0$	16	$\boxed{1}$ 13	$\boxed{4}$ 12	$\boxed{1}$ 13	6
$u_3 = -2$	$\boxed{3}$ 12	17	20	$\boxed{4}$ 11	7
	3	3	4	5	

i. ***Computation of variables u's and v's***

For all occupied cells

Let $u_2 = 0$ (the variable with highest number of allocations) and on solving we get

$u_1 + v_2 = 10$
$u_2 + v_2 = 13$
$u_2 + v_3 = 12$
$u_2 + v_4 = 13$
$u_3 + v_1 = 12$
$u_3 + v_4 = 11$

$u_1 = -3$
$u_3 = -2$
$v_1 = 14$
$v_2 = 13$
$v_3 = 12$
$v_4 = 13$

ii. ***Computation of net cell evaluations***

For all unoccupied cells calculate $C_{ij} - (u_i + v_j)$

For 15, $15 - (-3 + 14) = 4$

17, $17 - (-3 + 12) = 8$

18, $18 - (-3 + 13) = 8$

16, $16 - (0 + 14) = 2$

17, $17 - (-2 + 13) = 6$

20, $20 - (-2 + 12) = 10$

Since all $C_{ij} - (u_i + v_j) \geq 0$ the solution is optimum

The optimum solution is,

$2 \times 10 + 1 \times 13 + 4 \times 12 + 1 \times 13 + 3 \times 12 + 4 \times 11 =$ ₹.174 lakhs

It is observed that the initial basic feasible solution obtain by VAM itself is optimum.

4.6 Steps in Solving a Transportation Problem

Stage 1 Finding IBFS

i. *Formulate transportation table.*

ii. *Check for Σ Supply = Σ Demand*

iii. *Find Initial basics feasible solution*

Stage 2: Optimal solution

i. *Check for m + n − 1 = number of allocations*

ii. *Using MODI/stepping stone method obtain net cell evaluations and for optimality, all these evaluations must be greater than or equal to zero.*

3. *Solve the following transportation problem in which cell entries represent the unit costs (in lakhs of rupees) of transportation with usual notations.*

2	7	4	5
3	3	1	8
5	4	7	7
1	6	2	14
7	9	18	

Solution:

The given problem is balanced transportation problem as Σ supply = Σ demand (= 34)
Hence, it's basic feasible solution can be obtained, by using VAM.

$\boxed{5}$ 2	7	4	5	(2)	(2) ←		
3	3	$\boxed{8}$ 1	8	(2)	— ←		
5	$\boxed{7}$ 4	7	7	(1)	(1)	(1)	(1)
$\boxed{2}$ 1	$\boxed{2}$ 6	$\boxed{10}$ 2	14	(1)	(1)	(1)	(5) ←
7	9	18					
(1)	(1)	(1)					
(1)	(2)	(2)					
(4)	(2)	(5) ↑					
(4)	(2)	—					

IBFS cost $= 5 \times 2 + 8 \times 1 + 7 \times 4 + 2 \times 1 + 2 \times 6 + 10 \times 2$
$= 10 + 8 + 28 + 2 + 12 + 20 =$ ₹ 80/– lakhs

Optimality check

The prime condition is m + n − 1 = number of allocations is satisfied. (m = 4, n = 3 and number of allocations = 6).

Hence, optimality check can be performed.

	$v_1 = 1$	$v_2 = 6$	$v_3 = 2$	
$u_1 = 1$	$\boxed{5}$ 2	7	4	5
$u_2 = -1$	3	3	$\boxed{8}$ 1	8
$u_3 = -2$	5	$\boxed{7}$ 4	7	7
$u_4 = 0$	$\boxed{2}$ 1	$\boxed{2}$ 6	$\boxed{10}$ 2	14
	7	9	18	

For all occupied cells, calculate $C_{ij} - (u_i + v_j)$ (net cell evaluations)

i. ***Computation of variables u's and v's***

For all occupied cells form mathematical equations as below	Let $u_4 = 0$ (The variable with the highest number of allocations) and on solving
$u_1 + v_1 = 2$	$u_1 = 1$
$u_2 + v_3 = 1$	$u_2 = -1$
$u_3 + v_2 = 4$	$u_3 = -2$
$u_4 + v_1 = 1$	$v_1 = 1$
$u_4 + v_2 = 6$	$v_2 = 6$
$u_4 + v_3 = 2$	$v_3 = 2$

ii. ***Computation net cell evaluations***

For all un–occupied cells, $C_{ij} - (u_i + v_j)$

For $\quad 7 - (1 + 6) = 0 \qquad 4 - (1 + 2) = 1$

$3 - (-1 + 1) = 3 \qquad 3 - (-1 + 6) = -2$

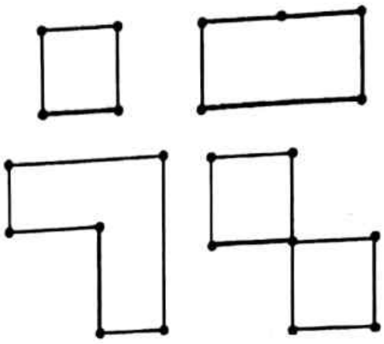
$5 - (-2 + 1) = 6 \qquad 7 - (-2 + 2) = 7$

As all $C_{ij} - (u_i + v_j)$ are not ≥ 0 the solution is not optimum.

Moving towards the optimality

Form a closed loop starting and ending at the negative net cell evaluation that is from 3 [C_{22}] adjusting x by adding, subtracting alternatively x units at the corners of the loop.

Note: *The loop may be of square or rectangle or ~~any other shape~~ having allocation only at each turn / corner. The possible shapes may be as shown.*



Following the above we get,

[5] 2	7	4	5
3	[x] 3	[8–x] 1	8
5	[7] 4	7	7
[2] 1	[2–x] 6	[10 +x] 2	14
7	9	18	

Let 8 – x = 0 and 2 – x = 0, x = min (8, 2) = 2

Put x = 2 and re–write the table

	$v_1 = 0$	$v_2 = 3$	$v_3 = 1$	
$u_1 = 2$	[5] 2	7	4	5
$u_2 = 0$	3	[2] 3	[6] 1	8
$u_3 = 1$	5	[7] 4	7	7
$u_4 = 1$	[2] 1	6	[12] 2	14
	7	9	18	

Again check for optimality,

For all occupied cells form mathematical equations as below	On solving the mathematical equations assuming $u_2 = 0$, we get,
$u_1 + v_1 = 2$	$v_2 = 3$
$u_2 + v_2 = 3$	$v_3 = 1$
$u_2 + v_3 = 1$	$u_4 = 1$
$u_3 + v_2 = 4$	$v_1 = 0$
$u_4 + v_1 = 1$	$u_3 = 1$
$u_4 + v_3 = 2$	$u_2 = 0$
	$u_1 = 2$

For all un–occupied cells,

$7 - (2 + 3) = 2 \qquad 4 - (2 + 1) = 1$

$3 - (0 + 0) = 3 \qquad 5 - (1 + 0) = 4$

$7 - (1 + 1) = 5 \qquad 6 - (1 + 3) = 2$

As all $C_{ij} - (u_i + v_j) \geq 0$

The solution is optimum, the optimum cost is,

$5 \times 2 + 2 \times 3 + 6 \times 1 + 7 \times 4 + 2 \times 1 + 12 \times 2 =$ ₹ 76/– lakhs

4. *A company is spending ₹ 1,000 everyday on transportation of its units from three plants to four distribution centers. The supply and demand units with unit cost of transportation are given as*

Distribution Centre

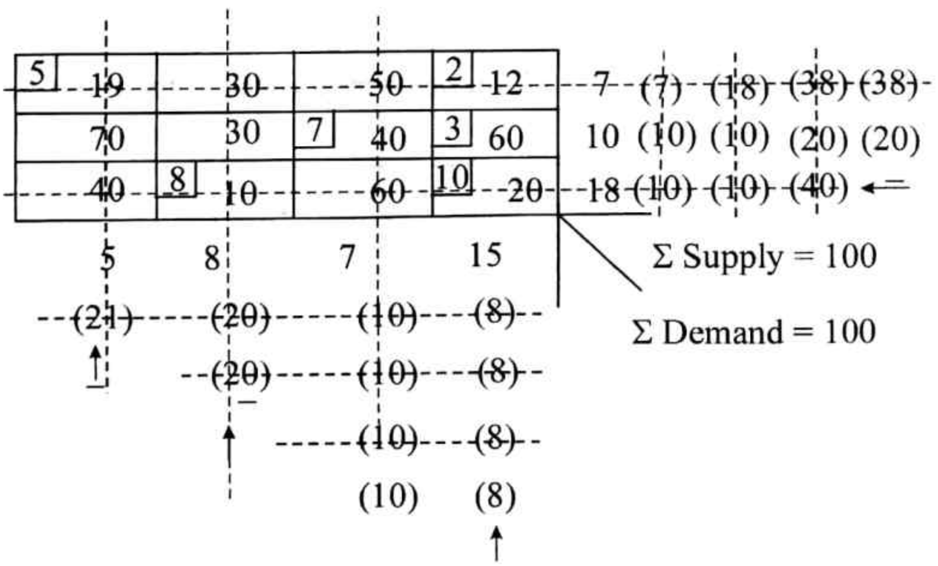
Plant	D_1	D_2	D_3	D_4	*Capacity*
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Demand →	5	8	7	15	

What can be the maximum saving to the company by optimum scheduling?

Solution:

Ignoring the amount that the company is spending, let us find the optimum transportation cost as usual [By using VAM and optimality check]

As Σ supply = Σ demand, VAM can be used to find IBFS.



Optimality check (rewrite the matrix with allocations)

	$v_1 = 19$	$v_2 = 2$	$v_3 = -8$	$v_4 = 12$	
$u_1 = 0$	$\boxed{5}$ 19	30	50	$\boxed{2}$ 12	7
$u_2 = 48$	70	30	$\boxed{7}$ 40	$\boxed{3}$ 60	10
$u_3 = 8$	40	$\boxed{8}$ 10	60	$\boxed{10}$ 20	18
	5	8	7	15	

i. ***Computation of variables u's and v's***

For all occupied cells form mathematical equations as below	On solving the mathematical equations assuming $u_1 = 0$, we get,
$u_1 + v_1 = 19$	$v_1 = 19$
$u_1 + v_4 = 12$	$v_4 = 12$
$u_2 + v_3 = 40$	$v_3 = -8$
$u_2 + v_4 = 60$	$u_2 = 48$
$u_3 + v_2 = 10$	$v_2 = 2$
$u_3 + v_4 = 20$	$u_3 = 8$

ii. ***Computation of net cell evaluations***

For all unoccupied cells,

Let us compute $C_{ij} - (u_i + v_j)$

For 30, $\quad 30 - (0 - 2) = 28 \qquad 50 - (0 + 8) = 58$

$70 - (48 + 19) = 3 \qquad 30 - (48 + 2) = -20$

$40 - (8 + 19) = 13 \qquad 60 - (8 - 8) = 60$

The solution is not optimum as all $C_{ij} - (u_i + v_j)$ are not ≥ 0

Moving towards optimality

Form a closed loop starting from the element corresponding to negative net cell evaluation and find x.

$\boxed{5}$ 19	30	50	$\boxed{2}$ 12	7
70	$\boxed{x}$ 30	$\boxed{7}$ 40	$\boxed{3-x}$ 60	10
40	$\boxed{8-x}$ 10	60	$\boxed{10+x}$ 20	18
5	8	7	15	

Let $\quad 3 - x = 0, \; 8 - x = 0$, which gives $x = 3, 8$, $x = \min(3,8) = 3$

Substituting the value of x in the loop we get,

	$v_1 = 19$	$v_2 = 2$	$v_3 = 12$	$v_4 = 12$	
$u_1 = 0$	$\boxed{5}$ 19	30	50	$\boxed{2}$ 12	7
$u_2 = 28$	70	$\boxed{3}$ 30	$\boxed{7}$ 40	60	10
$u_3 = 8$	40	$\boxed{5}$ 10	60	$\boxed{13}$ 20	18
	5	8	7	15	

Again checking for optimality,

For all occupied cells form mathematical equations as below	On solving the mathematical equations assuming $u_1 = 0$, we get,
$u_1 + v_1 = 19$	$v_1 = 19$
$u_1 + v_4 = 12$	$v_4 = 12$
$u_2 + v_2 = 30$	$u_2 = 28$
$u_2 + v_3 = 40$	$v_3 = 12$
$u_3 + v_2 = 10$	$v_2 = 2$
$u_3 + v_4 = 20$	$u_3 = 8$

For all unoccupied cells, $C_{ij} - (u_i + v_j)$

For 30, $30 - (0 + 2) = 28$ $\quad 50 - (0 + 12) = 38$

$70 - (28 + 19) = 23$ $\quad 60 - (28 + 12) = 20$

$40 - (8 + 19) = 13$ $\quad 60 - (8 + 12) = 40$

As all the net cell evaluations ≥ 0, the solution is optimum. The optimum cost is,

$19 \times 5 + 12 \times 2 + 30 \times 3 + 40 \times 7 + 10 \times 5 + 20 \times 13 = 95 + 24 + 90 + 280 + 50 + 260$

$= ₹\ 799/–$

It is given that the company is spending ₹ 1000/– for transportation. Hence, the savings to the company by using optimization technique is 1000 – 799 = ₹201/– (amount spending by company–optimum cost).

