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By K B Hemanth Raj

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Game Theory and Metaheuristics

Learning objectives

After Studying this chapter, you should be able to

- structure the pay-off matrix for a game
- understand the terminology used in theory of games
- solve a game having saddle point
- use dominance rules to reduce size of the game and then solve
- solve 2 x M and N x 2 games by graphical method.

5.1 Introduction

Competition is a watch word of modern life. A great variety of competitive situations are seen commonly in everyday life. This includes business, military operations, advertising about a product, marketing etc., In the competitive world, it is essential for an executive to guess the activities or actions of his opponent or competitor.

is a decision theory in where one's choice of action is determined after taking into account all possible alternatives available to an opponent playing in a same field (game).

Game is defined as an activity between two or more persons (players) involving activities according to a set of rules. A competitive situation is called game in this context, competitors are referred as players. In this chapter we discuss formulation of the games and their solutions.

5.2 Basic Terms used in Game Theory

Player: A competitor in the game is known as a player. A player may be individual or group of individuals or organisations.

Strategy: A set of alternative courses of action (choices) available to a player based on advanced knowledge is known as strategy.

Pure Strategy: If the player selects / uses the same strategy each time, then it is referred to as pure strategy. In this case the situation is deterministic and the objective is to maximize the gain (or) minimize the loss. (Using only one type of technique / strategy while playing in the game.)

Mixed Strategy: If the player selects / uses his course of action in accordance with some fixed probability, then it is referred to as mixed strategy. It is probabilistic situation where the player uses a combination of strategies.

(Using the blend of strategies while playing in the game.)

Optimum Strategy: A course of action (choice) which puts the player in the most preferred position, irrespective of the strategy of his competitors, is referred as an optimum strategy.

Two person zero-sum game: When only two players are involved in the game and if the gain made by the one player is equal to the loss of the other, then it is called two persons zero – sum game.

Example

Assuming that there are only two types of beverages, tea and coffee, any market share gained by the tea will be equal to the loss of market share of coffee.

Sum of gains and losses is zero, hence the situation is referred as zero sum game.

Pay off matrix: The representation of gains and losses resulting from different actions of the competitors and is represented in the form of a matrix is pay off matrix.

Value of game: It is the expected payoff of the player when all the players of the game follow their optimum strategies.

Fair Game: If the value of the game is zero, it is referred as a fair game.

Note: Pay off is also referred as outcome and in solving the problems payoff matrix and value of game is with reference to the row player. That is, a positive sign indicates the gain to row player, a negative sign indicates loss to him and for column player it is converse.

5.3 Formulation of Two Persons-Zero Sum Game

When the players select their particular strategies, the pay offs (gains or losses) can be represented in the form of a matrix called the pay off matrix. In general we assume the game as zero – sum game. That is, the gain of one player is equal to the loss of other and vice versa, in other words, one player's pay off table would contain the same elements in pay off table of the other player with the sign changed. Thus, it is sufficient to construct / formulate pay off only for one of the players.

Let player A have 'm' strategies A_1, A_2, – – – A_m and player B have n strategies B_1, B_2, – – – B_n. Assuming that each player has his choices from amongst the pure strategies. Also assuming that player A is always the winner and player B is always the loser. That is, all pay offs are assumed in terms of player A. Let a_{ij} be the payoff, which player A gains from the player B, if player A chooses strategy A_i and player B chooses strategy B_j. Then the payoff matrix to player A is

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & a_{3n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

The pay off matrix to player B is = $(-a_{ij})$ i.e., same pay off elements with opposite sign

5.4 Properties (Characteristics) of a Game

i. *There are a finite number of competitors called players.*

ii. *Each player has a finite number of possible courses of action called strategies.*

iii. *The gain or loss is shown as the outcome of strategies in a matrix form called pay off matrix.*

iv. *The game is said to be a fair game if its value is zero.*

v. *The pay-off is fixed and determined in advance.*

5.5 Assumptions made in Game Theory

i. *For each player a finite number of courses of action (strategies) are available. The list may not be the same for each player.*

ii. *Player 'A' (row player) attempts to maximise the gains and player 'B' attempts to minimise the losses.*

iii. *The decisions of both the players are made individually, prior to the play with no communication between them*

iv. *The decisions are made, announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.*

v. *Both the players know not only the pay-offs to themselves but also of each other.*

5.6 Applicatons of Game Theory

Game theory has come to play an increasingly important role in logic and in computer science. Several logical theories have a basis in game semantics.

Few applications of Game theory:

i. *These are used to model the interactive computations in the field of computer science.*

ii. *Army generals make use of this concept to plan the war strategies.*

iii. *To provide theoretical basis to the field of multi-agent systems.*

iv. *To evaluate / resolve the conflicting situations between the individuals and organisations.*

v. *To determine the best course of action for a firm in view of the expected counter moves from the competetors.*

Worked Examples

1. *A and B are playing a game with 2 coins according to the following rules.*
   a. When both are heads (H) it is a benefit or gain of ₹ 1 to player A
   b. When there is one head and tail (T) then it is a loss of ₹ 1 to player A, and
   c. When there are two tails, there is no loss or gain to any player. Formulate this as pay off matrix.

Solution:

It is given that when there are two heads, ₹1 is gain to the player A and when there is one head and tail. It is ₹1 loss to the player A and when there are two tails, no loss, no profit to any player. The pay off matrix for the player A is shown below which is nothing but formulation of the game.

(H, T indicates turning of the coin as head or tail. These are nothing but strategies for the players)

		B: H	B: T
A	H	1	– 1
	T	– 1	0

Pay off matrix

2. *In a game played by fingers two players A and B are simultaneously showing 2 or 3 fingers. When the sum of the fingers is odd A gets the points equal to the sum. When the sum of fingers is even A loses the points equal to the sum. Write the pay off matrix.*

Solution:

The pay off matrix for the player A is shown below.

		2	3
A	2	– 4	5
	3	5	– 6

Where 2 and 3 are the number of fingers (strategies) that can be shown by the players.

5.7 Max. Min Principle

The row player (say A), will select the maximum out of the minimum gains (Max. min). A minimu value in each row represents the least gain (payoff) to him, if he selects this particular strateg These values are written in the matrix by row minima. He will then select the strategy that maximiz his minimum gains. This choice of the player is called Max. min principle.

Example

B

				Row minima	
A	4	5	8	$\boxed{4}$	Max. min
	7	– 8	6	– 8	
	1	9	0	0	

Row minima indicates minimum gains of the row player against of his each strategy.

Max. min is the maximum benefit out of these minimum gains

5.8 Min. max Principle

The column player (say player B) will always try to minimize his maximum lasses (Min. max). For column player, the maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum losses. This choice of player 'B' is called the *Mini max* principle.

Example

In the pay off matrix

B

A	5	4	6
	0	3	2
	7	8	1
Column maxima	7	8	$\boxed{6}$
			Min. max

Column maximum indicates his maximum losses for his different strategies. Min. max is the minimum loss out of the maximum losses.

5.9 Procedure to Determine Saddle Point

Saddle point

If the Maxi min value is equal to the Mini max value, then the game is said to have a saddle point or equilibrium point.

Step 1

Select the minimum element of each row of the pay off matrix and write it against that particular row, identify the maximum out of these and round it (Max. min).

Step 2

Select the maximum element of each column of the pay off matrix and write it against that particular column, identify the minimum element out of these and round it (Min. max).

Step 3

If it appears Max. min = Min. max, the position of that element is a saddle point of the payoff matrix.

5.10 Solution of a Game

Determining the optimal strategy / strategies for each player and the value of game is referred as solution of the game. A game can be solved by using the following three methods / concepts, based on the nature of the problem.

i. *Saddle point concept / Min. max and Max. min principle.*

ii. *Dominance rule / concept.*

iii. *Graphical method.*

Each of these are discussed in detail with examples.

5.10.1 Solution of the game having saddle point (Pure Strategy)

If the Max. min value equals the Min. max value, then the game is said to have a saddle (equilibrium) point and the corresponding strategies are called optimum strategies. The amount of pay off at saddle point is known as the value of game (The intersection element of Max. min and Min. max).

3. *Solve the following two person zero sum game with the following 3×2 pay off matrix for player A.*

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 9 & 2 \\ 8 & 6 \\ 6 & 4 \end{bmatrix} \end{array}$$

Solution:

Step 1

Write the row minima against each strategy (A_1, A_2, A_3) and identify the highest among these (That is 6).

$$\begin{array}{lccl} & B_1 & B_2 & \text{Row minima} \\ A_1 & 9 & 2 & 2 \\ A_2 & 8 & 6 & \boxed{6}\ \text{Max. min} \\ A_3 & 6 & 4 & 4 \\ \text{Column maxima} & 9 & \boxed{6}\ \text{Min. max} & \end{array}$$

Step 2

Write the column maxima against each strategy (B_1, B_2) and identify the minimum element between these two (that is 6).

Thus, Min. max = Max. min

Hence, the game is having saddle point. Identify the optimal strategies for the both the players corresponding to these values by drawing horizontal and vertical lines. The intersection of these two lines gives the value of game.

i. ***Optimal strategy for player A is A_2.***

ii. ***Optimal strategy for player B is B_2 and***

iii. ***Value of the game = 6***

Observation

Though player *A* is having three strategies and player *B* 2 strategies, they are using a single strategy. Hence the nature of the game is deterministic / strictly determinable in nature (pure strategy game).

4. ***Solve the game whose pay off matrix is given below. Give the value of game and strategies by A and 'B'.***

	B_1	B_2	B_3	B_4
A_1	–5	2	0	7
A_2	5	6	4	8
A_3	4	0	2	–3

Solution:

Given

	B_1	B_2	B_3	B_4	Row minima
A_1	–5	2	0	7	–5
A_2	5	6	4	8	4 Max–min
A_3	4	0	2	–3	–3
Column maxima	5	6	4	8	
			Min.Max		

The game having saddle point as max min = min. max

The solution is,

Best strategy for player A : A_2

Best strategy for player B : B_3 , value of game = 4

Observation: Though player A is having three strategies and player B is having two strategies, they are using a single strategy. Hence the nature of the game is deterministic / strictly determinable in nature (pure strategy game).