5. *A company has three car manufacturing factories located in cities C_1, C_2, C_3 which can supply cars to four showrooms located in towns T_1, T_2, T_3, T_4. Each plant can supply 6, 1 and 10 truckload of cars daily respectively. The daily requirements of showrooms are 7, 5, 3 and 2 truck loads respectively. The transportation costs per truckload of cars (in thousands of rupees) from each factory to each show room are as follows. Find the optimum distribution schedule and cost.*

2	3	11	7
1	0	6	1
5	8	15	9

Solution:

Let us solve the problem in the following sequence of steps

Step 1:

Formulate the problem as transportation table (TT). is one which contains transportation costs, supply and demand conditions.

O_1 to O_3 are sources/factories

	D_1	D_2	D_3	D_4	Supply
O_1	2	3	11	7	6
O_2	1	0	6	1	1
O_3	5	8	15	9	10
Demand	7	5	3	2	

Σ Supply = 17

Σ Demand = 17

D_1 to D_4 are destinations / showrooms

Step 2:

Check for $\sum$ Supply = $\sum$ Demand Here, $\sum$ Supply = 6 + 1 + 10 = 17

$\sum$ Demand = 7 + 5 + 3 + 2 = 17, the problem is a balanced transportation problem.

Step 3:

Find the initial basic feasible solution by using VAM.

$\boxed{1}$ 2	$\boxed{5}$ 3	11	7	6	(1)	(1)	(5) ←
1	0	6	$\boxed{1}$ 1	1	(1)	–	–
$\boxed{6}$ 5	8	$\boxed{3}$ 15	$\boxed{1}$ 9	10	(3)	(3)	(4)
7	5	3	2				
(1)	(3)	(5)	(6) ↑				
(3)	(5) ↑	(4)	(2)				
(3)	–	(4)	(2)				

IBFS Cost = 1 × 2 + 5 × 3 + 1 × 1 + 6 × 5 + 3 × 15 + 1 × 9

= Rs102/–

Optimality Check (Using U–V / MODI Method)

To perform optimality test, the prime condition is m + n – 1 = number of allocations

where, m = number of rows

n = number of columns

in this case, m = 3, n = 4 Hence, m + n – 1 = 3 + 4 –1 = 6

Number of allocations = 6

Thus, optimality check can be performed for the given problem.

	$v_1 = 5$	$v_2 = 6$	$v_3 = 15$	$v_4 = 9$	
$u_1 = -3$	$\boxed{1}$ 2	$\boxed{5}$ 3	11	7	6
$u_2 = -8$	1	0	6	$\boxed{1}$ 1	1
$u_3 = 0$	$\boxed{6}$ 5	8	$\boxed{3}$ 15	$\boxed{1}$ 9	10
	7	5	3	2	

Step 4:

i. ***Computation of variables u's and v's***

Now, for all occupied cells,

$u_1 + v_1 = 2$ (Sum of u, v = Cost of the respective occupied cell)

$u_1 + v_2 = 3$

$u_2 + v_4 = 1$

$u_3 + v_1 = 5$

$u_3 + v_3 = 15$

$u_3 + v_4 = 9$

Let $u_3 = 0$ (the variable with highest number of allocations)

On solving we get,

$u_1 = -3$ $u_2 = -8$

$v_1 = 5$ $v_2 = 6$, $v_3 = 15$, $v_4 = 9$

Step 5:

ii. ***Computation net cell evaluations***

For all unoccupied cells calculate $C_{ij} - (u_i + v_j)$ [net cell evaluations]

For unoccupied cell 11, $11 - (-3 + 15) = -1$

For unoccupied cell 7, $7 - (-3 + 9) = 1$

For unoccupied cell 1, $1 - (-8 + 5) = 4$

For unoccupied cell 0, $0 - (-8 + 6) = 2$

For unoccupied cell 6, $6 - (-8 + 15) = -1$

For unoccupied cell 8, $8 - (0 + 6) = 2$

If all the net cell evaluations ≥ 0 then the solution is optimum. But it is not so, hence the solution is not yet optimum.

Step 6:

Moving towards the optimality

Form a closed loop starting from the element corresponding to the negative net cell evaluation assigning x units to it. (If only one negative net cell evaluation exists) and from the cell corresponding to most negative net cell evaluation, if more than one negative net cell evaluations are obtained.

When, two negative net cell evaluations are equal any one may be considered to form the loop (arbitrary choice).

$\boxed{1-x}$ 2	$\boxed{5}$ 3	$\boxed{x}$ 11	7	6
1	0	6	$\boxed{1}$ 1	1
$\boxed{6+x}$ 5	8	$\boxed{3-x}$ 15	$\boxed{1}$ 9	10
7	5	3	2	

Allocate x units alternatively adding and subtracting to the element where the loop has started, subtract and add this x where the loop turns / takes the deviation so that the rim conditions are not altered.

Find the value of x as follows

Set $1 - x = 0$ as $x = 0$

$3 - x = 0$ $6 + x$ gives $x = -3$, are meaningless

Then, $x = 1$ or 3

Select x as minimum of the above two values that is $x = 1$. Substitute the value of x in the loop so that the allocations will be modified as,

The loop may be of square or rectangle or any other shape having allocation at each turn / corner.

	$v_1 = 5$	$v_2 = 7$	$v_3 = 15$	$v_4 = 9$	
$u_1 = -4$	2	$\boxed{5}$ 3	$\boxed{1}$ 11	7	6
$u_2 = -8$	1	0	6	$\boxed{1}$ 1	1
$u_3 = 0$	$\boxed{7}$ 5	8	$\boxed{2}$ 15	$\boxed{1}$ 9	10
	7	5	3	2	

Again, check for optimality by u – v method

Let $u_3 = 0$

For all occupied cells we can write

$$u_1 + v_2 = 3$$
$$u_1 + v_3 = 11$$
$$u_2 + v_4 = 1$$
$$u_3 + v_1 = 5$$
$$u_3 + v_3 = 15$$
$$u_3 + v_4 = 9$$

On solving we get,

$$v_2 = 7$$
$$u_1 = -4$$
$$u_2 = -8$$
$$v_1 = 5$$
$$v_3 = 15$$
$$v_4 = 9$$

Now, for all unoccupied cell, compute $C_{ij} - (u_i + v_j)$

For	2,	$2 - (-4 + 5) = 1$
	7,	$7 - (-4 + 9) = 2$
	1,	$1 - (-8 + 5) = 4$
	0,	$0 - (-8 + 7) = 1$
	1,	$1 - (-8 + 15) = -1$
	8,	$8 - (0 + 7) = 1$

As all the net cell evaluations are not greater than or equal to zero the solution is not yet optimum. Again move towards optimality by forming a closed loop from 6 that is from negative net cell evaluation.

2	$\boxed{5}$ 3	$\boxed{1}$ 11	7	6
1	0	$\boxed{x}$ 6	$\boxed{1-x}$ 1	1
$\boxed{7}$ 5	8	$\boxed{2-x}$ 15	$\boxed{1+x}$ 9	10
7	5	3	2	

Set $\quad 1 - x = 0 \qquad 2 - x = 0$

which gives $x = 1, 2 \qquad x = \text{min. of } (1, 2) = 1$

Hence, modified matrix/allocations are,

	$v_1 = 5$	$v_2 = 7$	$v_3 = 15$	$v_4 = 9$	
$u_1 = -4$	2	$\boxed{5}$ 3	$\boxed{1}$ 11	7	6
$u_2 = -9$	1	0	$\boxed{1}$ 6	1	1
$u_3 = 0$	$\boxed{7}$ 5	8	$\boxed{1}$ 15	$\boxed{2}$ 9	10
	7	5	3	2	

Check for optimality by u – v method

For all occupied cells,

$u_1 + v_2 = 3$

$u_1 + v_3 = 11$

$u_2 + v_3 = 6$

$u_3 + v_1 = 5$

$u_3 + v_3 = 15$

$u_3 + v_4 = 9$

Let $u_3 = 0$

on solving we get

$v_2 = 7$

$u_1 = -4$

$u_2 = -9$

$v_1 = 5$

$v_3 = 15$

$v_4 = 9$

For all unoccupied cells calculate $C_{ij} - (u_i + v_j)$

For	2,	$2 - (-4 + 5) = 1$	7,	$7 - (-4 + 9) = 2$
	1,	$1 - (-9 + 5) = 5$	0,	$0 - (-9 + 7) = 2$
	1,	$1 - (-9 + 9) = 1$	8,	$8 - (0 + 7) = 1$

Since all $C_{ij} - (u_i + v_j) \geq 0$ the solution is optimum

The optimum distribution and cost is,

Source	Destination	Number of units	Cost
C_1	T_2	5	15
C_1	T_3	1	11
C_2	T_3	1	6
C_3	T_1	7	35
C_3	T_3	1	15
C_3	T_4	2	18

That is $5 \times 3 + 1 \times 11 + 1 \times 1 + 7 \times 5 + 2 \times 15 + 1 \times 9 =$ ₹ 100×1000

(As it is in thousands of rupees) = ₹1,00,000

4.7 Variations (Special Cases) in a Transportation Problem

Following are the various special cases of a transportation problem.

i. ***Unbalanced transportation problem***

ii. ***Degenerate transportation problem***

iii. ***Restricted transportation problem***

iv. ***Maximization problem***

The above listed cases are explained hereunder.

4.7.1 Unbalanced Transportation Problem

Whenever Σ supply = Σ demand, the problem is stated as balanced transportation problem.

Where as if Σ supply > Σ demand or Σ demand is > Σ supply, then the problem is referred as unbalanced transportation problem.

To balance this a dummy row with zeros as coefficients is introduced whenever Σ demand is > Σ supply that is to meet the demand an artificial origin is assumed to balance the short fall units. On the other hand a dummy column with zeros as coefficients is introduced wheneverΣ supply is > Σ demand that is a dummy market is assumed to balance the excess units produced.

Examples for an Unbalanced Transportation Problem

i. ***Demand > supply***

	D_1	D_2	D_3	Supply
O_1	4	7	8	10
O_2	1	2	4	20
O_3	3	1	5	10
Demand	10	15	25	40 / 50

Σ Supply = 40

Σ Demand = 50

It is observed that Σ supply = 40, Σ demand = 50

It is an unbalanced problem, as we can notice that supply is less than demand, to balance the rim conditions (supply, demand conditions) it is required to increase the supply. Supply can be increased by producing more units, in other words by adding a dummy row and allocating 10 units to it, the problem can be balanced as shown.

	D_1	D_2	D_3	
O_1	4	7	8	10
O_2	1	2	4	20
O_3	3	1	5	10
Du	0	0	0	10
	10	15	25	50 / 50

ii. ***Supply > Demand***

Let us consider the following example

	D_1	D_2	D_3	Supply
O_1	4	6	5	10
O_2	1	5	7	15
O_3	2	7	8	25
Demand	10	20	10	50 / 40

Σ Supply = 50

Σ Demand = 40

Here Σ supply = 50, Σ demand = 40

When supply is more than the demand, it can be balanced by assuming / adding a dummy market that is by adding a dummy column / demand, it can be balanced as shown below,

	D_1	D_2	D_3	Du	
O_1	4	6	5	0	10
O_2	1	5	7	0	15
O_3	2	7	8	0	25
	10	20	10	10	50 / 50

6. *Find initial basic feasible solution to the following transportation problem*

Destination

Origin	D_1	D_2	D_3	D_4	
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	12	12	4	7	70
	85	35	50	45	

Solution:

$\sum$ Supply = 195, $\sum$ demand = 215

To balance the problem add a dummy row with zeros as costs and write the difference in units as its rim condition (215–195 = 20 units for dummy row)

Now the problem is balanced, hence applying VAM to find IBFS

	D_1	D_2	D_3	D_4						
O_1	$\boxed{65}$ 6	$\boxed{5}$ 1	9	3	70	(2)	(2)	(2)		
O_2	11	$\boxed{30}$ 5	$\boxed{25}$ 2	8	55	(3)	(3)	(3)	(3)	(6) ⟵
O_3	12	12	$\boxed{25}$ 4	$\boxed{45}$ 7	70	(3)	(3)	(3)	(3)	(3)
Du	$\boxed{20}$ 0	0	0	0	20	(0)				
	85	35	50	45						
	(6) ↑	(1)	(2)	(3)						
	(5) ↑	(4)	(2)	(4)						
	↑	(4) ↑	(2)	(4)						
		(7) ↑	(2)	(1)						
			(2)	(1)						

IBFS cost = 65 × 6 + 5 × 1 + 30 × 5 + 25 × 2 + 25 × 4 + 45 × 7 + 20 × 0

= 390 + 5 + 150 + 50 + 100 + 315 + 0

= ₹1010/–

As it is mentioned to find only IBFS, it is not required to find optimal solution.

7. *Consider the following unbalanced problem. Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations 1, 2 and 3 respectively. Find the optimal solution.*

	To			
	A	*B*	*C*	*Supply*
1	*5*	*1*	*7*	*10*
2	*6*	*4*	*6*	*80*
3	*3*	*2*	*5*	*15*
	75	20	50	

Solution:

$\sum$Supply = 105, $\sum$Demand = 145

The problem is unbalanced. To balance the problem introduces a 'dummy source' whose transportation costs are given as 5, 3 and 2 respectively (as these are given as penalty costs do not use zeros). Add 145 – 105 = 40 units to 'dummy source'. Now, the modified problem is,

5	1	7	10
6	4	6	80
3	2	5	15
5	3	2	40
75	20	50	

Initial Basic Feasible Solution by using VAM

	A	B	C	Supply			
1	5	[10] 1	7	10	(4)		
2	[60] 6	[10] 4	[10] 6	80	(2)	(2)	(2)
3	[15] 3	2	5	15	(1)	(1)	(1)
Du	5	3	[40] 2	40	(1)	(1)	
	75	20	50				
	(2)	(1)	(3)				
	(2)	(1)	(3)				
	(3)	(2)	(1)				

The IBFS is,

1 – B: 10 units　　2 – A: 60 units　　2 – B : 10 units,

2 – C : 10 units　　3 – A : 15 units　　Du– C : 40 units

Optimality check

m + n – 1 = 4 + 3 – 1 = 6

Number of allocations = 6

Hence, optimality check can be performed

MODI method

	$v_1=6$	$v_2=4$	$v_3=6$	
$u_1=-3$	5	[10] 1	7	10
$u_2=0$	[60] 6	[10] 4	[10] 6	80
$u_3=-3$	[15] 3	2	5	15
$u_4=-4$	5	3	[40] 2	40
	75	20	50	

i) ***For all occupied cells, Let $u_2 = 0$***

$u_1 + v_2 = 1 \;\; = u_1 = 1 - v_2 = -3$

$u_2 + v_1 = 6 = v_1 = 6$

$u_2 + v_2 = 4 = v_2 = 4$

$u_2 + v_3 = 6 = v_3 = 6$

$u_3 + v_1 = 3 = u_3 = 3 - v_1 = -3$

$u_4 + v_3 = 2 = u_4 = 2 - \; v_3 = -4$

ii) ***For all un-occupied cells,***

For $C_{ij} - (u_i + v_j)$

5,	$5 - (-3 + 6) = 2$
7,	$7 - (-3 + 6) = 4$
2,	$2 - (-3 + 4) = 1$
5,	$5 - (-3 + 6) = 2$
5,	$5 - (-4 + 6) = 2$
3,	$3 - (-4 + 4) = 3$

As all net cell evaluations ≥ 0, the solution is optimum. In other words, the solution obtained by 'VAM' itself is optimum.

The optimum distribution and cost

	Units	Cost
1 – B	10	$10 \times 1 = 10$
2 – A	60	$60 \times 6 = 360$
2 – B	10	$10 \times 4 = 40$
2 – C	10	$10 \times 6 = 60$
3 – A	15	$15 \times 3 = 45$
Du – C	40)	(ignore)
		Total cost: Rs. 515

4.7.2 Degenerate Transportation Problem

A basic feasible solution for the general transportation problem must consist of exactly m + n – 1 (number of rows + number of columns –1) positive allocations in independent positions in the transportation table. It may happen some times that the number of occupied cells is lesser than m + n – 1. Such a solution is called a *degenerate solution*. In such cases, the current solution cannot be improved (MODI method cannot applied or optimality check cannot be performed.) Thus, we need to remove the degeneracy to improve the given solution. It can be done by introducing a dummy allocation (→ 0) to a least un–occupied cell which does not form a closed loop. The degeneracy in the transportation problem may occur at two stages.

i. ***It may arise in the first instance where an IBFS is obtained.***

ii. ***At any stage while moving towards optimal solution.***

8. *Solve the following transportation problem for minimization*

Destination

		D_1	D_2	D_3	↓ *Supply*
	O_1	2	2	3	10
Source	O_2	4	1	2	15
	O_3	1	3	1	40
	Demand →	20	15	30	

Solution:

Transportation table is,

2	2	3	10
4	1	2	15
1	3	1	40
20	15	30	

Σ Supply = 65

Σ Demand = 65

$\sum$ Supply = $\sum$ Demand

Hence, VAM can be applied to find basic feasible solution.

$\boxed{10}$ 2	2	3	10	(0)	(1)
4	$\boxed{15}$ 1	2	15	(1)	
$\boxed{10}$ 1	3	$\boxed{30}$ 1	40	(0)	(0)
20	15	30			
(1)	(1)	(1)			
(1)		(2)			

The basic feasible solution cost by VAM,

$10 \times 2 + 15 \times 1 + 10 \times 1 + 30 \times 1 \quad = 20 + 15 + 10 + 30 \quad = ₹\ 75/–$

Stage 2 Optimality check

[10] 2	2	3	10
4	[15] 1	2	15
[10] 1	3	[30] 1	40
20	15	30	

Prime Condition

$m + n - 1$ = number of allocations is to be satisfied, where, m = 3, n = 3 (m, n are number of rows and columns respectively) number of allocations = 4. $m + n - 1 = 5 \neq$ number of allocations.

[10] 2	[ε] 2	3	10
4	[15] 1	2	15
[10] 1	3	[30] 1	40
20	15	30	

Now, $m + n - 1$ = number of allocations. Hence optimality check can be performed either by u – v method or steeping stone method.

Steeping Stone Method: It an alternative method to MODI/UV method to obtain the optimal solution. The optimality check will be performed as explained below.

i. ***For all unoccupied cells, calculate the net cell evaluations directly by forming the closed loop and, adding/subtracting the cost elements at the corners of the loop alternatively.***

For unoccupied cell 3, the net cell evaluation is $3 - 1 + 1 - 2 = 0$

For unoccupied cell 4, the net cell evaluation is $4 - 1 + 2 - 2 = 3$

For unoccupied cell 2, the net cell evaluation is $2 - 1 + 2 - 2 + 1 - 1 = 1$

For unoccupied cell 3, the net cell evaluation is $3 - 1 + 2 - 2 = 2$

ii. ***As all the net cell evaluations are ≥ 0, the solution is optimum. Hence, the optimum cost is***

$10 \times 2 + \varepsilon \times 2 + 15 \times 1 + 10 \times 1 + 30 \times 1$ $(\varepsilon \to 0)$

$20 + 0 + 15 + 10 + 30 =$ ₹75/–

Note:

i. *In stepping stone method directly we get the net cell evaluations as shown above.*

ii. *If the net cell evaluations are not ≥ 0 then, like in u v method move towards optimality.*

If it is not specified to use the stepping stone method, it is adviced to use u v / MODI method to obtain the optimal solution.

4.7.3 Restricted route (Prohibited transportation problem).

In a transportation problem, in some instances due to some practical reasons it may not be possible to transport the goods/units by using particular routes. Such routes are termed as *prohibited routes*. For which no allocation of units should be made.

We write M(a big value) or – (blank space) ∞ (very big value) for the prohibited locations, to avoid the allocations in such cells. While finding the difference between the least and next least element in 'VAM' the particular cell will be ignored (It is treated as ∞ and will never be considered as least or next least element).

9. *Consider the transportation problem having the following table*

Destination

		1	2	3	4	5	
Source	1	8	6	3	7	5	20
	2	5	M	8	4	7	30
	3	6	3	9	6	8	30
(0) Dummy	4	0	0	0	0	0	20
		25	25	20	10	20	

After several iterations a basic feasible solution is obtained that has the following basic variables.

$x_{13} = 20$ $\quad$ $x_{32} = 25$

$x_{21} = 25$ $\quad$ $x_{34} = 5$

$x_{24} = 5$ $\quad$ $x_{45} = 20$

Continue the transportation method for two more iterations and state whether the solution is optimum or not (M indicates prohibited / restricted route).

Solution:

As the allocations are given, there is no need to find IBFS, optimality check (considering the given allocations) can be performed directly.

$m + n - 1 = 4 + 5 - 1 = 8$ Number of allocations = 6

Hence, $m + n - 1 \neq$ number of allocations it is degenerate transportation problem.

	1	2	3	4	5	Supply
1	8	6	$\boxed{20}$ 3	7	5	20
2	$\boxed{25}$ 5	M	8	$\boxed{5}$ 4	7	30
3	6	$\boxed{25}$ 3	9	$\boxed{5}$ 6	8	30
4	0	0	0	0	$\boxed{20}$ 0	20
Demand	25	25	20	10	20	

Σ Supply = 100

Σ Demand = 100

To apply MODI / stepping stone method it should be non–degenerate.

Introducing/considering two dummy allocations ε_1, ε_2 (which tends to zero) for the least un–occupied cells which doesn't form closed loop (If least un–occupied cell is forming a loop, then consider next least unoccupied cell).

		$v_1 = 0$ 1	$v_2 = -4$ 2	$v_3 = 0$ 3	$v_4 = -1$ 4	$v_5 = 0$ 5	
$u_1 = 3$	1	8	6	$\boxed{20}$ 3	7	5	20
$u_2 = 5$	2	$\boxed{25}$ 5	M	8	$\boxed{5}$ 4	7	30
$u_3 = 7$	3	6	$\boxed{25}$ 3	9	$\boxed{5}$ 6	8	30
$u_4 = 0$ (D)	4	$\boxed{\varepsilon_2}$ 0	0	$\boxed{\varepsilon_2}$ 0	0	$\boxed{20}$ 0	20
		25	25	20	10	20	

For all occupied cells form mathematical equations as below

$$u_1 + v_3 = 3$$
$$u_2 + v_1 = 5$$
$$u_2 + v_4 = 4$$
$$u_3 + v_2 = 3$$
$$u_3 + v_4 = 6$$
$$u_4 + v_1 = 0$$
$$u_4 + v_3 = 0$$
$$u_4 + v_5 = 0$$

On solving the mathematical equations assuming $u_4 = 0$, we get,

$$u_1 = 3$$
$$u_2 = 5$$
$$v_4 = -1$$
$$v_2 = -4$$
$$u_3 = 7$$
$$v_1 = 0$$
$$v_3 = 0$$
$$v_5 = 0$$

For all un–occupied cells, $C_{ij} - (u_i + v_j)$

For 8,	$8 - (3 - 0) = 5$
For 6,	$6 - (3 + 3) = 0$
For 7,	$7 - (3 + 1) = 3$
For 5,	$5 - (3 - 0) = 2$
For 8,	$8 - (5 + 0) = 3$
For 7,	$7 - (5 + 0) = 2$
For 6,	$6 - (7 + 0) = -1$
For 9,	$9 - (7 + 0) = 2$
For 8,	$8 - (7 + 0) = 1$

As all, $C_{ij} - (u_i + v_j)$ are not ≥ 0 the solution is not optimum.

Moving towards optimality

8	6	$\boxed{20}$ 3	7	5
$\boxed{25-x}$ 5	M	8	$\boxed{5+x}$ 4	7
$\boxed{x}$ 6	$\boxed{25}$ 3	9	$\boxed{5-x}$ 6	8
$\boxed{\varepsilon_1}$ 0	0	$\boxed{\varepsilon_2}$ 0	0	$\boxed{20}$ 0

Forming a closed loop starting from the cell of negative net cell evaluation, we have $5 - x = 0$, $25 - x = 0$, $x = 5$ that is min of (5, 25)

Substituting the value of x and writing the table.

	$v_1 = 0$	$v_2 = -3$	$v_3 = 0$	$v_4 = -1$	$v_5 = 0$	
$u_1 = 3$	8	6	$\boxed{20}$ 3	7	5	20
$u_2 = 5$	$\boxed{20}$ 5	M	8	$\boxed{10}$ 4	7	30
$u_3 = 6$	$\boxed{5}$ 6	$\boxed{25}$ 3	9	6	8	30
$u_4 = 0$	$\boxed{\varepsilon_2}$ 0	0	$\boxed{\varepsilon_2}$ 0	0	$\boxed{20}$ 0	20
	30	25	20	10	20	

Optimality check

For all occupied cells form mathematical equations as below	On solving the mathematical equations assuming $u_4 = 0$, we get,,
$u_1 + v_3 = 3$	$u_1 = 3$
$u_2 + v_1 = 5$	$u_2 = 5$
$u_2 + v_4 = 4$	$v_4 = -1$
$u_3 + v_1 = 6$	$u_3 = 6$
$u_3 + v_2 = 3$	$v_2 = -3$
$u_4 + v_1 = 0$	$v_1 = 0$
$u_4 + v_3 = 0$	$v_3 = 0$
$u_4 + v_5 = 0$	$v_5 = 0$

For all un–occupied cells, $C_{ij} - (u_i + v_j)$

For 8, $8 - (3 - 0) = 5$

For 6, $6 - (3 + 3) = 0$

For 7, $7 - (3 + 1) = 3$

For 5, $5 - (3 - 0) = 2$

For 8, $8 - (5 + 0) = 3$

For 7, $7 - (5 - 0) = 2$

For 9, $9 - (6 - 0) = 3$

For 6, $6 - (6 - 1) = 1$

For 8, $8 - (6 - 0) = 2$

For 0, $0 - (0 - 3) = 3$

For 0, $0 - (0 - 1) = 1$

As all the net cell evaluation ≥ 0, the solution is optimum. The optimum schedule is,

Source	**Destination**	**Quantity**
1	3	20
2	1	20
2	4	10
3	1	5
3	2	25
4	1	0
4	3	0
4	5	20

Optimum cost is, $20 \times 3 + 20 \times 5 + 10 \times 4 + 5 \times 6 + 25 \times 3 + \varepsilon_1 \times 0 + \varepsilon_2 \times 0 + 20 \times 0$

$$= 60 + 100 + 40 + 30 + 75 \quad = ₹305/–$$

Observation: The solution is optimum after two iterations.