5. ***Solve the game whose payoff matrix is given by,***

Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Solution:

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ \text{Column maxima} \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \\ \begin{array}{ccc} \boxed{1} & 5 & \boxed{1} \end{array} \end{array} \begin{array}{l} \text{Row minima} \\ \boxed{1} \; \text{Max. min} \\ -4 \\ -1 \\ \text{Min. max} \end{array}$$

Min. max = 1

Max. min = 1

Hence, the game is having saddle point

i. The optimal strategy for player A is A_1

ii. The optimal strategy for player 'B' is B_1 or B_3

iii. The value of game = + 1

	B_1	B_2	B_3	
A_1	1	−3	1	$\boxed{1}$
A_2	0	−4	−3	−4
A_3	1	5	−1	−1
	$\boxed{1}$	5	1	

or

	B_1	B_2	B_3	
A_1	1	−3	1	1
A_2	0	−4	−3	−4
A_3	1	5	−1	−1
	1	5	$\boxed{1}$	

This game is having alternative optimal solution as optimal strategy for player B can be B_1 or B_3.

6. *Solve the following game whose pay off matrix is given below*

		B		
		I	*II*	*III*
A	*I*	*− 3*	*− 2*	*6*
	II	*2*	*0*	*2*
	III	*5*	*− 2*	*− 4*

Solution:

	I	II	III	Row minima
I	−3	−2	6	−3
II	2	$\boxed{0}$	2	$\boxed{0}$ Max. min
III	5	−2	−4	−4
Column maxima	5	$\boxed{0}$ Min. max	6	

We have Maxi min = Mini max, therefore saddle point exists. The game is *deterministic in* nature.

Best strategy for A = II

Best strategy for B = II, Value of game = 0

The above game is referred as a fair game as the value of it is zero.

7. *Solve the following game whose pay off matrix is given in the following matrix.*

		B I	B II	B III
A	I	2	– 1	8
	II	– 4	– 3	4
	III	– 8	– 4	0
	IV	1	– 6	– 2

Solution:

	I	II	III	Row minima
I	2	–1	8	$\boxed{-1}$ Maxi min
II	–4	–3	4	–4
III	–8	–4	0	–8
IV	1	–6	–2	–6
Column maxima	2	$\boxed{-1}$ Mini max	8	

Best strategy for A = I, Best strategy for B = II, Value of game = – 1

Whenever, Maxi min = Mini max, there exists saddle point or equilibrium point and the game is deterministic or pure strategy game.

Note: *Player A will try to maximize his minimum gains and player B will try to minimize his maximum losses. Hence Maxi min is applicable to player A and Mini max is applicable to player B*

8. *Consider the game G with the following pay off. Determine the value of game ignoring the value of λ.*

	B_1	B_2
A_1	2	6
A_2	– 2	λ

Solution:

Ignoring the value of λ,

			Row minima
	2	6	$\boxed{2}$ Max. min
	– 2	λ	– 2
Column maxima	$\boxed{2}$ Min. max	6	

As Maxi min = Mini max, saddle point exists. Value of the game = 2

Best strategy for row player = A_1, Best strategy for column player = B_1

9. *For what value of λ, the game with the following pay off matrix is strictly determinable (pure strategy).*

		B I	B II	B III
A	I	λ	6	2
	II	-1	λ	-7
	III	-2	4	λ

Solution:

Ignoring the value of λ whatever we get Maxi min = 2, Mini max = − 1.

				Row minima
	λ	6	2	$\boxed{2}$ *Maxi min*
	-1	λ	-7	-7
	-2	4	λ	-2
Column maxima	$\boxed{-1}$	6	2	
	Mini max			

For strictly determinable the value of λ is $-1 \leq \lambda \leq 2$.

That is in the range between Maxi min and Mini max values

10. *Determine the range of values of p and q that will make the pay off matrix (a_{ij}) given below, a deterministic game in nature.*

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{bmatrix} 2 & 4 & 7 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} \end{array}$$

Solution:

Let us first consider the game to determine Maxi. min and Mini. max ignoring the values p and q

$$\begin{array}{ccc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{Row minima} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 2 & 4 & 7 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} & \begin{array}{l} 2 \\ \boxed{7}\ \text{Maxi min} \\ 4 \end{array} \\ \text{Column maxima} & \begin{array}{ccc} 10 & \boxed{7} & 8 \end{array} & \\ & \text{Mini max} & \end{array}$$

Thus the Maxi min = Mini max = 7

Thus there exists a saddle point at position (2, 2).

This imposes / requires the condition on 'p' as $p \leq 7$ and q as $q \geq 7$

Hence, $p \leq 7$, $q \geq 7$ is the range of 'p' and 'q'.

Note: *Check for thwesaddle point assuming value of 'p' as ≤ 7 and the value of 'q' ≥ 7.*

5.10.2 Games without Saddle Point (Mixed Strategy)

In certain cases, no pure strategy solution exists for the game. In other words, saddle point do not exists. In such cases, both the players may adopt on optimal blend of strategies called mixed strategies. In case, where there is no saddle point the game must be reduced to 2 × 2, either by dominance property or graphical means (method) which can be solved as explained below

A 2 × 2 game without saddle point

Solution of a 2 × 2 game without saddle point

$$\begin{array}{cc} & \text{B} \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \text{A} \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

Let x_1, x_2 be the probabilities of using 1st and 2nd strategies by player A such that, $x_1 + x_2 = 1$

Let y_1, y_2 be the probabilities of using 1st and 2nd strategies by player B such that, $y_1 + y_2 = 1$, (sum of the probabilities is = 1).

then

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}; \qquad x_2 = 1 - x_1$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}; \qquad y_2 = 1 - y_1$$

$$v = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

***11.** Two players A and B are playing a game of tossing a coin simultaneously. Player 'A' wins 1 unit of value when there are two heads, wins nothing when there are two tails and looses $\frac{1}{2}$ unit of value when there is one head and one tail. Determine the pay off matrix, the best strategies for each player and value of the gone.*

Solution:

Let H, T are the two strategies for the players of showing Head and Tail of the coin respectively. Following the rules as per the data given in the problem.

The pay-off matrix/formulation of the game is,

$$\begin{array}{ccc} & \multicolumn{2}{c}{\boldsymbol{B}} \\ & \boldsymbol{H} & \boldsymbol{7} \end{array}$$

$$A \quad \begin{array}{c} \boldsymbol{H} \\ \boldsymbol{T} \end{array} \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

The above matrix can be written as,

$$\begin{array}{ccc} & \multicolumn{2}{c}{\boldsymbol{B}} \\ & \boldsymbol{H} & \boldsymbol{T} \end{array}$$

$$\boldsymbol{A} \quad \begin{array}{c} \boldsymbol{H} \\ \boldsymbol{T} \end{array} \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$ (multiplying all the elements by 2 to avoid the fractions)

Solution:

Let x_1, x_2 and y_1, y_2 be the probabilities of using H, T by the players:

Solving the pay-off matrix we get,

$$x_1 = \frac{0-(-1)}{2-(-2)} = \frac{1}{4}, \; x_2 = \frac{3}{4} \qquad (x_2 = 1 - x_1)$$

$$y_1 = \frac{0-(-1)}{2-(-2)} = \frac{1}{4}, \; y_2 = \frac{3}{4} \qquad (y_2 = 1 - y_1)$$

$$v = \frac{0-(+1)}{2-(-2)} = \frac{-1}{4}$$

12. *Solve the following game whose pay off matrix is*

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \end{array}$$

14. *In a game of matching coins, player 'A' wins ₹ 8, if both coins show heads and ₹ 1 if both are tails. Player B wins ₹ 3 when coins do not match. Given the choice of being Player A or Player B, which would you choose and what would be your strategy?*

Solution:

The formulation of the game is,

$$\begin{array}{cc} & \begin{array}{cc} & B \\ & \begin{array}{cc} H & T \end{array} \end{array} \\ A \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Let showing head, tail of the coin be the strategies as H, T

If both are heads 'A' will gain 8 points, if there is one head and one tail player 'A' will get 1 point and if both are the tail 'A' looses 3 points.

Let x_1, x_2 be the probabilities of showing head, tail by player 'A', Let y_1, y_2 be the probabilities of showing head, tail by player 'B' respectively

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad x_2 = 1 - x_1 \quad y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad y_2 = 1 - y_1$$

and the value of game, $v = \dfrac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the above equations we get,

$$x_1 = \frac{(1)-(-3)}{(8+1)-(-3-3)} = \frac{4}{9-(-6)} = \frac{4}{15}, \quad x_2 = 1 - x_1 = 1 - \frac{4}{15} = \frac{11}{15}$$

$$y_1 = \frac{(1)-(-3)}{(8+1)-(-3-3)} = \frac{1+3}{9-(-6)} = \frac{4}{15}, \quad y_2 = 1 - \frac{4}{15} = \frac{11}{15}$$

$$v = \frac{(8)(1)-(-3\times-3)}{(8+1)-(-3+-3)} = \frac{8-9}{9+6} = \frac{-1}{15}$$

Hence, the solution of the game is

Player	Probabilities of using	
	Head	Tail
A	4/15	11/15
B	4/15	11/15

with the value of game as $-1/15$.

Choice: *is to be player 'B' as 'A' is the looser as per the value of game. Strategies: The probabilities of using head and tail of the coin are 4/15, 1/15.*

5.11 Dominance Rule

The size of a game can be reduced by eliminating a strategy, which is inferior to another. Such a strategy is said to be dominated by the other. The concept of dominance is especially useful for the evaluation of two person zero – sum games where a saddle point does not exist. In general the following rules (or properties) of dominance are used to reduce the size of payoff matrix.

i. **Row dominance:** ***If every element in a particular row is less than or equal to the corresponding element of another row, then the former row is said to be inferior or dominated by the later row. Therefore, the player will never employ the former row. Hence, this can be deleted.***

Example

B

A		C_1	C_2	C_3
	R_1	4	6	− 5
	R_2	5	6	− 3

R_2 dominates R_1 as all the elements of $R_2 \geq$ all the elements of R_1. Hence, R_1 can be deleted (As it is inferior to R_2).

ii. **Column dominance:** ***If every element in a particular column is greater than or equal to corresponding elements of another column, the former column never yields a better result than the later and therefore the former column will be used never. Hence, former column can be deleted.***

Example

B

A		C_1	C_2
	R_1	4	2
	R_2	0	− 1
	R_3	− 3	− 3

C_2 dominates C_1 as all the elements of $C_2 \leq$ the elements of C_1

iii. **Modified Row Dominance:** ***When no single row strategy has dominance over the other, then the comparison may be made between a row and the average of group of rows. If every element*** *is less than or equal to the average of corresponding elements of the group, then the former row can be deleted. If a row dominates over the average of group of rows, then the group may be discarded.*

Example

		B	
		B_1	B_2
A	A_1	2	4
	A_2	5	0
	A_3	0	8

There is no pure row dominance considering the average of 2nd, 3rd rows we have $\left(\frac{5+0}{2}, \frac{0+8}{2}\right) = (2.5, 4)$.

Therefore, the average of second and third rows is dominating over the first row hence, deleting the first row, the reduced matrix is

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_2 \\ A_3 \end{array} & \begin{bmatrix} 5 & 0 \\ 0 & 8 \end{bmatrix} \end{array}$$

Modified Column Dominance

If there is no single column dominance over the other, then the comparison may be made between a column and the average of group of columns. If every element is greater than or equal to the average of corresponding elements of the group, then the former column can be deleted.

If a column dominates over the average of group of columns, then the group may be discarded.

Example

		B		
		B_1	B_2	B_3
A	A_1	8	15	1
	A_2	3	− 1	4

There is no pure column dominance considering the average of 2nd, 3rd columns.

We have $\left(\frac{15+1}{2} = 8, \frac{-1+4}{2} = 1.5\right)$

As all the elements of the average $\leq B_1$, B_1 is inferior. Hence, delete B_1 and the reduced matrix is

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		B_2	B_3
A	A_1	15	1
	A_2	− 1	4

15. *Solve the following game by using the dominance concept*

		Player B		
		B_1	B_2	B_3
Player A	A_1	4	5	8
	A_2	6	4	6
	A_3	4	2	4

Solution:

Let us find the row minima and column maxima

		Player B			
		B_1	B_2	B_3	Row minima
Player A	A_1	4	5	8	4
	A_2	6	4	6	$\boxed{4}$ Maxi min
	A_3	4	2	4	2
column maxima		6	8	$\boxed{5}$	Mini max

Since Maxi min ≠ Mini max, we can't use the method of pure strategy (saddle point concept) to solve the game.

It can be seen that row A_2 dominates row A_3 as every element of row $A_2 \geq$ row A_3, thus row A_3 can be deleted. The resulting matrix is

	B_1	B_2	B_3
A_1	4	5	8
A_2	6	4	6

Column B_1 dominates column B_3 as the element values of $B_1 \leq$ element of B_3

Hence B_3 can be deleted. The reduced matrix is,

	B_1	B_2
A_1	4	5
A_2	6	4

Let the probabilities of using the strategies A_1, A_2, A_3, be x_1, x_2, x_3 respectively and the probabilities of using B_1, B_2, B_3 strategies by player 'B' be y_1, y_2, y_3 respectively such that $x_1 + x_2 + x_3 = 1$

$y_1 + y_2 + y_3 = 1$

In the reduced matrix $a_{11} = 4, a_{12} = 5, a_{21} = 6, a_{22} = 4$

Then using the formula,

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \qquad x_2 = 1 - x_1$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \qquad y_2 = 1 - y_1$$

$v = \dfrac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$, Substituting the values of $a_{11}, a_{12}, a_{21}, a_{22}$ in the above equations, we get,

$$x_1 = \frac{2}{3},\ x_2 = \frac{1}{3},\ x_3 = 0,\ y_1 = \frac{1}{3},\ y_2 = \frac{2}{3},\ y_3 = 0$$

and the value of game $v = \dfrac{14}{3}$

i. *Probabilities of using I, II, III strategies by the player A*

$$x_1 = \frac{2}{3},\ x_2 = \frac{1}{3},\ x_3 = 0$$

ii. *Probabilities of using I, II, III strategies by the player B*

$$y_1 = \frac{1}{3},\ y_2 = \frac{2}{3},\ y_3 = 0$$

iii. *The value of game* $v = \dfrac{14}{3}$

16. *Solve the following game by dominance concept*

$$\begin{array}{ccc} & & B \\ & & \begin{array}{ccc} I & II & III \end{array} \\ A & \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix} \end{array}$$

Solution:

Step 1:

All the elements of row II ≥ row III and row II is superior to row III. Hence delete row III.

1	7	2
6	2	7

Step 2:

As all the elements of column I ≤ column III, column I is superior than column III. Hence, delete column III.

	I	II
I	1	7
II	6	2

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{2-6}{(3)-(6+7)} = \frac{-4}{-10} = \frac{2}{5}, \qquad x_2 = \frac{3}{5}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{2-7}{-10} = \frac{-5}{-10} = \frac{1}{2}, \qquad y_2 = \frac{1}{2}$$

$$v = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{2-42}{-10} = \frac{-40}{-10} = 4$$

Probabilities of using I, II and III strategies by player A (x_1, x_2, x_3) is $\left(\frac{2}{5}, \frac{3}{5}, 0\right)$

Probabilities of using I, II and III strategies by player B (y_1, y_2, y_3) is $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Value of the game is = 4

17. *Use the dominance principle, solve the following game.*

$$\begin{array}{cc|ccc|}
 & & & B & \\
 & & I & II & III \\
\hline
 & 1 & 1 & -3 & -2 \\
A & 2 & 0 & -4 & 2 \\
 & 3 & -5 & 2 & 3 \\
\hline
\end{array}$$

Solution:

		B			
		I	II	III	Row minima
A	1	1	– 3	– 2	– 3
	2	0	– 4	2	– 4
	3	– 5	2	3	– 5
Column maxima		1	2	3	

As there is no saddle point, as Max min is not equal to Min max. It is a mixed strategy game

Column Dominance

Column II dominates column III, eliminating column II we get,

		B	
		I	II
A	1	1	– 3
	2	0	– 4
	3	– 5	2

Row dominance

Row 1 dominates row 2, eliminating row 2 we get,

	I	II
1	1	– 3
3	– 5	2

Comparing the formulated game with

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we have,

Let x_1, $x_{2,}$ $x_3$be the probabilities for player 'A', Let y_1, $y_{2,}$ y_3 be the probabilities for player 'B' respectively

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the formulae we get,

$$\text{Strategies of } A = \begin{pmatrix} 1 & 2 & 3 \\ \frac{7}{11} & 0 & \frac{4}{11} \end{pmatrix}$$

$$\text{Strategies of } B = \begin{pmatrix} I & II & III \\ \frac{5}{11} & \frac{6}{11} & 0 \end{pmatrix} = \frac{1\times7+(-5)4}{7+4} = \frac{-13}{11}$$

18. *Determine nature of the following game whose pay off matrix is,*

		B		
		I	II	III
A	I	−1	2	1
	II	1	−2	2
	III	3	4	−3

Solution:

	I	II	III	Row minima
I	−1	2	1	$\boxed{-1}$ Maxi min
II	1	−2	2	−2
III	3	4	−3	−3
Column maxima	3	4	$\boxed{2}$ Mini max	

Since the Mini.max ≠ Maxi.min, there is no saddle point that is, it is probabilistic game or mixed strategy game.

19. *Obtain the optimal strategies for both persons and the value of game whose pay-off matrix is as follows:*

		B				
		I	II	III	IV	V
A	I	2	5	10	7	2
	II	3	3	6	6	4
	III	4	4	8	12	1

Solution:

By observation: No pure row dominance exists when compared one row with another row.

By comparing the columns, Column V dominates simultaneously Column III, Column IV (as the elements of the Column V ≤ elements of Columns III and IV). Deleting the columns III and IV we get,

		B		
		I	II	IV
A	I	2	5	2
	II	3	3	4
	III	4	4	1

Again by observation all the elements of Column I ≤ elements of Column II. Hence, Column II. Hence, Column I dominates Column II, deleting Column II we get,

		B	
		I	IV
A	I	2	2
	II	3	4
	III	4	1

In the above matrix/game, Row II dominates R-I as the elements of R II ≥ R I. Hence, deleting R I we get

		B	
		I	IV
A	II	3	4
	III	4	1

Let x_1, x_2, x_3 be the probabilities of using I, II and III strategies by player A and y_1, y_2, y_3, y_4 and y_5 be the probabilities of using I, II, III, IV and V strategies by player B.

$x_1 = 0$, $y_2 = y_3 = y_5 = 0$ (as the corresponding rows / columns are deleted)

On solving the above reduced game we get,

$$x_2 = \frac{1-4}{4-8} = \frac{-3}{-4} = \frac{3}{4}, \qquad x_3 = \frac{1}{4} \qquad (x_3 = 1 - x_2)$$

$$y_1 = \frac{1-4}{4-8} = \frac{3}{4}, \; y_4 = \frac{1}{4} \quad (y_4 = 1 - y_1)$$

$$v = \frac{3-16}{4-8} = \frac{-13}{-4} = \frac{13}{4}$$

20. *Use the dominance principle to solve the following game.*

		B			
		I	II	III	IV
A	1	20	15	12	35
	2	25	14	8	10
	3	40	2	19	5
	4	5	4	11	0

Solution:

There is no saddle point as max. min is not equal to min. max.

Therefore it is mixed strategy problem, the value of the game lies between 12 and 15 [in between max. min and min. max value]

Row dominance: Therefore row one is set dominate row – 4, eliminating row – 4 we get

		B I	II	III	IV
A	1	20	15	12	35
	2	25	14	8	10
	3	40	2	19	5

Column dominance: Column II is set to be dominating column I. Therefore eliminating column I we get,

	II	III	IV
1	15	12	35
2	14	8	10
3	2	19	5

Again row 1 is dominates row– 2, deleting row –2 we get

	II	III	IV
1	15	12	35
3	2	19	5

Column 2 is set to dominate column 4, eliminating column 4 we get,

	II	III
1	15	12
3	2	19

Comparing the reduced game with

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Let x_1, x_2, x_3, x_4 be the probabilities for player 'A', Let y_1, y_2, y_3, y_4 be the probabilities of for the player 'B' of the strategies respectively

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the formulae we get,

$$\text{Strategies of } A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{17}{20} & 0 & \frac{3}{20} & 0 \end{pmatrix}$$

Strategies of $B = \begin{pmatrix} I & II & III & IV \\ 0 & \frac{7}{20} & \frac{13}{20} & 0 \end{pmatrix}$ and the value of game

$$v = \frac{(12 \times 17) + (19 \times 3)}{17 + 3} = \frac{204 + 57}{20} = \frac{261}{20}$$

21. *Solve the following 2 person zero sum game based on the concept of dominance.*

$$\begin{array}{c|ccc} & I & II & III \\ \hline I & -4 & 6 & 3 \\ II & -3 & -3 & 4 \\ III & 2 & -3 & 4 \end{array}$$

Solution:

Let x_1, x_2 and x_3 be the probabilities of I, II and III strategies for player A; y_1, y_2 and y_3 be the probabilities for player 'B'.