4.7.4 Maximization Problem

When the elements of transportation table (matrix) represents profits instead of cost, the objective is to maximize instead of minimize. This kind of transportation problem is known as maximization transportation problem. To solve this problem, transform the profits to relative costs. This is done by subtracting all the elements from the largest element in the matrix. In this way the largest profit (element) shows a zero relative cost. The solution procedure is same as that of minimization problem but the total profit is calculated using the profit matrix / transportation table.

10. *There are three factories A, B and C supplying goods to four dealers D_1, D_2 D_3 and D_4. The production capacities of these factories are 1000, 700 and 900 units respectively. The requirements from the dealers are 900, 800, 500 and 400 units per month respectively. The per unit return (excluding transportation cost) are ₹8/–, ₹7/– and ₹9/– at the three factories. The following table gives the unit transportation costs from the factories to dealers. Determine the optimum solution to maximize the total returns.*

	D_1	D_2	D_3	D_4
A	2	2	2	4
B	3	5	3	2
C	4	3	2	1

Solution:

Profit = selling price – cost

To obtain the transportation table it is required to subtract the transportation cost from the selling price at the concerned/respective factory. The obtained matrix will be profit matrix which is to be maximized.

The transportation table is, (with profit)

	D_1	D_2	D_3	D_4
A	6	6	6	4
B	4	2	4	5
C	5	6	7	**8**

The above table is obtained by subtracting the transportation costs from the respective profits 8,7 and 9 are profits at A, B and C respectively.

Converting the maximization problem into minimization (by subtracting all the elements from the highest element) and applying VAM to find IBFS we get,

[Σ supply = Σ demand]

$\boxed{200}$ 2	$\boxed{800}$ 2	2	4	1000	(0)	(0)	(0)
$\boxed{700}$ 4	6	4	3	700	(1)	(0)	(2)
3	2	$\boxed{500}$ 1	$\boxed{400}$ 0	900	(1)	(1)	
900	800	500	400				
(1)	(0)	(1)	(3)				
(1)	(0)	(1)					
(2)	(4)						

Optimality Check

m + n – 1 = 3 + 4 – 1 =6, Number of allocations = 5,

Hence, it is a degenerate transportation problem

To perform optimality test, it is required to introduce a dummy allocation $\varepsilon \to 0$ to a least unoccupied cell which doesn't form a closed loop.

	$v_1 = 2$	$v_2 = 2$	$v_3 = 0$	$v_4 = 0$	
$u_1 = 0$	$\boxed{200}$ 2	$\boxed{800}$ 2	$\boxed{\varepsilon}$ 2	4	1000
$u_2 = 2$	$\boxed{700}$ 4	6	4	3	700
$u_3 = 0$	3	2	$\boxed{500}$ 1	$\boxed{400}$ 0	900
	900	800	500	400	

For all occupied cells form mathematical equations as below	On solving the mathematical equations assuming $u_1 = 0$, we get,
$u_1 + v_1 = 2$	$v_1 = 2$
$u_1 + v_2 = 2$	$v_2 = 2$
$u_1 + v_3 = 0$	$v_3 = 0$
$u_2 + v_1 = 4$	$u_2 = 4 - 2 = 2$
$u_3 + v_3 = 0$	$u_3 = 0$
$u_3 + v_4 = 0$	$v_4 = 0$

For all unoccupied cells

Compute $C_{ij} - (u_i + v_j)$

For	4,	$4 - (0 + 0) = 4$
	6,	$6 - (2 + 2) = 2$
	4,	$4 - (2 + 0) = 2$
	3,	$3 - (2 + 0) = 1$
	3,	$3 - (0 + 2) = 1$
	2,	$2 - (0 + 2) = 0$

As all the net cell evaluations ≥ 0, the solution is optimum.

The optimum (in this case maximum, as it is profit) solution is,

$200 \times 6 + 800 \times 6 + \varepsilon \times 6 + 700 \times 4 + 500 \times 7 + 400 \times 8 =$ ₹ 15,500

[While calculating the profit, consider the profit matrix with obtained allocations]

11. *A product is produced by 4 factories F_1, F_2, F_3 and F_4. Their unit production costs are ₹ 2, 3, 1 and 5 respectively. Production capacity of the factories are 50, 70, 30 and 50 units respectively. The product is supplied to 4 stores S_1, S_2, S_3 and S_4, the requirements of which are 25, 35, 105 and 20 respectively. Unit costs of transportation are given below.*

Factory / Store	*1*	*2*	*3*	*4*
A	*2*	*4*	*6*	*11*
B	*10*	*8*	*7*	*5*
C	*13*	*3*	*9*	*12*
D	*4*	*6*	*8*	*3*

Find the transportation plan such that the total production and transportation cost is minimum.

Solution:

It is given that production costs at A, B, C and D are ₹ 2, ₹ 3, Re. 1 and ₹ 5 respectively. Adding these costs to the given unit transportation costs we get,

	1	2	3	4
A	2 + 2	4 + 2	6 + 2	11 + 2
B	10+ 3	8 + 3	7 + 3	5 + 3
C	13+ 1	3 + 1	9 + 1	12 + 1
D	4 + 5	6 + 5	8 + 5	3 + 5

Thus, the total cost matrix with supply, demand conditions is,

				Supply
4	6	8	13	50
13	11	10	8	70
14	4	10	13	30
9	11	13	8	50
25	35	105	20	Σ Supply=200, Σ Demand=185

As demand < supply, let us add a dummy column to balance the problem.

	1	2	3	4	Du	
A	4	6	8	13	0	50
B	13	11	10	8	0	70
C	14	4	10	13	0	30
D	9	11	13	8	0	50
	25	35	105	20	15	

Now, $\sum$supply = $\sum$demand = 200, the problem is balanced. Hence, we can apply any one of the methods (preferably VAM) to find IBFS.

Initial basic feasible solution by VAM.

[25] 4	[5] 6	[20] 8	13	0	50	(4)	(2)	(2)	(2)	(5)	
13	11	[55] 10	8	[15] 0	70	(8)	(2)	(2)	(2)	(2)	(2)
14	[30] 4	10	13	0	30	(4)	(6)				
9	11	[30] 13	[20] 8	0	50	(8)	(1)	(1)	(3)	(5)	(5)
25	35	105	20	15							
(5)	(2)	(0)	(0)	(0)							
(5)	(2)	(2)	(0)								
(5)	(5)	(2)	(0)								
-	(5)	(2)	(0)								
-	-	(3)	(0)								
-	-	(3)	(0)								

Thus, the allocations by VAM

[25] 4	[5] 6	[20] 8	13	0	50
13	11	[55] 10	8	[15] 0	70
14	[30] 4	10	13	0	30
9	11	[30] 13	[20] 8	0	50
25	35	105	20	15	

Initial Basic Feasible Solution :

$25 \times 4 + 5 \times 6 + 20 \times 8 + 55 \times 10 + 15 \times 0 + 30 \times 4 + 30 \times 13 + 20 \times 8$

$= 100 + 30 + 160 + 550 + 0 + 120 + 390 + 160 = ₹\ 1510$

Optimal Solution

$m + n - 1 = 4 + 5 - 1 = 8$, No. of allocations = 8

Hence, the optimality test can be performed. For all occupied cells,

	$v_1=4$	$v_2=6$	$v_3=8$	$v_4=3$	$v_5=-2$	
$u_1 = 0$	[25] 4	[5] 6	[20] 8	13	0	50
$u_2 = 2$	13	11	[55] 10	8	[15] 0	70
$u_3 = -2$	14	[30] 4	10	13	0	30
$u_4 = 5$	9	11	[30] 13	[20] 8	0	50
	25	35	105	20	15	

Let $u_1 = 0$

$u_1 + v_1 = 4 \Rightarrow v_1 = 4$

$u_1 + v_2 = 6 \Rightarrow v_2 = 6$

$u_1 + v_3 = 8 \Rightarrow v_3 = 8$

$u_2 + v_3 = 10 \Rightarrow u_2 = 10 - v_3 = 2$

$u_2 + v_5 = 0 \Rightarrow v_5 = 0 - u_2 = -2$

$u_3 + v_2 = 4 \Rightarrow u_3 = 4 - v_2 = -2$

$u_4 + v_3 = 13 \Rightarrow u_4 = 13 - v_3 = 5$

$u_4 + v_4 = 8 \Rightarrow v_4 = 8 - 5 = 3$

For all un–occupied cells, $C_{ij} - (u_i + v_j)$

13,	$13 - (0 + 3) = 10$
0,	$0 - (0 - 2) = 2$
13,	$13 - (2 + 4) = 7$
11,	$11 - (2 + 6) = 3$
8,	$8 - (2 + 3) = 3$
14,	$14 - (-2 + 4) = 12$
10,	$10 - (-2 + 8) = 4$
13,	$13 - (-2 + 3) = 12$
0,	$0 - (-2 - 2) = 4$
9,	$9 - (5 + 4) = 0$
11,	$11 - (5 + 6) = 0$
0,	$0 - (5 - 2) = -3$

The solution is not optimum as all the net–cell evaluations $\ngeq 0$

Moving towards optimality

[25] 4	[5] 6	[20] 8	13	0	50
13	11	[55] +x 10	8	[15] −x 0	70
14	[30] 4	10	13	0	30
9	11	[30] −x 13	[20]	[x] 0	50
25	35	105	20	15	

Form a closed loop starting from the negative net cell evaluation consider

$15 - x = 0$, $30 - x = 0$, $x = 15$, $x = 30$, $x = \min(15, 30) = 15$, substitute the value of 'x' and re–write the table.

$\boxed{25}$ 4	$\boxed{5}$ 6	$\boxed{20}$ 8	13	0	50
13	11	$\boxed{70}$ 10	8	0	70
14	$\boxed{30}$ 4	10	13	0	30
9	11	$\boxed{15}$ 13	$\boxed{20}$ 8	$\boxed{15}$ 0	50
25	35	105	20	15	

Again let us check for the optimality, for all occupied cells,

	$v_1=4$	$v_2=6$	$v_3=8$	$v_4=3$	$v_5=-5$	
$u_1 = 0$	$\boxed{25}$ 4	$\boxed{5}$ 6	$\boxed{20}$ 8	13	0	50
$u_2 = 2$	13	11	$\boxed{70}$ 10	8	0	70
$u_3 = -2$	14	$\boxed{30}$ 4	10	13	0	30
$u_3 = 5$	9	11	$\boxed{15}$ 13	$\boxed{20}$ 8	$\boxed{15}$ 0	50
	25	35	105	20	15	

$u_1 + v_1 = 4 \Rightarrow v_1 = 4$

$u_1 + v_2 = 6 \Rightarrow v_2 = 6$

$u_1 + v_3 = 8 \Rightarrow v_3 = 8$

$u_2 + v_3 = 10 \Rightarrow u_2 = 10 - v_3 = 2$

$u_3 + v_2 = 4 \Rightarrow u_3 = 4 - v_2 = -2$

$u_4 + v_3 = 13 \Rightarrow u_4 = 13 - v_3 = 5$

$u_4 + v_4 = 8 \Rightarrow v_4 = 8 - u_4 = 3$

$u_4 + v_5 = 0 \Rightarrow v_5 = 0 - u_4 = -5$

a) ***For all un–occupied cells,*** $C_{ij} - (u_i + v_j)$,

$13 - (0 + 3) = 10$

0, $\quad 0 - (0 - 5) = 5$

13, $\quad 13 - (2 + 4) = 7$

11, $\quad 11 - (2 + 6) = 3$

8,	$8 - (2 + 3) = 3$
0,	$0 - (2 - 5) = 3$
10,	$10 - (-2 + 8) = 4$
13,	$13 - (-2 + 3) = 12$
0,	$0 - (-2 - 5) = 7$
9,	$9 - (5 + 4) = 0$
11,	$11 - (5 + 6) = 0$

As all the net–cell evaluations ≥ 0, the solution is optimum. The optimum solution / allocations.

Store

Factory	1	2	3	4	Du	
A	[25] 4	[5] 6	[20] 8	13	0	50
B	13	11	[70] 10	8	0	70
C	14	[30] 4	10	13	0	30
D	9	11	[15] 13	[20] 8	[15] 0	50
	25	35	105	20	15	

The optimum cost $= 25 \times 4 + 5 \times 6 + 20 \times 8 + 70 \times 10 + 30 \times 4 + 15 \times 13 + 20 \times 8 + 15 \times 0 = 100 + 30 + 160 + 700 + 120 + 195 + 160 =$ ₹ 1465

12. *A firm produces a component and distributes them to 5 wholesalers at a fixed price of Rs. 10/unit. Sales forecast indicate that monthly demand will be 3000, 3000, 1000, 5000 and 4000 units at wholesale dealers a, b, c, d and e respectively. The monthly production capacities are 5,000, 1,000, and 10,000 at plants A, B and C respectively. The production cost per unit are Rs. 2, Rs. 1 and Rs. 3 at plants and A, B and C respectively. The unit transportation cost in Rs. between the plants and wholesalers are given in the given table:*

Whole salers

Plant	*a*	*b*	*c*	*d*	*e*
A	*0.5*	*0.5*	*1.0*	*1.5*	*1.5*
B	*1.0*	*1.5*	*1.0*	*1.0*	*1.5*
C	*1.0*	*1.0*	*0.5*	*1.5*	*1.0*

Determine the transportation schedule between plants and wholesalers in order to maximize the total profit per month. Use VAM to obtain the initial basic feasible

Solution:

Profit = Selling price – Total cost

In the present problem the selling price is Rs. 10, fixed at all the 3 plants.

Total cost = Production cost + Transportation cost.

(If nothing is given about any cost other than transportation cost, then the total cost is transportation cost itself)

Production cost per unit is given as Rs. 2, Re. 1 and Rs. 3 at plants A, B and C respectively.

Hence, these costs are to be added to the transportation costs at A, B and C respectively.

	a	b	c	d	e
A	0.5+2	0.5+2	1.0+2	1.5+2	1.5+2
B	1.0+1	0.5+1	1.0+1	1.0+1	1.5+1
C	1.0+3	1.0+3	0.5+3	1.5+3	1.0+3

The total cost matrix is,

	a	b	c	d	e
A	2.5	2.5	3.0	3.5	3.5
B	1.5	1.5	2.0	2.0	2.5
C	4.0	4.0	3.5	4.5	4.0

Profit Matrix:

Subtracting all the elements from the selling price we get, the profit matrix is

	a	b	c	d	e
A	7.5	7.5	7.0	6.5	6.5
B	8.0	**8.5**	8.0	8.0	7.5
C	6.0	6.0	6.5	5.5	6.0

Minimization form (standard form):

Convert the above matrix into minimization form (standard form) by subtracting all the elements from the highest element. The minimization form with given rim condition is,

1.0	1.0	1.5	2.0	2.0	5000
0.5	0	0.5	0.5	1	1000
2.5	2.5	2.0	3.0	2.5	10000
3000	3000	1000	5000	4000	

To avoid the decimals, let us multiply by all the elements by 10,

	a	b	c	d	e	
A	10	10	15	20	20	5000
B	5	0	5	5	10	1000
C	25	25	20	30	25	10000
	3000	3000	1000	5000	4000	

$\sum$ Supply = 16,000 = $\sum$ Demand

Hence, IBFS can be obtained by using VAM.

	a	b	c	d	e				
A	[3000] 10	[2000] 10	15	20	20	5000	(0)	(0)	(5)
B	5	0	5	[1000] 5	10	1000	(5)	-	-
C	25	[1000] 25	[1000] 20	[4000] 30	[4000] 25	10000	(5)	(5)	(5)
	3000	3000	1000	5000	4000				
	(5)	(10)	(10)	(15)	(10)				
	(15)	(15)	(5)	(10)	(5)				
	-	(15)	(5)	(10)	(5)				

A – a: 3000 units

A – b: 2000 units

B – d :1000 units

C – d :1000 units

C – c :1000 units

C – d :4000 units

C – e :4000 units

Optimal solution

$m + n - 1 = 3 + 5 - 1 = 7$

Number of allocations = 7

Hence, optimality check can be performed.

MODI method: Let us rewrite the matrix with allocation obtained by VAM

	$v_1=25$	$v_2=25$	$v_3=20$	$v_4=30$	$v_5=25$	
$u_1=-15$	10 [3000]	10 [3000]	15	20	20	5000
$u_2=-25$	5	0	5	5 [3000]	10	1000
$u_3=0$	25	25 [3000]	20 [3000]	30 [3000]	25 [3000]	10000
	3000	3000	1000	5000	4000	

i) ***For all occupied cells,***

Let $u_3 = 0$, as it is having max-allocations.

$u_1 + v_1 = 10 = v_1 = 10 - u_1 = 25$

$u_1 + v_2 = 10 = u_1 = 10 - v_2 = -15$

$u_2 + v_4 = 5 = u_2 = 5 - v_4 = -25$

$u_3 + v_2 = 25 = v_2 = 25$

$u_3 + v_3 = 20 = v_3 = 20$

$u_3 + v_4 = 30 = v_4 = 30$

$u_3 + v_5 = 25 = v_5 = 25$

ii) ***For all un-occupied cells, net cell evaluations,*** $C_{ij} - (u_i + v_j)$

$15,\ 15 - (-15 + 20) = 10$

$20,\ 20 - (-15 + 30) = 5$

$20,\ 20 - (-15 + 25) = 10$

$5,\ 5 - (-25 + 25) = 5$

$0,\ 0 - (-25 + 25) = 0$

$5,\ 5 - (-25 + 20) = 10$

For $10,\ 10 - (-25 + 25) = 10$

$25,\ 25 - (0 + 25) = 0$

It is observed that all $C_{ij} - (u_i + v_j) \geq 0$, Hence the solution is optimum. In other words, the solution obtained by VAM (IBFS) itself is optimal solution.

The optimal solution:

Consider the allocations obtained for the profit matrix.

Units	Profit
A – a: 3000	3000 × 7.5 = 22,5000
A – b: 2000	2000 × 7.5 = 15,000
B – d: 1000	1000 × 8 = 8,000
C – b: 1000	1000 × 6 = 6,000
C – c: 1000	1000 × 6.5 = 6,500
C – d: 4000	4000 × 5.5 = 22,000
C – e: 4000	4000 × 6.0 = 24,000

Total profit: Rs. 1, 04, 000

The profit matrix was not multiplied by any factor. Hence, no need to divide the profit obtained by any factor.

4.8 The Assignment Problem

An assignment problem is a special type of problem where the objective is to assign a number of origins equal to number of destinations at a minimum cost or maximum profit. The assignment problem is solved by a method know as Hungarian method.

Let there be m new computers that are to be arranged in a shop floor. Let there be m vacant places (S_1, S_2...........S_m). The cost of installing the computer C_i at place S_i is C_{ij}.

This problem can be represented by the following m × m matrix.

$$
\begin{array}{c}
 \\ C_1 \\ C_2 \\ C_3 \\ \cdot \\ \cdot \\ \cdot \\ C_m
\end{array}
\begin{array}{c}
\begin{array}{ccccc} S_1 & S_2 & S_3 - - - - - & & S_m \end{array} \\
\left[\begin{array}{ccccc}
C_{11} & C_{12} & C_{13} \cdots\cdots & & C_{1m} \\
C_{21} & C_{22} & C_{23} \cdots\cdots & & C_{2m} \\
 & & & & \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
C_{m1} & C_{m2} & C_{m3} \cdots\cdots & & C_{mm}
\end{array}\right]
\end{array}
$$

4.9 Applications of Assignment Problems

Following are the few applications of Assignment problems in general.

i. *To minimize the total time / cost required to complete group of tasks.*

ii. *To assign a best job to the best person.*

iii. *To assign vehicles to the routes.*

iv. *To assign sales repreasenatives to the sales territories.*

v. *In minimizing the time of arrival and departure of theairlines.*

4.10 Algorithm for an Assignment Problem (Hungarian Method)

The following procedure is used to solve an assignment problem whenever it is a square matrix / balanced assignment problem.

Step 1: Row reduction

Select the minimum element in each row and subtract it from the other elements of the row.

Step 2: Column reduction

Select the least element in each column and subtract it from other elements of the column.

The matrix thus obtained is called as reduced matrix.

Reduced matrix is a matrix consisting of at least one 0 in each row and column.

Step 3: Allocations

Scan the elements row wise and assign for zero in a row having single zero and cancel the respective column. If none of the rows are having single zero then scan the elements of columns and assign for a column having single zero canceling the respect row. Continue this procedure and if number of allocations = order of matrix, the solution is optimum and write the optimum schedule

If number of allocations is not equal to the order of matrix then modify the reduced matrix in the next step.

Step 4: Modification of the reduced matrix

i. *Identity the least uncrossed element.*

ii. *Add this to the elements, which are double–crossed.*

iii. *Subtract this from uncrossed elements.*

iv. *Retain the single crossed elements as they are*

Step 5:

Repeat the steps 3 and 4 until number of allocations = order of matrix is obtained.

Note:

i. ***Whenever the number of allocations = order of matrix, the solution is optimum.***

ii. ***When the matrix is of not square type the problem is stated as an unbalanced assignment problem. To apply the above steps the matrix must be square type.***

iii. ***An unbalanced assignment problem can be balanced by using a dummy row or column.***

iv. ***If the problem is of maximization type, covert it into minimization form and then solve.***

13 *Solve the following Assignment problem.*

$$\begin{array}{c|ccc|} & J_1 & J_2 & J_3 \\ \hline P_1 & 10 & 20 & 30 \\ P_2 & 20 & 10 & 40 \\ P_3 & 50 & 30 & 20 \\ \hline \end{array}$$

Solution:

As the number of rows = number of columns (square matrix) the problem is balanced assignment problem. Hence, Hungarian method can be used to solve.

Step 1: Row reduction

Selecting the least / minimum elements in each row and subtracting the other elements of the respective rows we get,

$$\begin{array}{|ccc|} \hline 0 & 10 & 20 \\ 10 & 0 & 30 \\ 30 & 10 & 0 \\ \hline \end{array}$$

Step 2: Column reduction

Selecting the least / minimum elements in each column and subtracting the other elements of the respective columns we get,

Same Matrix will be obtained as each column contains '0'

Step 3: Allocations

Scanning the elements row wise and assigning for zero in a row having single zero. After assigning cancelling the respective column. We get

$$\begin{array}{c|ccc|} & J_1 & J_2 & J_3 \\ \hline P_1 & \boxed{0} & 10 & 20 \\ P_2 & 10 & \boxed{0} & 30 \\ P_3 & 30 & 10 & \boxed{0} \\ \hline \end{array}$$

Solution is optimal as number of allocations is equal to order of the Matrix

Hence, the optimal solution is.

$$P_1 - J_1 \ : \ 10$$
$$P_2 - J_2 \ : \ 10$$
$$P_3 - J_3 \ : \ \underline{20}$$
$$40 \text{ units of cost}$$

Before proceeding to apply Hungarian Method, Check whether the given problem is balanced or not. i.e., whether $\sum$ supply $=$ $\sum$ demand or not.

14. *Solve the following assignment problem with the usual notations. The entries of the cells represent the assignment costs.*

Job

		J_1	J_2	J_3
Person	P_1	18	17	16
	P_2	15	13	14
	P_3	19	20	21

Solution:

As the number of rows = number of columns (square matrix) the problem is balanced assignment problem. Hence, Hungarian method can be used to solve.

Step 1: Row reduction

Selecting the least / minimum elements in each row and subtracting the other elements of the respective rows we get,

2	1	0
2	0	1
0	1	2

Step 2: Column reduction

Selecting the least / minimum elements in each column and subtracting the other elements of the respective columns we get,

2	1	0
2	0	1
0	1	2

Step 3: Allocations

Scanning the elements row wise and assigning for zero in a row having single zero. After assigning cancelling the respective column. We get

$$
\begin{array}{c|ccc|}
 & J_1 & J_2 & J_3 \\
\hline
P_1 & 2 & 1 & \boxed{0} \\
P_2 & 2 & \boxed{0} & 1 \\
P_3 & \boxed{0} & 1 & 2 \\
\hline
\end{array}
$$

$P_1 - J_3$
$P_2 - J_2$
$P_3 - J_3$

Optimal assignment cost: 16 + 13 + 19 = **48 units.**

15. *Solve the following assignment problem, the numbers of matrix indicate cost.*

	A	B	C
1	12	11	8
2	8	9	11
3	11	14	12

Solution:

As the number of rows = number of columns (square matrix) the problem is balanced assignment problem. Hence, Hungarian method can be used to solve.

Step 1: Row reduction

Selecting the least / minimum elements in each row and subtracting the other elements of the respective rows we get,

4	3	0
0	1	3
0	3	1

Step 2: Column reduction

Selecting the least / minimum elements in each column and subtracting the other elements of the respective columns we get,

4	2	0
0	0	3
0	2	1

The above matrix is the reduced matrix

Step 3: Allocations

i. *Scanning the first row, we have single zero (1-C) assigning it and cancel column C.*

ii. *Next the second row is having two zeros (2-A, 2-B) and hence, do not assign for it right now.*

iii. *Third row contains single zero (3-A) hence, assign for it and cancel the column A*

iv. *Now in the second round of scanning, second row is having single zero (2-B) and hence it can be assigned, cancelling the column B, we get*

	A	B	C
1	4	2	$\boxed{0}$
2	0	$\boxed{0}$	3
3	$\boxed{0}$	2	1

As number of allocations = order of matrix, the solution is optimum

Optimum assignment is

1 → C, 2 → B, 3 → A

Optimal cost is = 8 + 9 + 11 = ₹ 28/– (By adding the respective costs at the allocations.

16. *Four projects are to be done on four different computers. The cost (in thousands rupees) for completing P^{th} project on C^{th} computer is given below. Assign the projects to different computers so as to minimize the total cost.*

		Computer			
		C_1	C_2	C_3	C_4
Project	P_1	15	11	13	15
	P_2	17	12	13	13
	P_3	14	15	10	14
	P_4	16	13	11	17

Solution:

As the number of columns is equal to number of rows the problem is a balanced assignment problem.

Step 1:

Identify the least element in each row and subtract it from other elements.

4	0	2	4
5	0	1	1
4	5	0	4
5	2	0	6

Step 2:

Identify the least element in each column and subtract it from other elements.

0	0	2	3
1	0	1	0
0	5	0	3
1	2	0	5

This is the reduced matrix.