It can be observed that all the elements of $R_3 \geq$ corresponding elements of R_2. Hence, delete R_2.

The reduced matrix will be,

	I	II	III
I	− 4	6	3
III	2	− 3	4

Comparing column wise all the elements of 3rd column ≥ the corresponding elements of 1st column.

So, the player 'B' will never use 3rd strategy

Thus, the reduced matrix is,

	I	II
I	− 4	6
III	2	− 3

Comparing the reduced matrix with $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and on solving we get,

(As the reduced game is 2 × 2 game without saddle point)

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{12})} = \frac{1}{3},\ x_2 = 0,\ x_3 = \frac{2}{3} \text{ (Sum of the probabilities = 1)}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{12})} = \frac{3}{5}, y_2 = \frac{2}{5}, y_3 = 0$$

$$\text{and Value of game } v = \frac{a_{11}a_{22} - a_{21}a_{22}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = 0$$

$\therefore$ It is a fair game as the value of game is zero.

Thus, the optimal strategies are,

For player A (I, II, III) with probabilities $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$

For player B (I, II, III) with probabilities $\left(\frac{3}{5}, \frac{2}{5}, 0\right)$

Note: Dominance property is used when the saddle point does not exists. Prior to applying the dominance check for the existence of the saddle point.

Some times a game may have the saddle point but it may be given to solve by dominance concept. In such cases, the game can reduced to 1 × 1 game i.e., to a single element by dominance concept The following example illustrates the concept of solving such games.

22. *Using the dominance concept, obtain the optimal strategies for both the players and determine the value of game. The pay off matrix for player A is given.*

		B: I	II	III	IV	V
A	I	2	4	3	8	4
	II	5	6	3	7	8
	III	6	7	9	8	7
	IV	4	2	8	4	3

Solution:

This problem is having saddle point (min max = max min = 6). But as it is given to use the dominance property, let us solve it by dominance rule only. By inspection of rows. It is clear that row III dominates row IV as all the elements of row III are ≥ row IV. Hence, row IV can be deleted. The resulting matrix is

$$
\begin{array}{cc|ccccc|}
 & & & & B & & \\
 & & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\
\cline{3-7}
 & \text{I} & 2 & 4 & 3 & 8 & 4 \\
A & \text{II} & 5 & 6 & 3 & 7 & 8 \\
 & \text{III} & 6 & 7 & 9 & 8 & 7 \\
\cline{3-7}
\end{array}
$$

We can see that column I dominates column IV as all the elements of column I are ≤ column IV. Hence, column IV can be deleted. The resulting matrix is,

$$
\begin{array}{cc|cccc|}
 & & & & B & \\
 & & \text{I} & \text{II} & \text{III} & \text{V} \\
\cline{3-6}
 & \text{I} & 2 & 4 & 3 & 4 \\
A & \text{II} & 5 & 6 & 3 & 8 \\
 & \text{III} & 6 & 7 & 9 & 7 \\
\cline{3-6}
\end{array}
$$

It can be seen that row III dominates row I. Hence, delete row I.
The resulting matrix is

$$
\begin{array}{cc|cccc|}
 & & & & B & \\
 & & \text{I} & \text{II} & \text{III} & \text{V} \\
\cline{3-6}
A & \text{II} & 5 & 6 & 3 & 8 \\
 & \text{III} & 6 & 7 & 9 & 7 \\
\cline{3-6}
\end{array}
$$

Column I dominates column V

Hence, deleting column V we get

$$
\begin{array}{cc|ccc|}
 & & & B & \\
 & & \text{I} & \text{II} & \text{III} \\
\cline{3-5}
A & \text{II} & 5 & 6 & 3 \\
 & \text{III} & 6 & 7 & 9 \\
\cline{3-5}
\end{array}
$$

Column I dominates column II

Hence delete column II. The resulting matrix is

$$
\begin{array}{cc|cc|}
 & & B & \\
 & & \text{I} & \text{III} \\
\cline{3-4}
A & \text{I} & 5 & 3 \\
 & \text{III} & 6 & 9 \\
\cline{3-4}
\end{array}
$$

Row I dominates row II, deleting it we get,

		B	
		I	II
A	III	6	9

Again column I dominates column II

Hence, delete column II

Thus

		B
		I
A	III	6

is the reduced matrix (element).

Best strategy for player – A III, Best strategy for player – B I, Value of game = 6

Note: ***i. The same answer can be verified with saddle point concept***

ii. If the dominance principle is applied to the pay–off matrix having a saddle point, then we get a single element reduced matrix only.

23 *Solve the following game by using the concept of dominance*

		B			
		I	*II*	*III*	*IV*
A	*I*	*3*	*2*	*4*	*0*
	II	*3*	*4*	*2*	*4*
	III	*4*	*2*	*4*	*0*
	IV	*0*	*4*	*0*	*8*

Solution:

					Row minima	
	3	2	4	0	0	
	3	4	2	4	$\boxed{2}$	Maxi min
	4	2	4	0	0	
	0	4	0	8	0	
Column maxima	$\boxed{4}$	4	4	8		

Min. max

As there is no saddle point let us use the dominance concept

Step 1:

All the elements of $R_3 \geq R_1$. R_3 is superior, delete R_1

	I	II	III	IV
II	3	4	2	4
III	4	2	4	0
IV	0	4	0	8

All the elements of $C_3 \leq C_1$, C_1 is inferior, C_3 is superior, delete C_1.

	II	III	IV
II	4	2	4
III	2	4	0
IV	4	0	8

There is no pure dominance of rows or columns. Hence, average of two rows or columns can be considered.

The average is C_3 and C_4 $\left(\frac{2+4}{2}, \frac{4+0}{2}, \frac{0+8}{2}\right) \leq C_2$

Therefore, C_2 is inferior. Hence, delete C_2.

	II	IV
II	2	4
III	4	0
IV	0	8

The average of R_3 and $R_4 \geq R_2$, R_2 is inferior, hence deleting R_2 we get,

	III	IV
III	4	0
IV	0	8

Let x_1, x_2, x_3, x_4 be the probability of using I, II, III & IV strategies by player A and y_1, y_2, y_3, y_4 be the probabilities of using I, II, III & IV by player B. As strategies, I, II of both the players are deleted we get,

$$x_3 = \frac{8-0}{12-0} = \frac{8}{12} = \frac{2}{3}, \qquad x_4 = \frac{1}{3}$$

$$y_3 = \frac{8-0}{12-0} = \frac{8}{12} = \frac{2}{3}, \qquad y_4 = \frac{1}{3}$$

$$v = \frac{32-0}{12} = \frac{32}{12} = \frac{8}{3}$$

	Strategies			
	I	II	III	IV
Probabilities for A	0	0	2 / 3	1 / 3
Probabilities for B	0	0	2 / 3	1 / 3

Value of game is = 8 / 3

5.12 Graphical Method

Graphical method is useful for games of order 2 × M or N × 2 that is, when one player has two dominant strategies to mix while other has many to play. The advantage of this method is that it is relatively faster.

The graphical method enables to reduce the original 2 × M or N × 2 game to a 2 × 2 game. Then mixed strategy principle (formula) can be applied to solve the game.

Graphical method of 2 × *M* game

In this case, row player usually (player A) will have two strategies to play while the column player will have 'M' strategies. The graphical method is used to find which two strategies of the column player are to be used.

Graphical method of *N* × 2 game

In this case, column player will have two strategies while row player will have 'N' strategies. The graphical method is used to find which two strategies of the row player are to be used that is in both the concepts the game must be reduced to 2 × 2.

Steps involved in graphical method

Step 1

The game must be reduced to such a sub–game that at least one of the player's has only two strategies (if the game is of order M × N)

Step 2

Draw two parallel lines to include the boundaries of two strategies of first player, say 'A' (Any convenient distance may be taken to draw parallel lines).

Step 3

Pay off lines of the strategies of 'A' for different strategies of player 'B' are to be plotted on the graph as straight lines.

Step 4

i. *If the game is of 2 × M type, identified the lower boundary (formed with the straight lines) and locate highest point on it (Maxi min principle) say it is* P_{max}

ii. *If the game formed is of N × 2 type, identify the upper boundary (formed with the straight lines) and locate the lowest point on it (Mini max principle) say the point is* P_{min}.

Step 5:

Consider the strategies passing through the identified point (if more than two strategies are passing through the point, any two strategies can be considered to solve the game) and solve the 2 × 2 game as usual.

***2 × M* game Games:** The following examples illustrates solving 2 × M games.

24. *Solve the following 2 × 3 game by graphical method*

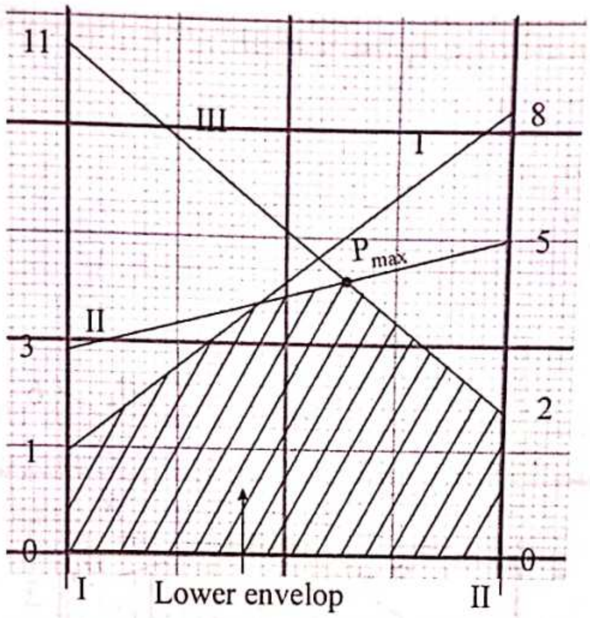
$$\begin{array}{cc|ccc|} & & & B & \\ & & I & II & II \\ \hline A & I & 1 & 3 & 11 \\ & II & 8 & 5 & 2 \\ \hline \end{array}$$

Solution:

For the given 2 × 3 game, the graph is plotted as shown.