Step 3:

Allocations

i. *Scannig the first row, we have two zeros (P_1-C_1, P_1-C_2) and hence, do not assign for it right now.*

ii. *In the similar way second and third rows are also having two zeros, hence do not assign for them.*

iii. *Fourth row is having single zero and hence assign for it. (P_4-C_3)*

iv. *In the second round of scanning, proceeding in the same way, we get,*

	C_1	C_2	C_3	C_4
P_1	0	$\boxed{0}$	2	3
P_2	1	0	1	$\boxed{0}$
P_3	$\boxed{0}$	5	0	3
P_4	1	2	$\boxed{0}$	5

The solution is optimum as number of allocations = order of matrix, optimum distribution is,

$$P_1 \rightarrow C_2, P_2 \rightarrow C_4, P_3 \rightarrow C_1, P_4 \rightarrow C_3$$

Total cost is 11 + 13 + 14 + 11 = 49 × 1000 = ₹ 49,000/–

17. *Solve the following assignment problem by Hungarian Assignment Method (HAM)*

Jobs

		J_1	J_2	J_3	J_4
Men	*1*	*12*	*30*	*21*	*15*
	2	*18*	*33*	*9*	*31*
	3	*44*	*25*	*24*	*21*
	4	*23*	*30*	*28*	*14*

Solution:

After the steps 1 & 2, the reduced matrix is,

0	14	9	3
9	20	0	22
23	0	1	0
9	12	14	0

Allocations:- Scanning the elements row wise and assigning for zero in a row having single zero and canceling the respective column we get,

	J_1	J_2	J_3	J_4
1	$\boxed{0}$	14	9	3
2	9	20	$\boxed{0}$	22
3	23	$\boxed{0}$	1	0
4	9	12	14	$\boxed{0}$

The optimum Assignments are,

$1 - J_1$ $2 - J_3$ $3 - J_2$ $4 - J_4$.

The assignment cost is , 12 + 9 + 25 + 14 = 60

18. *Jumbo company has four machines and four jobs to be completed. Each machine should be assigned to complete one job. The time required to set up each machine for completing each job is shown in the following table. Jumbo company wants to minimize the total set up time needed to complete the four jobs. Solve by Hungarian method.*

Job

Machine		J_1	J_2	J_3	J_4
	M_1	14	5	8	7
	M_2	2	12	6	5
	M_3	7	8	3	9
	M_4	2	4	6	10

Solution:

Step 1: Row reduction:- Identify the least element in each row and subtract it from other elements.

9	0	3	2
0	10	4	3
4	5	0	6
0	2	4	8

Step 2: Column reduction:- Identify the least element in each column and subtract it from other elements.

9	0	3	0
0	10	4	1
4	5	0	4
0	2	4	6

Step 3: Allocations: - Scanning the elements row wise and assigning for zero in a row having single zero and cancel the respective column. When none of the rows are having single zero for the allocation, then consider column having single zero and cancel the respective row by which we get,

9	$\boxed{0}$	3	0
$\boxed{0}$	10	4	1
4	5	$\boxed{0}$	4
0	2	4	6

Number of allocations ≠ order of matrix. Hence the solution is not optimum.

Modification of the reduced matrix,

Identify the least uncrossed element (1). Add this to the elements, which are double-crossed. Subtract this from uncrossed elements and retain the single crossed elements as they are and re-allocating we get,

	J_1	J_2	J_3	J_4
M	10	$\boxed{0}$	4	0
M_2	0	9	4	$\boxed{0}$
M_3	4	4	$\boxed{0}$	3
M_4	$\boxed{0}$	1	4	3

The solution is optimum, as number of allocations is equal to order of the matrix.

The optimum Assignments are,

$M_1 - J_2$, $M_2 - J_4$, $M_3 - J_3$, $M_4 - J_1$

The assignment cost is , 5 + 5 + 3 + 2 = 15

19. *Four Professors are each capable of teaching any one of the four different subjects. Class preparation time (in hours) for different topics varies from Professor to Professor and is given in the table. Each professor should be assigned only one subject. Find the schedule so as to minimize the total subject preparation time for all subjects / Professors.*

Subject

Professor		S_1	S_2	S_3	S_4
	P_1	2	10	9	7
	P_2	15	4	14	8
	P_3	13	14	16	11
	P_4	3	15	13	8

Solution:

Step 1: Row reduction

0	8	7	5
11	0	10	4
2	3	5	0
0	12	10	5

Step 2: Column reduction

0	8	2	5
11	0	5	4
2	3	0	0
0	12	5	5

Step 3: Allocations

$\boxed{0}$	8	2	5
11	$\boxed{0}$	5	4
2	3	$\boxed{0}$	0
0	12	5	5

Number of allocations ≠ order of matrix. Hence the solution is not optimum.

Step 4: Modification of the reduced matrix

	S_1	S_2	S_3	S_4
P_1	0	8	2	5
P_2	11	0	5	4
P_3	2	3	0	0
P_4	0	12	5	5

Step 5: Optimal solution

	S_1	S_2	S_3	S_4
P_1	0	8	$\boxed{0}$	3
P_2	11	$\boxed{0}$	3	2
P_3	4	5	0	$\boxed{0}$
P_4	$\boxed{0}$	12	3	3

Number of allocations = the order of matrix. Hence, the optimum scheduling is,

$P_1 \rightarrow S_3$, $P_2 \rightarrow S_2$, $P_3 \rightarrow S_4$, $P_4 \rightarrow S_1$

Total duration = 9 + 4 + 11 + 3 = 27 hours

20. *Solve the following assignment problem with the following cost matrix.*

		Job J_1	J_2	J_3	J_4	J_5
Person	P_1	11	17	8	16	20
	P_2	9	7	12	6	15
	P_3	13	16	15	12	16
	P_4	21	24	17	28	26
	P_5	14	10	12	11	15

Solution:

Step 1: Row reduction

3	9	0	8	12
3	1	6	0	9
1	4	3	0	4
4	7	0	11	9
4	0	2	1	5

Step 2: Column reduction

$$\begin{bmatrix} 2 & 9 & 0 & 8 & 8 \\ 2 & 1 & 6 & 0 & 5 \\ 0 & 4 & 3 & 0 & 0 \\ 3 & 7 & 0 & 11 & 5 \\ 3 & 0 & 2 & 1 & 1 \end{bmatrix}$$

Step 3: Allocations

$$\begin{bmatrix} 2 & 9 & \boxed{0} & 8 & 8 \\ 2 & 1 & 6 & \boxed{0} & 5 \\ \boxed{0} & 4 & 3 & 0 & 0 \\ 3 & 7 & 0 & 11 & 5 \\ 3 & \boxed{0} & 2 & 1 & 1 \end{bmatrix}$$

No. of allocations $\neq$ order of the Matrix. Hence, the solution is not optimal.

Modifying the reduced Matrix:

$$\begin{bmatrix} 1 & 9 & \boxed{0} & 8 & 7 \\ 1 & 1 & 6 & \boxed{0} & 4 \\ \boxed{0} & 5 & 4 & 1 & 0 \\ 2 & 7 & 0 & 11 & 4 \\ 2 & \boxed{0} & 2 & 1 & 0 \end{bmatrix}$$

Solution is not optimal as no. of allocations is not equal to order of the matrix.

Modifying the reduced Matrix again,

$$\begin{array}{c|ccccc} & J_1 & J_2 & J_3 & J_4 & J_5 \\ \hline P_1 & \boxed{0} & 8 & 0 & 8 & 6 \\ P_2 & 0 & 0 & 6 & \boxed{0} & 3 \\ P_3 & 0 & 5 & 5 & 2 & \boxed{0} \\ P_4 & 1 & 6 & \boxed{0} & 11 & 3 \\ P_5 & 2 & \boxed{0} & 3 & 2 & 0 \end{array}$$

Solution is optimal as no. of allocations = order of the Matrix:

$P_1 - J_1$: 11

$P_2 - J_4$: 6

$P_3 - J_5$: 16

$P_4 - J_3$: 17

$P_5 - J_2$ 10

50

21. *A car company has one car at each of the 5 depots (A, B, C, D and E). A customer requires a car in each town namely (P, Q, R, S and T). Distance between depots and towns (in kilometers) are given in the following matrix. How should the cars be assigned to the customer to minimize the distance traveled.*

	A	B	C	D	E
P	160	130	175	190	200
Q	135	130	130	160	175
R	140	110	155	170	185
S	50	50	180	80	110
T	55	35	70	80	105

Solution:

Step 1: Row reduction

30	0	45	60	70
5	0	0	30	45
30	0	45	60	75
0	0	130	30	60
20	0	35	45	70

Stepe 2: Reduced matrix and the allocations

30	$\boxed{0}$	45	30	25
5	0	$\boxed{0}$	0	0
30	0	45	30	30
$\boxed{0}$	0	130	0	15
20	0	35	15	25

Number of allocations ≠ order of matrix

15	$\boxed{0}$	30	15	10
5	15	$\boxed{0}$	0	0
15	0	30	15	15
$\boxed{0}$	15	130	0	15
5	0	20	$\boxed{0}$	10

Modified reduced matrix is,

15	0	20	15	$\boxed{0}$
15	25	$\boxed{0}$	10	0
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	120	0	5
5	0	10	$\boxed{0}$	0

Number of allocations = order of matrix = 5

Assignment of the cars is,

P → E, Q → C, R → B, S → A, T → D

Total distance = 200 + 130 + 110 + 50 + 80 = 570 kilometers

4.11 Types of Assignment Problem

The types of an assignment problem are as below

i. *Unbalanced assignment Problem*

ii *Maximization problem*

iii *Prohibited route (restricted assignment problem)*

iv *Alternative optimal solution*

4.11.1 Unbalanced Assignment Problem

Whenever number of rows = number of columns, the problem is stated as balanced assignment problem. On the other hand if number of rows is not equal to number of columns then the problem is referred as unbalanced assignment problem. To balance this dummy row or column with zero coefficients is to be introduced so as to get number of rows = number of columns.

That is *always an assignment matrix must be converted to be as a square matrix to solve it.*

Examples:

Case 1: Number of rows is less than number of columns

		J_1	J_2	J_3	J_4
	P_1	4	2	0	1
Person	P_2	2	5	4	6
	P_3	7	5	6	9

In the above matrix, the number of rows is not equal to the number of columns. In other words, though there are 4 jobs only 3 persons are available to perform them.

Hence, this is treated as unbalanced assignment problem. This can be balanced by adding a dummy person, so that number of rows is equal to the number of columns as shown below, after adding a dummy row.

Person		J_1	J_2	J_3	J_4
	P_1	4	2	0	1
	P_2	2	5	4	6
	P_3	7	5	6	9
	Du	0	0	0	0

Case 2: Number of rows is greater than number of columns

		Job		
Person		J_1	J_2	J_3
	P_1	3	4	2
	P_2	1	3	5
	P_3	6	7	9
	P_4	2	1	4

Here, there are 4 persons but only 3 jobs are available. Hence, it is unbalanced problem. This can be balanced by assuming/adding a dummy job, so that number of columns is equal to the number of rows as shown in the diagram, after adding a dummy column.

Person		J_1	J_2	J_3	Du
	P_1	3	4	2	0
	P_2	1	3	5	0
	P_3	6	7	9	0
	P_4	2	1	4	0

22. *The manager of a depot has four subordinates to perform three tasks. Since the efficiency of each subordinate is different, the estimated time taken by each subordinate to perform these four tasks would be different. These estimated times are given in the matrix below. Suggest the optimal assignment of the tasks to each person so as to minimize the total time to perform these three tasks.*

Worker	*Task*		
	T_1	T_2	T_3
W_1	19	36	25
W_2	23	37	16
W_3	45	30	25
W_4	28	40	30

Solution:

Step 1: **Row reduction:-** As the given matrix is unbalanced, by adding a dummy row we get directly the row reduced matrix.

		T_1	T_2	T_3	Du
From	W_1	19	36	25	0
	W_2	23	37	16	0
	W_3	45	30	25	0
	W_4	28	40	30	0

(To)

Step 2: Column reduction:- Identify the least element in each column and subtract it from other elements

0	6	9	0
4	7	0	0
26	0	9	0
9	10	14	0

Step 3: Allocations :- Scanning the elements row wise and assigning for zero in a row having single zero and canceling the respective column we get,

		T_1	T_2	T_3	Du
From	W_1	$\boxed{0}$	6	9	0
	W_2	4	7	$\boxed{0}$	0
	W_3	26	$\boxed{0}$	9	3
	W_4	9	10	14	$\boxed{0}$

(To)

Optimal assignements are, $W_1 - T_1$

$W_1 - T_1$, $W_2 - T_3$, $W_3 - T_2$, $W_4 -$ Du

23 *A batch of 4 jobs can be assigned to 5 different machines. The following table gives the installation time in hours for each job on various machines. Find the optimal assignment of jobs to machines which will minimize the total installation time. Comment on your answer.*

		Machine				
		M_1	M_2	M_3	M_4	M_5
Job	J_1	10	11	4	2	8
	J_2	7	11	10	14	12
	J_3	5	6	9	12	14
	J_4	13	15	11	10	7

Solution:

The given problem is an unbalanced problem and it can be balanced by adding a dummy job (row) we get,

	M_1	M_2	M_3	M_4	M_5
J_1	10	11	4	2	8
J_2	7	11	10	14	12
J_3	5	6	9	12	14
J_4	13	15	11	10	7
Du	0	0	0	0	0

Reducing the matrix and by assigning we get,

8	9	2	$\boxed{0}$	6
$\boxed{0}$	4	3	7	5
0	1	4	7	9
6	8	4	3	$\boxed{0}$
0	$\boxed{0}$	0	0	0

Number of allocations $\neq$ order of matrix. Hence, solution is not optimal modifying reduced matrix by the steps given,

8	8	1	$\boxed{0}$	6
$\boxed{0}$	3	2	7	5
0	$\boxed{0}$	3	7	9
6	7	3	3	$\boxed{0}$
1	0	$\boxed{0}$	0	1

Number of allocations = order of matrix. The optimum scheduling is,

$J_1 \rightarrow M_4, J_2 \rightarrow M_1, J_3 \rightarrow M_2, J_4 \rightarrow M_5, Du \rightarrow M_3$

Total time 2 + 7 + 6 + 7 + 0 = 22 hours

Observation: Since M_3 consumes more installation time no job is assigned to it in other words machine M_3 is dummy / idle.

4.11.2 Maximization Problem

Like in transportation problem when the elements of an assignment problem represents *profits instead of costs*, the objective is to maximize instead of minimize. This kind of assignment problem is known as *maximization assignment problem*. To solve this problem, transform the profits to relative costs. This is done by subtracting the all other elements from the *largest element* in the matrix. In this way the largest profit (element) shows a zero relative cost. The solution procedure is same as that of minimization problem but the total profit is calculated using the profit matrix assignment matrix.

24. *A company has 5 tasks and 5 persons to perform the same. The matrix shows the returns (profit) in hundreds of rupees. For assigning jobs to the persons. Assign the 5 tasks to 5 persons to maximize the total return.*

		Person				
		P_1	P_2	P_3	P_4	P_5
Task	J_1	5	11	10	12	4
	J_2	2	4	6	3	5
	J_3	3	12	5	14	6
	J_4	6	14	4	11	7
	J_5	7	9	8	12	5

Solution:

As it is maximization problem like in transportation problem, it is to be converted into minimization form. Select the highest and subtract all other elements from it.

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

Step 1 :Row reduction

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	7

Step 2 :Column reduction

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	6

Step 3: Allocations

3	1	2	$\boxed{0}$	7
$\boxed{0}$	2	0	3	0
7	2	9	0	7
4	$\boxed{0}$	10	3	6
1	3	4	0	6

Number of allocations $\neq$ order of matrix. Hence, solution is not optimal modifying reduced matrix by the steps given, we get,

Step 4: Modification of the reduced matrix

2	1	1	0	6
0	3	0	4	0
6	2	8	0	6
3	0	9	3	5
0	3	3	0	5

Allocations

2	1	1	$\boxed{0}$	6
0	3	$\boxed{0}$	4	0
6	2	8	0	6
3	$\boxed{0}$	9	3	5
$\boxed{0}$	3	3	0	5

Number of allocations ≠ order of matrix. Hence, solution is not yet optimum again modifying reduced matrix and assigning we get,

Step 5: Modified allocations:

2	1	$\boxed{0}$	0	5
1	4	0	5	$\boxed{0}$
6	2	7	$\boxed{0}$	5
3	$\boxed{0}$	8	3	4
$\boxed{0}$	3	2	0	4

Number of allocations = order of matrix. Hence, the optimum scheduling is,

$$J_1 \rightarrow P_3, J_2 \rightarrow P_5, J_3 \rightarrow P_4, J_4 \rightarrow P_2, J_5 \rightarrow P_1$$

Total profit = 10 + 5 + 14 + 14 + 7 = 50 units. That is, 50 × 100 = ₹ 5000/–

Note: The elements of original matrix are considered as per the assignments obtained to get profit.

25. *Solve the following assignment problem*

	J_1	J_2	J_3	J_4
P_1	*11*	*3*	*5*	*8*
P_2	*9*	*9*	*8*	*4*
P_3	*10*	*3*	*5*	*10*
P_4	*4*	*13*	*12*	*11*
P_5	*8*	*9*	*10*	*4*

The entries indicate the profits by assigning jobs to persons. Who will be idle person.

Solution:

The given problem is maximization problem to convert this into minimization problem select the highest element and subtract all other elements from it. Moreover, as the number of rows is not equal to the number of columns the problem is unbalanced. To make it as balanced problem, add a dummy column. Then we get,

2	10	8	5	0
4	4	5	9	0
3	10	8	3	0
9	0	1	2	0
5	4	3	9	0

The allocations for the reduced matrix by Hungarian method are,

$\boxed{0}$	10	7	3	0
2	4	4	7	$\boxed{0}$
1	10	7	1	0
7	$\boxed{0}$	0	0	0
3	4	2	7	0

Number of allocations $\neq$ order of matrix, the modified allocations are,

$\boxed{0}$	9	6	2	0
2	3	3	6	$\boxed{0}$
1	9	6	$\boxed{0}$	0
8	$\boxed{0}$	0	0	1
3	3	1	6	0

Number of allocations $\neq$ order of matrix. Hence the solution is not optimal modifying reduced matrix by the steps given, we get,

	J_1	J_2	J_3	J_4	Du
P_1	$\boxed{0}$	8	5	2	0
P_2	2	2	2	6	$\boxed{0}$
P_3	1	8	5	$\boxed{0}$	0
P_4	9	$\boxed{0}$	0	1	2
P_5	3	2	$\boxed{0}$	6	0

Solution is optimal as number of allocations = order of matrix

The optimum assignment is,

$P_1 \to J_1$, $P_2 \to Du$, $P_3 \to J_4$, $P_4 \to J_2$, $P_5 \to J_3$

P_2 is the idle person, as he is not assigned with any job.

26 *The owner of a small firm has four operators available to assign the jobs. Five jobs are offered with the expected profit in rupees for each machinist on each job as shown below*

		Job				
		A	*B*	*C*	*D*	*E*
	1	*6.20*	*7.80*	*5.00*	*10.10*	*8.20*
Operator	*2*	*7.10*	*8.40*	*6.10*	*7.30*	*5.90*
	3	*8.70*	*9.20*	*11.10*	*7.10*	*8.10*
	4	*4.80*	*6.40*	*8.70*	*7.70*	*8.00*

Find the assignment of operators to jobs that will result in a maximum profit. Which job should be declined?

Solution:

Note: *If the numbers in the table are the fractions like 2.5, 3.5, etc, then multiply them by suitable factor to convert into whole numbers and finally divide the answer by the same factor.*

Convert the decimals into whole numbers multiplying the entire matrix by a suitable factor. In the present problem let us multiply all the elements by the factor 10 then we get,

62	78	50	101	82
71	84	61	73	59
87	92	111	71	81
48	64	87	77	80

As the above elements represent the profit, it is a maximization problem which is to be converted into minimization (standard form) by subtracting all the elements from the highest element of the matrix. The highest element is 111, The standard form is,

49	33	61	10	29
40	27	50	38	52
24	19	0	40	30
63	47	24	34	31

The problem is un–balanced as number of rows is not equal to no of columns.

By adding a dummy row so it is balanced as shown.

49	33	61	10	29
40	27	50	38	52
24	19	0	40	30
63	47	24	34	31
0	0	0	0	0

Row reduction:– Identify the least element in each row and subtract it from other elements.

39	23	51	0	19
13	0	23	11	25
24	19	0	40	30
39	23	0	10	7
0	0	0	0	0

Reduced matrix and Allocations:–

When none of the rows are having single zero for the allocation, then consider column having single zero and cancel the respective row by which we get,

39	23	51	$\boxed{0}$	19
13	$\boxed{0}$	23	11	25
24	19	$\boxed{0}$	40	30
39	23	0	10	7
$\boxed{0}$	0	0	0	0

Number of allocations is not equal to order of the matrix. Hence, the solution is not optimum.

Modifying the reduced matrix,

	A	B	C	D	E
1	32	23	51	$\boxed{0}$	12
2	6	$\boxed{0}$	23	11	18
3	17	19	$\boxed{0}$	40	23
4	32	23	0	10	$\boxed{0}$
5	$\boxed{0}$	1	1	1	0

No of allocations = order of the matrix. Hence, the solution is optimum

Optimum schedule is

1→ D ,2→ B ,3→ C , 4→ E , 5→ A

Adding the respective profits we get, + 10.10 + 8.40 + 11.10 + 8 + 0 = ₹ 37.60

4.11.3 Prohibited route (restricted assignment problem)

In an assignment problem, due to some practical reasons it may not be possible to assign few jobs to the few persons or vice versa. Such problems are termed as restricted assignment problems .We write M or $-\infty$ for the restricted locations, to avoid the allocations in such cells. In obtaining the reduced matrix these allocations are ignored. The problem will be solved by using 'HAM' as usual.

27. *Four new computers (C_1, C_2, C_3, C_4) are to be installed in a computer center. There are 5 vacant places (A, B, C, D and E) available. Because of limited space C_2 cannot be placed at C and C_3 cannot be placed at A. The assignment cost of the computers to the places is given below. Find the optimal assignment schedule.*

	A	B	C	D	E
C_1	4	6	10	5	6
C_2	7	4	–	5	4
C_3	–	6	9	6	2
C_4	9	3	7	2	3

Solution:

The problem is an unbalanced problem as the number of rows is not equal to number of columns, it can be balanced by adding a dummy row (*Du*) as shown below. To avoid the allocation in the restricted place use ∞

	4	6	10	5	6
	7	4	∞	5	4
	∞	6	9	6	2
	9	3	7	2	3
Du	0	0	0	0	0

After introducing a dummy row each column will be having a zero, hence, no need to operate the columns. Operating the rows we get the reduced matrix and the assignments as shown below.

$\boxed{0}$	2	6	1	2
3	$\boxed{0}$	∞	1	0
∞	4	7	4	$\boxed{0}$
7	1	5	$\boxed{0}$	1
0	0	$\boxed{0}$	0	0

Solution is optimal as number of allocations = order of matrix = 5

The optimum scheduling is,

$C_1 \rightarrow$ A, $C_2 \rightarrow$ B, $C_3 \rightarrow$ E, $C_4 \rightarrow$ D, $Du \rightarrow$ C

The place 'C' is to be kept vacant.

Adding the cost of respective allocations made. Total cost = 4 + 4 + 2 + 2 + 0 = 12 units of cost.

28. *Four workers are available to work on four machines and the respective costs associated with each machine worker assignment is given below:*

	Machine M_1	M_2	M_3	M_4
W_1	12	3	6	-
W_2	4	10	-	5
W_3	7	2	8	9
W_4	-	7	8	6

The sign (-) indicates that the particular worker machine assignment is not permitted. Determine the optimum assignment

Solution:

Re-writing the given problem and allocating ∞ to the prohibited locations

	M_1	M_2	M_3	M_4
W_1	12	3	6	∞
W_2	4	10	∞	5
W_3	7	2	8	9
W_4	∞	7	8	6

Step 1: Row reduction:- Identify the least element in each row and subtract it from other elements.

9	0	1	∞
0	6	∞	1
5	0	6	7
∞	1	2	0

Step 2: Column reduction:-Identify the least element in each column and subtract it from other elements

9	0	1	∞
0	6	∞	1
5	0	5	7
∞	1	0	0

Step 3: *Allocations:-* Scanning the elements row wise and assigning for zero in a row having single zero and canceling the respective column and selecting the columns of single zero and canceling the respective row we get,

9	$\boxed{0}$	1	∞
$\boxed{0}$	6	∞	1
5	0	5	7
∞	1	$\boxed{0}$	0

No. of allocations ≠ order of Matrix. Hence, the solution is not optimum.

Modified reduced matrix and

Allocations

	M_1	M_2	M_3	M_4
W_1	9	0	$\boxed{0}$	∞
W_2	$\boxed{0}$	6	∞	0
W_3	5	$\boxed{0}$	4	6
W_4	∞	2	0	$\boxed{0}$

Solution is optimum or No. of allocations = order of the matrix
Optimum assignment is loss.

$W_1 - M_3$ $\quad$ $W_3 - M_2$ $\quad$ $W_2 - M_1$ $\quad$ $W_4 - M_4$

$6 + 4 + 2 + 6 = 18$

4.11.4 Alternative Optimal Solution

When none of the rows / columns are having single zero, then the problem will be having more than one solution. In other words, alternative optimal solutions. To solve this consider one zero ignoring the other zeros in arbitrary choice. Let us understand this concept through the following example.

29 *Solve the following assignment problem (elements in the matrix indicate units of the cost)*

	1	*2*	*3*	*4*
A	*4*	*1*	*0*	*1*
B	*1*	*3*	*4*	*0*
C	*3*	*2*	*1*	*3*
D	*2*	*2*	*3*	*0*

Solution:

Step 1: Row reduction

4	1	0	1
1	3	4	0
2	1	0	2
2	2	3	0

Step 2: Reduced matrix and Assignments

3	$\boxed{0}$	$0^{\times}$	1
$\boxed{0}$	2	4	0
1	0	$\boxed{0}$	2
1	1	3	$\boxed{0}$

Solution is optimum as number of allocations = order of matrix

The optimum scheduling is,

A → 2, B → 1, C → 3, D →4

The optimal cost = 1 + 1 + 1 + 0 = 3.