P_{max} is the point highest point in the lower envelop passing through this point the corresponding strategies of the column player are II and III hence, the reduced matrix is

$$\begin{array}{cc|cc|} & & B & \\ & & II & III \\ \hline A & I & 3 & 11 \\ & II & 5 & 2 \\ \hline \end{array}$$



Now, the game is of 2×2 with out saddle point

Comparing the reduced matrix with $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and applying the following formulae

Let x_1, x_2 are the probabilities of using I, II strategies by player A.

Let y_1, y_2, y_3 are the probabilities of using I, II, III strategies by player B.

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

$$v = \frac{a_{11} a_{22} - a_{21} a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

We get,

$$x_1 = \frac{2-5}{5-16} = \frac{-3}{-11} = \frac{3}{11}, \; x_2 = \frac{8}{11}, \; (\because x_1 + x_2 = 1)$$

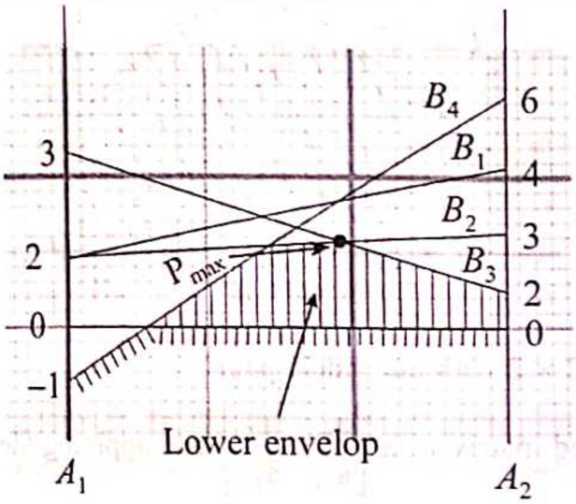
$$y_1 = \frac{2-11}{5-16} = \frac{-9}{-11} = \frac{9}{11}, \; y_2 = \frac{2}{11}, \; (\because y_1 + y_2 = 1)$$

$$v = \frac{6-55}{-11} = \frac{-49}{-11} = \frac{49}{11}$$

25. *Solve the following game by graphical method*

	B_1	B_2	B_3	B_4
A_1	2	2	3	-1
A_2	4	3	2	6

Solution:



B_2, B_3 and B_4 have the strategies at point P_{max}. Hence delete other strategies of B. Consider B_2, B_3. he reduced matrix will be

	B_2	B_3
A_1	2	3
A_2	3	2

The above game is a 2×2 game with out saddle point.

Let x_1, x_2 are the probabilities of using I, II strategies by player A.

Let y_1, y_2, y_3, y_4 are the probabilities of using I, II, III, IV strategies by player B.

$$x_1 = \frac{2-3}{4-6} = \frac{-1}{-2} = \frac{1}{2}, \qquad x_2 = \frac{1}{2}$$

$$y_2 = \frac{2-3}{4-6} = \frac{-1}{-2} = \frac{1}{2}, \qquad y_3 = \frac{1}{2}$$

$$v = \frac{4-9}{4-6} = \frac{-5}{-2} = \frac{5}{2}$$

$$(x_1, x_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$(y_1, y_2, y_3, y_4) = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$$

26. *Reduce the following (2 × n) game to (2 × 2) game by graphical method and hence solve.*

		B				
		I	*II*	*III*	*IV*	*V*
A	*I*	*2*	*–1*	*5*	*–2*	*6*
	II	*–2*	*4*	*–3*	*1*	*0*

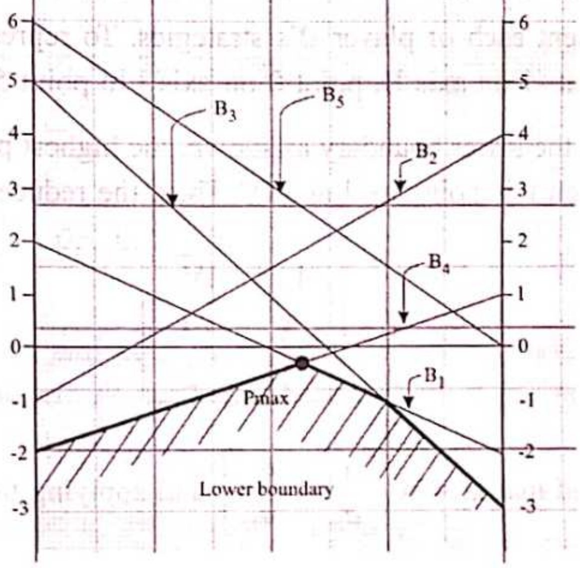
Solution:

The graph represent the pay–off lines, identify the lower boundary and locate the highest point P_{max} on it (Max. min principle). Selecting the strategies passing through this point we get the reduced matrix/game (2 × 2) as,

$$\begin{array}{ccrr} & & \multicolumn{2}{c}{B} \\ & & I & IV \\ A & I & 2 & -2 \\ & II & -2 & 1 \end{array}$$

On solving using the formulae we get, the probabilities of using the strategies for player 'A' are $\left(\frac{3}{7}, \frac{4}{7}\right)$, Probabilities of using the strategies I, II, III, IV, and V for player 'B'are $\left(\frac{3}{7}, 0, 0, \frac{4}{7}, 0\right)$

Value of game $V = -\frac{2}{7}$



Note: *Passing through $P_{max.}$ we have the strategies of $B{:}B_1, B_2$. Player 'A' has to choose two best strategies of player 'B' who is having 5 strategies. With respect to player 'A' Max min principle (he will try to maximize his minimum gains) is applicable. Hence, maximum point is selected in the lower boundary formed with the pay–off lines.*

27. Solve the following game by graphical method

		B				
		I	*II*	*III*	*IV*	*V*
A	*1*	*3*	*0*	*6*	*– 1*	*7*
	2	*– 1*	*5*	*– 2*	*2*	*1*

Solution:

The game does not have a saddle point. Hence it is probabilistic in nature. Player A is having two strategies while player B has three strategies. Player A has to choose two best strategies of B. let x_1, x_2 be two probabilities of using the I, II strategies respectively.

$x_1 . x_2 = 1$; $\quad$ $x_2 = 1 - x_1$ $\quad$ [sum of the probabilities is unity].

Step1*:* The given game is of 2 × 5 (one player is having 2 strategies), hence graphical method can be used.

Step 2*:* Draw two parallel lines to include the boundaries of two strategies of first player; say 'A' (Any convenient distance may be taken to draw parallel lines).

These two lines will represent two strategies available to the player "A".

Step3*:* Draw lines to represent each of player B's strategies. To represent the player B's strategy, join point 3 on axis I to point -1 on axis II, point 0 on axis I to point 5 on axis II and so on.

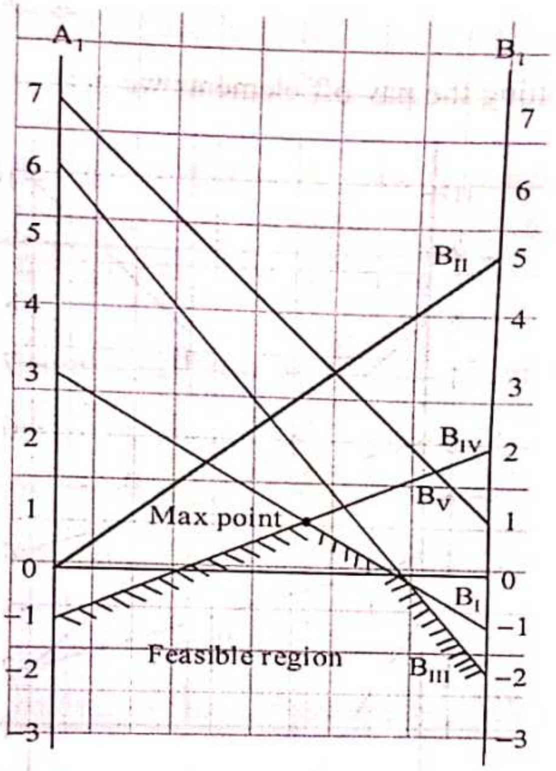
The feasible region is the lower boundary as shown, the highest point is shown by Max point. The strategies passing through this point are I and IV. Thus, the reduced game is,

	I	IV
1	3	– 1
2	– 1	2

Comparing the reduced matrix with $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and applying the following formulae

Let x_1, x_2 are the probabilities of using I, II strategies by player A.

Let y_1, y_2, y_3, y_4 are the probabilities of using I, II, III, IV strategies by player B.



$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \text{ and}$$

$$\text{Value of game } v = \frac{a_{11}a_{22} - a_{21}a_{22}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the above equations we get,

$$\text{Strategies of } A = \left(\frac{3}{4}, \frac{4}{7}\right)$$

$$\text{Strategies of } A = \left(\frac{3}{7}, \quad 0, \quad 0, \quad \frac{4}{7}, \quad 0\right), \; V = \frac{-1 \times 3 + 2 \times 4}{7 + 4} = \frac{5}{11}$$

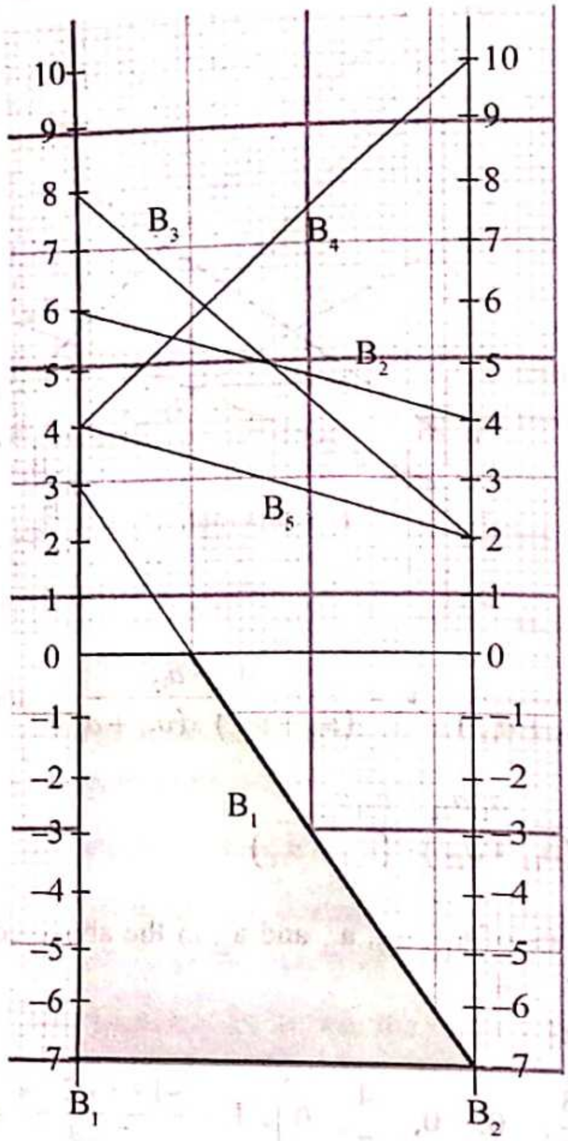
Some times only one strategy may pass through the lower boundary. In such cases no intersection point is obtained. In other words, the game will be of pure strategy. The following example illustrates such games.

28. *Solve the following game using graphical method.*

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & 6 & 8 & 4 & 4 \\ -7 & 4 & 2 & 10 & 2 \end{bmatrix} \end{array}$$

Solution:

With usual notations plotting the pay off elements we get,



From the plot only the strategy B_1 is located in the lower boundary. Hence, the game is reduced to single strategy game. In other words, the reduced game is

$$\begin{array}{cc} & B_1 \\ A_1 & \begin{bmatrix} 3 \\ -7 \end{bmatrix} \end{array}$$

By observation R_1 dominates R_2. Hence, deleting R_2 we get

$$\begin{array}{cc} & B_1 \\ A_1 & [\ 3\ ] \end{array}$$

Best strategy for player A : A_1

Best strategy for player B : B_1 and the value of game is 3.

Note: *When a game is reduced to single row or single column matrix, it is an indication that there exists saddle point and it involves pure strategies. Hence, direct solution will be obtained without using the arithmetic/algibraic (formula) approach.*

N × 2 games : The following examples illustrates solving N × 2 games.

29. Solve the following game using graphical method

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}\begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix}$$

Solution:

The game does not have a saddle point. Hence it is probabilistic in nature. Player B is having two strategies while player A has five strategies. Player B has to choose two best strategies of A. let y_1, y_2 be two probabilities of using the I, II strategies respectively.

$y_1 + y_2 = 1;$ $\qquad y_2 = 1 - y_1$ [sum of the probabilities is unity].

The given game is of 5 × 2 (one player is having 2 strategies), hence graphical method can be used.

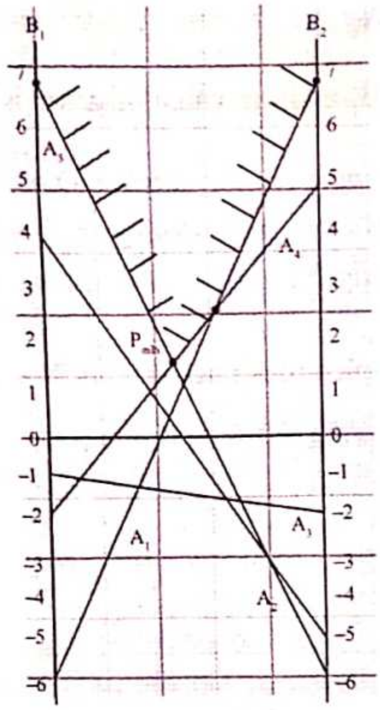
P_{min} is the lowest point in the upper boundary, passing through this point we have the strategies A_4, A_5.

The reduced game is

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_4 & -2 & 5 \\ A_5 & 7 & -6 \end{array}$$

Let x_1, x_2, x_3, x_4, x_5 be the probabilities of using I, II, III, IV, V strategies by player 'A' and y_1, y_2 be the probabilities of using I, II strategies by player B.

$x_1 = x_2 = x_3 = 0$, as they are eliminated from the graph.



By using the formulae we get,

$$x_4 = \frac{-6-7}{-8-12} = \frac{-13}{-20} = \frac{13}{20}, \; x_5 = 1 - x_4 = 1 - \frac{13}{20} = \frac{7}{20}$$

$$y_1 = \frac{-6-5}{-8-12} = \frac{-11}{-20} = \frac{11}{20}, \; y_2 = 1 - \frac{11}{20} = \frac{9}{20}$$

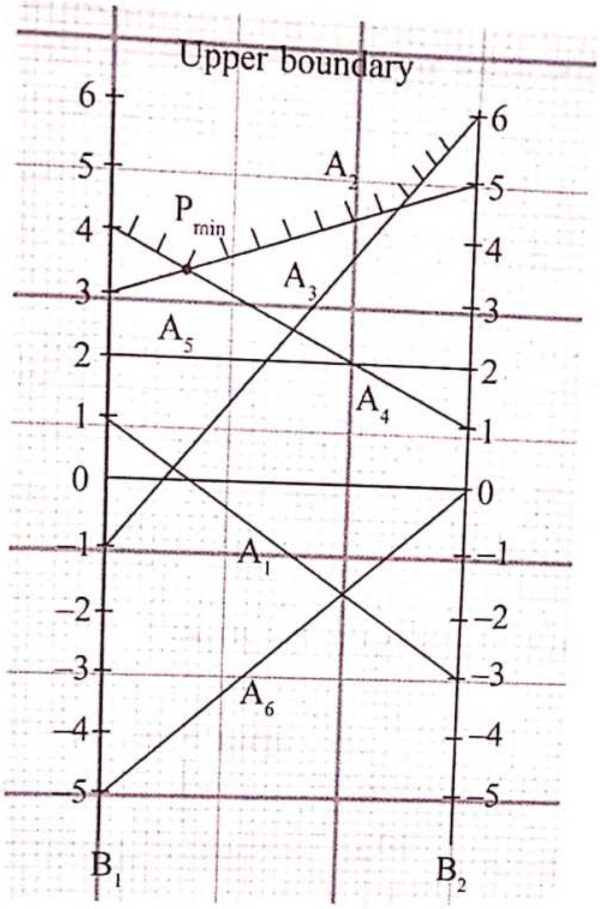
$$v = \frac{12-35}{-8-12} = \frac{-23}{-20} = \frac{23}{20}$$

30. ***Obtain the optimal strategies for both persons and the value of game whose pay-off matrix is as follows:***

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

Solution:

Assuming A_1, A_2, A_3, A_4, A_5 and A_6 as strategies for the player A, B_1 and B_2 for the player B; the plot is



P_{min} is the lowest point in the upper boundary, strategies passing through P_{min} are A_2 and A_4.

Hence, the reduced game is,

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_2 \\ A_4 \end{array} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

Let x_1, x_2, x_3, x_4, x_5 and x_6 be the probabilities of using I, II, III, IV, V and VI strategies by player 'A' and y_1, y_2 be the probabilities of using I, II strategies by player 'B'.

On solving the above reduced game we get,

$$x_2 = \frac{1-4}{4-9} = \frac{-3}{-5} = \frac{3}{5},\ x_4 = \frac{2}{5} \quad (x_4 = 1 - x_2) \quad (x_1 = x_3 = x_5 = x_6 = 0, \text{ as they are deleted})$$

$$y_1 = \frac{1-5}{4-9} = \frac{-4}{-5} = \frac{4}{5},\ y_2 = \frac{1}{5} \quad (y_2 = 1 - y_1)$$

$$v = \frac{3-20}{4-9} = \frac{-17}{-5} = \frac{17}{5}$$

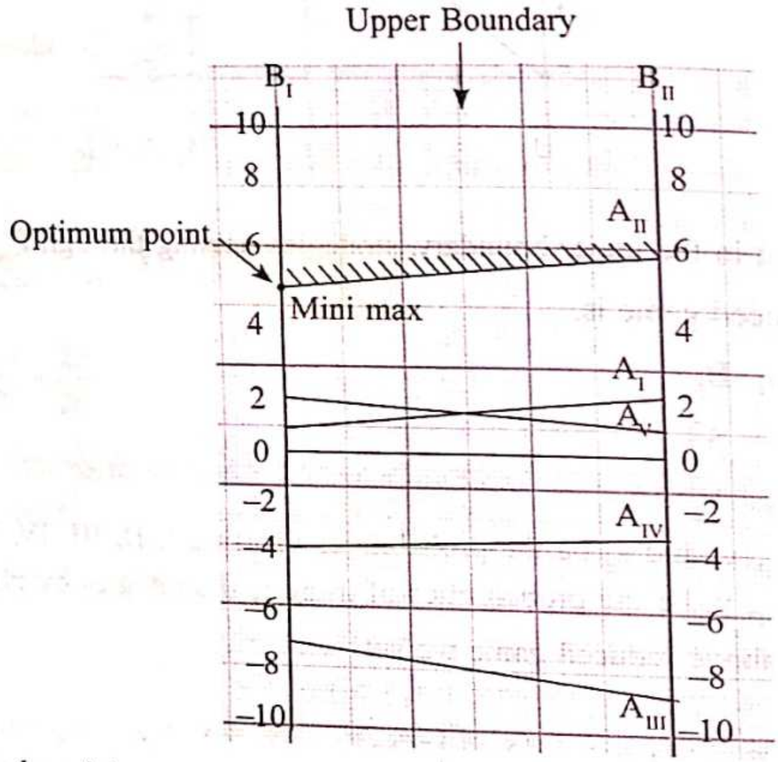
Some times only one strategy may pass through the upper boundary. In such cases no intersection point is obtained. In other words, the game will be of pure strategy. The following example illustrates such games.