Alternative optimal solution is,

3	$0^{\times}$	$\boxed{0}$	1
$\boxed{0}$	2	4	0
1	$\boxed{0}$	0	2
1	1	3	$\boxed{0}$

A → 3, B → 1, C → 2, D → 4 (Allocating for 0s marked with x.)

Alternative optimal solution = 0 + 1 + 2 + 0 = 3

30. *Solve the following Assignment problem. The entries in the cells represent the cost of assignments.*

	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

Solution:

Step 1: Row reduction

$$\begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix}$$

Step 2: Column reduction

$$\begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix}$$

Step 3: Allocations

$$\begin{bmatrix} \boxed{0} & 2 & 2 & 1 \\ 3 & \boxed{0} & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & \boxed{0} \end{bmatrix}$$

No. of allocations $\neq$ order of the Matrix and hence the solution is not optimum.

Modified Reduced Matrix and allocations

$$\begin{bmatrix} \boxed{0} & 1 & 1 & 1 \\ 4 & \boxed{0} & 0 & 2 \\ 0 & 2 & 1 & 3 \\ 7 & 0 & 0 & \boxed{0} \end{bmatrix}$$

The solution is not yet optimal as still number of allocations $\neq$ order of the Matrix.

Modifying the above reduced Matrix further

$$\begin{array}{c|cccc} & M_1 & M_2 & M_3 & M_4 \\ \hline J_1 & 0 & 0 & 0 & \boxed{0} \\ J_2 & 5 & \boxed{0} & 0\times & 2 \\ J_3 & \boxed{0} & 1 & 0 & 2 \\ J_4 & 8 & 0 & \boxed{0} & 0\times \end{array}$$

It is observed that none of the rows /columns are having single zero. (It is an indication for having alternative optimal solution).

Allocations for a row/column having two 0's (with arbitrary choice)

No. of allocations are equal to order of the Matrix.
Hence, the solution is optimal.
The allocations are.

$$\begin{array}{lcr} J_1 - M_4 & : & 6 \\ J_2 - M_2 & : & 5 \\ J_3 - M_1 & : & 4 \\ J_4 - M_3 & : & \underline{8} \\ & & 23 \end{array}$$

(or) The alternative optimal solution is (Allocating for 0s marked with x.)

$$\begin{array}{lcr} J_1 - M_4 & : & 6 \\ J_2 - M_2 & : & 5 \\ J_3 - M_1 & : & 4 \\ J_4 - M_3 & : & \underline{8} \\ & & 23 \end{array}$$

31. *A company has 4 territories open and 4 salesmen available for assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring in the following annual sales*

Territory	*I*	*II*	*III*	*IV*
Annual sales	*60,000*	*50,000*	*40,000*	*30,000*

Four salesmen are also considered to differ in their ability; it is estimated that, working under the same conditions, their yearly sales would be proportional as follows.

Salesman	*A*	*B*	*C*	*D*
Proportion	*7*	*5*	*5*	*4*

If the criterion is maximum expected sales, then the initiative answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on. Verify this answer by the assignment technique.

Solution:

Let us convert the given data as an effective matrix.

The total proportion of all the salesmen is, $7 + 5 + 5 + 4 = 21$

The profit by assigning territory 1 to salesman A is, $\dfrac{7 \times 60,000}{21}$, territory 2 to salesman A is $\dfrac{7 \times 50,000}{21}$ and so on. To obtain simple/effective matrix, let us divide all the computations by 10,000 units and multiply by 21 units (so that we will get simple whole numbers)

	T_1	T_2	T_3	T_4
A	$\frac{7\times 30000}{21}$	$\frac{7\times 50000}{21}$	$\frac{7\times 40000}{21}$	$\frac{7\times 30000}{21}$
B	$\frac{5\times 60000}{21}$	$\frac{5\times 50000}{21}$	$\frac{5\times 40000}{21}$	$\frac{5\times 30000}{21}$
C	$\frac{5\times 60000}{21}$	$\frac{5\times 50000}{21}$	$\frac{5\times 40000}{21}$	$\frac{5\times 30000}{21}$
D	$\frac{4\times 60000}{21}$	$\frac{4\times 50000}{21}$	$\frac{4\times 40000}{21}$	$\frac{4\times 30000}{21}$

=

	T_1	T_2	T_3	T_4
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

Is the profit (maximization problem) matrix, Converting it into minimization problem and applying Hungarian method. We get

$\boxed{0}$	3	6	9
0	1	2	3
0	1	2	3
0	$\boxed{0}$	0	0

The order of matrix $\neq$ number of allocations.

Hence, identify the least uncrossed element add this to double crossed elements subtract this form uncrossed element. We get the following sequence of solutions

$$\begin{array}{|c|c|c|c|} \hline \boxed{0} & 2 & 5 & 8 \\ \hline 0 & \boxed{0} & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ \hline 1 & \boxed{0} & 0 & 0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline \boxed{0} & 2 & 4 & 7 \\ \hline 0 & \boxed{0} & 0 & 1 \\ \hline 0 & 0 & \boxed{0} & 1 \\ \hline 2 & 1 & 0 & \boxed{0} \\ \hline \end{array}$$

The problem is having alternative optimal solution. The optimal assignment is

Territory	Sales man
T_1	S_1
T_2	S_2
T_3	S_3
T_4	S_4

or

Territory	Sales man
T_1	S_1
T_2	S_3
T_3	S_2
T_4	S_4

The answer is verified by using the assignment technique that is it is proved.

4.12 DIFFERENCES BETWEEN A TRANSPORTATION PROBLEM AND AN ASSIGNMENT PROBLEM

Transportation Problem	Assignment Problem
i. The matrix may or may not be square matrix to solve the problem	*i. Always the matrix must be square matrix to solve the problem*
ii. Supply and demand will be known	*ii. Supply, demand will be unity*
iii. When Supply is not equal to the demand the problem is unbalanced	*iii. When number of allocations is not equal to the order of the matrix, the problem is unbalanced*
iv. The number of allocations in a row / column may exceed more than one	*iv. Only one allocation is made in each row and column*
v. When all the net cell evaluations are greater than or equal to zero, the solution is optimum.	*v. When the number of allocations is equal to the order of the matrix, the solution is optimum.*
vi. May be degenerate /non degenerate problem	*vi. Always it is inherently degenerate*

4.13 The Traveling Salesman Problem (Routing Problem)

Suppose a salesman has to visit many cities, needs to start from a particular city, visit each city once, and then return to his starting point. The objective is to select the sequence in which the cities are visited in such a way that his total traveling time is minimised. Starting from a given city the salesman will have a total of $(n - 1)!$ different sequences (possible round trips).

If number of cities is only two, obviously there is no choice. If number of cities become three say P, Q and R one of them (say P) is the home base, then there are two possible ways (or) routes.

$A \rightarrow B \rightarrow C$ and $A \rightarrow C \rightarrow B$

For 4 cities A, B, C and D there are $3!= 6$ possible routes.

If it is having 6 cities,$5\times 4 \times 3 \times 2 = 120$, $5!= 120$ routes

Thus, it is practically impossible to find the best route by trying each one. In general, if there are 'n' cities, there are $(n- 1)!$ possible routes. Traveling sales man problem gives the best route without trying each one.

Applications of traveling salesman problem

i. ***Postal deliveries***

ii. ***Cable connections***

iii. ***Inspection***

iv. ***School bus routes***

4.13.1 Formulation of the traveling sales man problem

Suppose c_{ij} is the cost / inspection distance or time from city i to city j and $x_{ij}= 1$ if the sales man goes directly from city i to city j; and $x_{ij}= 0$ otherwise. Then minimum $\Sigma x_{ij}c_{ij}$ with the additional restriction that the x_{ij} must be chosen that no city is visited twice before the tour of all cities is completed. In particular, he cannot go directly from city i to j itself. This possibility may be avoided adopting the convention $c_{ij}= \infty$

The distance (or cost / time) matrix the problem is given by

$$\begin{array}{cc|ccccc|}
 & & & & \text{To} & & \\
 & & C_1 & C_2 & \cdots & C_n \\
\hline
 & C_1 & \infty & C_{12} & \cdots & C_{1n} \\
 & C_2 & C_{21} & \infty & \cdots & C_{2n} \\
\text{From} & \vdots & & & & \\
 & \vdots & & & & \\
 & C_n & C_{n1} & C_{n2} & \cdots & \infty \\
\hline
\end{array}$$

Further, since the salesman has to visit all the n cities (C_1, C_2, C_3 --- C_n) the optimal solution remains independent of selection of the starting point. Let us understand the traveling sales man problem through the following worked examples.

32. *Solve the following Routing (travelling salesman) problem.*

	1	2	3
1	–	8	4
2	8	–	10
3	4	6	–

Solution:

Replacing the diagonal elements with ∞ we get,

∞	8	4
8	∞	10
4	6	∞

Treating/considering the problem as assignment problem on solving we get,

Step 1: Row reduction

∞	4	0
0	∞	2
0	2	∞

Step 2: Column reduction

∞	2	0
0	∞	2
0	0	∞

Step 3: Assignments:

	1	2	3
1	∞	2	$\boxed{0}$
2	$\boxed{0}$	∞	2
3	0	$\boxed{0}$	∞

Solution is optimal as number of allocations = order of the Matrix

Further, if we form/write the route with the assignments obtained.

$1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ (feasible as the condition of routing problem is satisfied).

(**Note:** The assignment problem solution itself is optimal and feasible solution).

Hence, the optimal and feasible solution:

is $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$, cost: $4 + 6 + 8 = 18$ units.

33. *Solve the following assignment problem. If it is treated as a salesman problem and the cell entries represent cost in rupees, find the least cost route such that salesman does not visit any city twice.*

	A	B	C	D	E
A	–	2	5	7	1
B	6	–	3	8	2
C	8	7	–	4	7
D	12	4	6	–	5
E	1	3	2	8	–

Solution:

Reduced Matrix: Identifying the least element in each row and subtract it from other elements and identify the least element in each column and subtract it from other elements we get,

∞	1	4	6	0
	∞	1	6	0
4	3	∞	0	3
8	0	2	∞	1
0	2	1	7	∞

$\rightarrow$

∞	1	3	6	0
4	∞	0	6	0
4	3	∞	0	3
8	0	1	∞	1
0	2	0	7	∞

Assignments: Scanning the elements row wise and assigning for zero in a row having single zero and canceling the respective column we get,

To

From	A	B	C	D	E
A	∞	1	3	6	$\boxed{0}$
B	4	∞	$\boxed{0}$	6	0
C	4	3	∞	$\boxed{0}$	3
D	∞	$\boxed{0}$	1	∞	1
E	$\boxed{0}$	2	0	7	∞

No. of allocations = order of the matrix. Hence, the solution is optimum. Optimum solution is,

A – E – 1, B – C – 3 , C – D – 4, D – B – 4 , E – A – 1 with the total cost = ₹ 13

The solution is optimum for an assignment problem as number of allocations = order of matrix. But it is violating constraint of travelling salesmen problem as the routing A → E → A indicates that he visits E and will come back to city A with out covering in between cities.

Now examine the matrix for some of the next best solution to the assignment problem, and try to find out one solution which satisfies the additional constraint. The smallest element other than zero is 1, so try the effect of allocation to it. Start by making unity assignment in the cell (1, 2) instead of zero–assignment in the cell (1, 5) as shown in the above matrix.

Thus, the route satisfying the additional constraint of the problem is

A→B→C→D→E→A as the optimal and feasible route (solution).

The optimal cost is = 2+3+4+5+1 = ₹ 15.

34. *Solve the traveling salesman problem given by the following data,*

$C_{12} = 4,\ C_{13} = 7,\ C_{14} = 3,\ C_{23} = 6,\ C_{24} = 3$ *and* $C_{34} = 7$ *where* $C_{ij} = C_{ji}$

Solution:

The matrix form of the given data,

	1	2	3	4
1	∞	4	7	3
2	4	∞	6	3
3	7	6	∞	7
4	3	3	7	∞

Step 1: Row reduction: Identify the least element in each row and subtract it from other elements.

	1	2	3	4
1	∞	1	4	0
2	1	∞	3	0
3	1	0	∞	1
4	0	0	4	∞

Step 2: Column reduction:– Identify the least element in each column and subtract it from other elements.

	1	2	3	4
1	∞	1	1	0
2	1	∞	0	0
3	1	0	∞	1
4	0	0	1	∞

Step 3: Assignments:– Scanning the elements row wise and assigning for zero in a row having single zero and canceling the respective column we get,

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & \boxed{0} \\ 2 & 1 & \infty & \boxed{0} & 0 \\ 3 & 1 & \boxed{0} & \infty & 1 \\ 4 & \boxed{0} & 0 & 1 & \infty \end{array}$$

$1 \rightarrow 4 \rightarrow 1$

The above routing shows that it is violating the additional constraint of the travelling salesman problem. Thus, though it is optimum for an assignment problem it is not feasible for travelling salesman problem.

The other possible routes satisfying additional constraint taking the next least element to zero are,

Route	Length (Units)
$1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1$	3+3+6+7 = 19
$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$	4+6+7+3 = 20
$2 \rightarrow 1 \rightarrow 4 \rightarrow 3 \rightarrow 2$	4+3+7+6 = 20
$3 \rightarrow 1 \rightarrow 4 \rightarrow 2 \rightarrow 3$	7_3+3+6 = 23
$3 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 3$	7+3+4+7 = 21

The optimum route in the above possible routes is

$1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1$ and the route length is 19 units.