31. *Using graphical method solve the following game.*

		B I	B II
A	1	1	2
	2	5	6
	3	– 7	– 9
	4	– 4	– 3
	5	2	1

Solution:

The pay off lines are drawn as shown in the graph



From the graph, the minimum point in the upper boundary is shown by Mini max. This point is optimum point and corresponds to II strategy. Hence, the game is reduced to a single element. In other words the game is having saddle point and hence is deterministic in nature

Strategy of A (0, 1, 0, 0, 0),Strategy of B (1, 0) and ,Value of the game= 5

Note:

i. *When game is 2 × M, select P_{max} that is a highest point in the lower boundary.*

ii. *When game is N × 2, select P_{min} that is lowest point in upper boundary.*

iii. ***When more than 2 strategies are passing through P_{max} or P_{min} then any two strategies passing though the point can be considered to solve the game. (The will be having alternative optimal solution). The game will have alternative optimal solution.***

Some times it may be given to reduce games of higher order to $2 \times M$ *or* $N \times 2$ and then it may be required to solve by graphical method. The following example illustrates this kind of problems.

32. *Use the dominance rule to reduce the following game either to $2 \times M$ or $N \times 2$ game and then solve by graphical method*

$$\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{B} \\
 & & I & II & III & IV \\ \hline
 & I & 19 & 6 & 7 & 5 \\ \hline
A & II & 7 & 3 & 14 & 6 \\ \hline
 & III & 12 & 8 & 18 & 4 \\ \hline
 & IV & 8 & 7 & 13 & -1 \\ \hline
\end{array}$$

Solution:

The given game is of M × N matrix let us reduce it either 2 × M or N × 2 by using dominance principle then graphical method can be used to reduce it to 2 × 2 game.

All the elements of row III ≥ row IV.

Hence row III is superior, delete row IV.

	I	II	III	IV
I	19	6	7	5
II	7	3	14	6
III	12	8	18	4

All the elements of column II ≤ column I, column II is superior, discard column I.

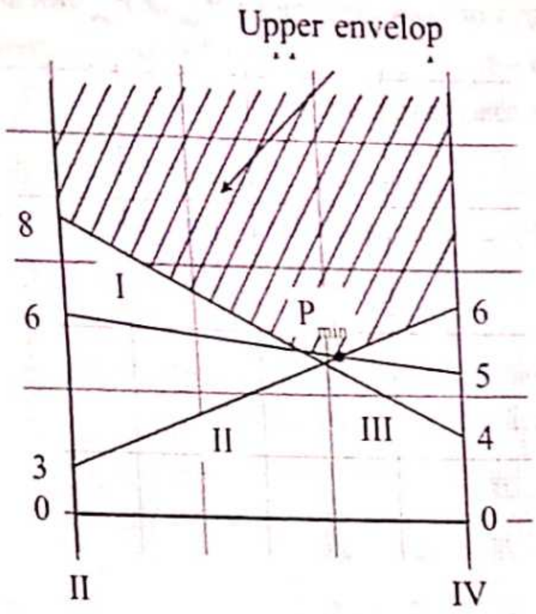
	II	III	IV
I	6	7	5
II	3	14	6
III	8	18	4

→

	II	IV
I	6	5
II	3	6
III	8	4

(All the elements of column II ≤ column III, column II is superior, discard column III.)

The game is reduced to 3×2 for which graph is plotted



From the graph we get,

		B II	B IV
A	I	6	5
	II	3	6

The above game is 2×2 with out saddle point.

Let x_1, x_2, x_3, x_4 are the probabilities of using I, II, III, IV strategies by player A.

Let y_1, y_2, y_3, y_4 are the probabilities of using I, II, III, IV strategies by player B. Solving by using the formulae we get,

$$x_1 = \frac{6-3}{4} = \frac{3}{4}, \quad x_2 = \frac{1}{4}$$

$$x_3 = x_4 = 0, \qquad (\because x_1 + x_2 + x_3 + x_4 = 1)$$

$$y_2 = \frac{6-5}{4} = \frac{1}{4}, \quad y_4 = \frac{3}{4}$$

$$y_1 = y_3 = 0, \qquad (\because y_1 + y_2 + y_3 + y_4 = 1)$$

$$v = \frac{6.6 - 5.3}{(6+6)-(5+3)} = \frac{36-15}{12-8} = \frac{21}{4}$$

Some games may have P_{min} or P_{max} at same level in the graph. In such cases the game will have an alternative optimal solution. The following example illustrates this kind of problems.

33. *Solve the following game by graphical method whose pay off matrix to 'A' is given,*

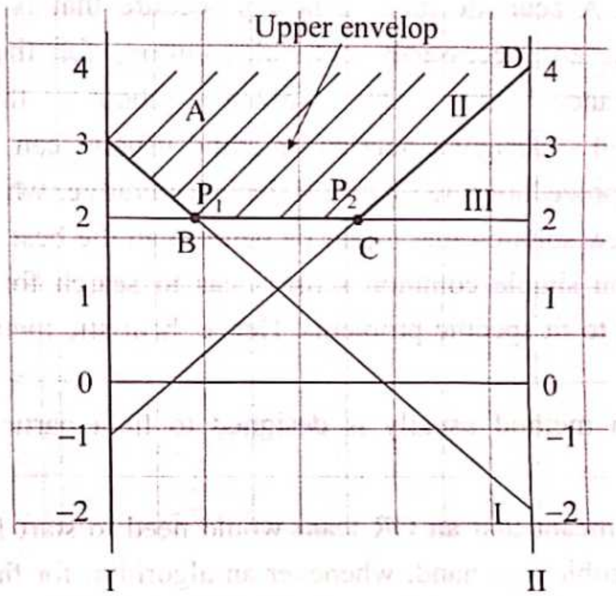
$$\begin{array}{cc} & B \\ & \begin{array}{ccc} I & II & III \end{array} \\ A \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

Solution:

To use graphical method the game should be either 2 × M or N × 2 type. By inspection column I dominates the column III. [As all the elements of column I ≤ column III].

Hence, the reduced matrix is

$$\begin{array}{cc} & B \\ & \begin{array}{cc} I & II \end{array} \\ A \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 3 & -2 \\ -1 & 4 \\ 2 & 2 \end{bmatrix} \end{array}$$



ABCD is the upper boundary, the lowest point in the upper boundary is same/ repeating at B, C. Hence, the reduced game is

$$\begin{array}{cc} & \begin{array}{cc} I & II \end{array} \\ A \begin{array}{c} I \\ III \end{array} & \begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} \end{array} \quad \text{or} \quad \begin{array}{cc} & \begin{array}{cc} I & II \end{array} \\ A \begin{array}{c} II \\ III \end{array} & \begin{bmatrix} -1 & 4 \\ 2 & 2 \end{bmatrix} \end{array}$$

The game is having alternative optimal solution, solving the first reduced game by using the formula, the optimal strategies and probabilities are A (I, II, III) = (4/5, 0, 1/5) and for B (I, II, III) = (4/5, 1/5, 0). The value of game v = 2.

5.13 METAHEURISTICS

Metaheuristics: In computer science and mathematical optimization, a metaheuristic is a higher-level procedure or heuristic designed to find, generate or select a heuristic (partial search algorithm) that may provide a sufficiently good solution to an optimization problem, especially with incomplete or imperfect information or limited computation capacity.

By using various methods, a OR model is solved in two stages namely.

i. ***Obtaining feasible solution and***

ii. ***Obtaining the optimal solution***

Some problems are so complicated that it may not be possible to solve for an optimal solution. Sometimes, the methods used may be invalid, for addressing a wide variety of practical problems. In such situations, it is still important to find a good feasible solution, that is at least close to the optimal solution (reasonably acceptable). Heuristic methods are commonly used to search for such a solution. A heuristic method is a procedure that is likely to discover a very good feasible solution, but not necessarily optimal solution, for the specific problem being considered. No assurance can be given about the quality of the solution obtained, but a well – designed heuristic method usually can provide a solution that is atleast nearly optimal. The procedure often is a full fledged iterative, where each iteration involves conducting a search for a new solution that might be better than the best solution found previously. These methods are based on simple common sense ideas to search for a best possible solution. These need not be designed to fit specific problems. Hence, heuristic methods can be considered to be and not in nature.

In other words, each method usually is designed to fit a particular problem rather than generalization.

For many years, this meant that an OR team would need to start from scratch to develop a heuristic method to fit the problem at hand, whenever an algorithm for finding an optimal solution was not available. This has changed in recent years with the development of powerful metaheuristics. A metaheuristics is a general solution method that provides both a general structure and strategy guidelines for developing a specific heuristic method to fit a particular kind of problem.

Metaheuristics have become one of the most important techniques in the toolkit of OR practitioners.

Three most commonly used metaheuristics are,

i. ***Tabu search*** ii. ***Simulated annealing and*** iii. ***Genetic algorithms***

These are discussed in detail in this chapter.

A non linear programming problem with multiple local optima can also be solved by using the concept of heuristics.

5.14 Applications of Metaheuristics

Metaheuristics may make few assumptions about the optimization problem being solved and so they may be usable for a variety of problems.

i. *Metaheuristics are used for combinatorial optimization in which an optimal solution is sought over a discrete search-space. An example problem is the travelling salesman problem where the search-space of candidate solutions grows faster than exponentially as the size of the problem increases, which makes an exhaustive search for the optimal solution infeasible.*

ii. *Used to tackle thye Multidimensional combinatorial problems, including most design problems in engineering such as form-finding and behavior-finding, suffer from the curse of dimensionality, which also makes them infeasible for exhaustive search or analytical methods.*

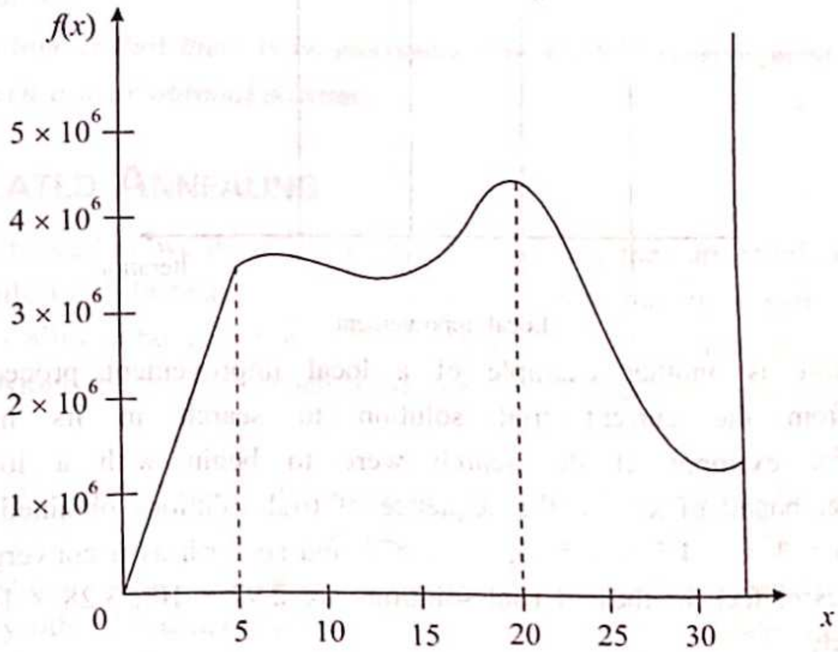
iii. *Metaheuristics are also widely used for job shop scheduling and job selection problems.*

34. *Max* $f(x) = 12x^5 - 975x^4 + 28{,}000\,x^3 - 345{,}000\,x^2 + 1{,}800{,}000\,x$

Subject to $0 \le x \le 31$

Solution:

The given problem is non linear programming problem having multiple local optima.

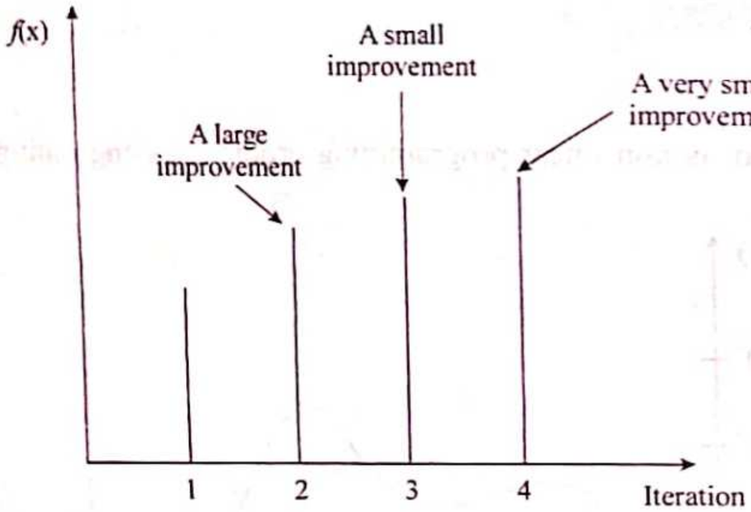


From the plot, the objective function f(x) over the feasible values of the single variable x, we can see that the problem has three local optima, one at x = 5, another at x = 20, and the third at x = 31, where the global optima is at x = 20.

The objective function f(x) is sufficiently complicated that it would be difficult to determine where the global optimum lies. Calculus could be used, but this would require solving a polynomial equation of the higher degree to determine where the critical points lie. It would be even difficult to ascertain f(x) with multiple local optima rather than just a global optimum.

For nonlinear programming problems of this type, a simple heuristic method is to conduct a local improvement procedure. Such a procedure starts with an initial trial solution and then, at each iteration searches in the neighborhood of the current trial solution to find a better trial solution. This process continues till it is not possible to improve the trial solution further. This procedure can be viewed as a hill climbing procedure that keeps climbing higher on the plot of the objective function, assigns the objective is minimization until it is essentially reaches the top of the hill. Figure shows a typical sequence of values of f(x) that would be obtained by such a local improvement procedure.

As studied in mathematics, the given problem having only one variable, bisection / regula falsi method can also be used to this particular problem.



Local improvement

This procedure is another example of a local improvement procedure, since each iteration starts from the current trial solution to search in its neighborhood for better solution. For example, if the search were to begin with a lower bounds for x = 0 and an upper bound of x = 6, the sequence of trial solutions obtained by the bisection method would be x = 3, x = 4.5, x = 5.25, x = 4.875 and so forth as it converges to x = 5. The corresponding values of f(x) for these 4 trial solutions are 2.97×10^6, 3.28×10^6, 3.3×10^6 and 3.3×10^6 respectively.

Thus, the second iteration provides a relatively large improvement over the first one, the third iteration gives a considerably small improvement, and the fourth iteration yields a very small improvement.

This is shown in the Figure.

With the bisection method, we may conclude that at x = 5 (local optima) the optimum solution is obtained. So it never would find the global optimum at x = 20. Thus, bisection method improves the current trial solutions within the local neighborhood of those solutions once it climbs the top of the hill, it will stop because it cannot climb any higher within the local neighborhood of the trial solution at the top of the hill. This indicates the drawback of any local improvement procedure like bisection method.

5.15 Nature of Metaheuristics

i. ***Metaheuristics is a general kind of solution method that orchestrates the interaction between local improvement procedures and higher level of strategies to create a process that is capable of escaping from local optima and performing a robust search of the feasible region.***

ii. ***A key feature of the metaheuristic is its ability to escape from a local optimum.***

iii. ***After reaching (or nearly reaching) a local optimum, different metaheuristics execute this escape in different ways.***

iv ***However, a common characteristic is that the trial solution that immediately follow a local optimum are allowed to be inferior to this local optimum.***

v. ***The advantage of a well–designed mataheuristic is that it tends to move relatively quickly toward very good solutions, so it provides a very efficient way of dealing with large and complicated problems.***

vi. ***The disadvantage is that there is no guarantee that the best solution found will be optimum solution or even a near optimal solution.***

5.16 Simulated Annealing

Local improvement procedure we described starts with climbing the current hill and coming down from it to find the tallest hill. Instead, simulated annealing focuses mainly on searching for the tallest hill. Since the tallest hill can be anywhere in the feasible region, the emphasis is on taking steps in random directions. Along the way, we reject some, but not all steps that would go downward rather than upward.

Since most of the accepted steps are going upward, the search will gradually gravitate toward those parts of feasible region containing the tallest hills. Therefore, the search process gradually increases the emphasis on climbing upward by rejecting an increasing proportion of steps that go downward. Like any other local improvement procedure, simulated annealing moves from current solution to an immediate neighbor in the local neighborhood of this solution.

How is an immediate neighbor is selected? Let, Z_c = objective function value for the current trial solution. Z_n = objective function value for the current candidate to be the next trial solution. T = a parameter that measures the tendency to accent the current candidate to be the next trial solution if this candidate is not an improvement on the current trial solution.

Move selection rule

i. *Among all the immediate neighbors of the current trial solution, select one randomly to become the current candidate to be the next trial solution.*

ii. *Assuming the objective is maximization of the objective function, accept or reject this candidate to be the next trial solution as per the following rule.*

  a. If $Z_n >= Z_c$ always accept this candidate.

  b. If $Z_n <= Z_c$ accept the candidate with the following probability: pro {acceptance} $= e^X$, where $X = \frac{(Z_n < Z_c)}{T}$

iii. ***If the candidate is rejected, repeat the process with a new randomly selected immediate neighbor of the current trial solution.***

iv. ***If no immediate neighbor remain, terminate the algorithm.***

v. *T = a parameter that measures the tendency to accept the current candidate to be the next trial solution if this candidate is not an improvement on the current trial solution.*

Outline of the basic simulated annealing algorithm

i. **Initialization** *Start with a feasible initial trial solution.*

ii. **Iteration** *Use the move selection rule to select the new trial solution. If none of the immediate neighbors of the current trial solution are accepted, the algorithm is terminated.*

iii. **Check the current temperature schedule** *When the desired number of iterations have been performed at the current value of T, decrease T to the next value in the temperature schedule and resume performing iterations at this next value.*

iv. **Stopping rule** *When the desired number of iterations have been performed at the smallest value of T in the temperature schedule,* **stop.** *Algorithm is also stopped when none of the immediate neighbors of the current trial solution are accepted.*

v. *Accept the best trial solution found at any iteration (including for larger values of T) as the final solution.*

5.17 The Traveling Sales Man Problem (Routing Problem)

Suppose a salesman has to visit many cities, needs to start from a particular city, visit each city once, and then return to his starting point. The objective is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized. Starting from a given city the salesman will have a total of (n – 1)! different sequences (possible round trips).

If number of cities is only two, obviously there is no choice. If number of cities become three say P, Q and R one of them (say P) is the home base, then there are two possible ways (or) routes.

$P \rightarrow Q \rightarrow R$ and $P \rightarrow Q \rightarrow R$

For 4 cities P, Q, R and S there are 3! = 6 possible routes!

If it is having 6 cities,

$5 \times 4 \times 3 \times 2 = 120$

5! = 120 routes

Thus, it is practically impossible to find the best route by trying each one.

In general, if there are 'n' cities, there are (n – 1)! possible routes. Travelling sales man problem gives the best route without trying each one.

Applications of travelling salesman problem

i. *Postal deliveries*

ii. *Cable connections*

iii. *Inspection*

iv. *School bus routes*

Formulation of travelling sales man problem

Suppose c_{ij} is the cost/inspection distance or time from city i to city j and $x_{ij} = 1$ if the sales man goes directly from city i to city j; and $x_{ij} = 0$ otherwise. Then minimum $\Sigma x_{ij} c_{ij}$ with the additional restriction that the x_{ij} must be chosen that no city is visited twice before the tour of all cities is completed. In particular, he cannot go directly from city i to j itself. This possibility may be avoided adopting the convention $c_{ij} = \infty$.

The distance (or cost/time) matrix the problem is given by

$$
\begin{array}{cc|cccc|}
 & & & \text{To} & & \\
 & & C_1 & C_2 & \ldots\ldots & C_n \\
\hline
 & C_1 & \infty & C_{12} & \ldots\ldots & C_{1n} \\
 & C_2 & C_{21} & \infty & \ldots\ldots & C_{2n} \\
\text{From} & : & & & & \\
 & : & & & & \\
 & : & & & & \\
 & C_n & C_{n1} & C_{n2} & \ldots\ldots & \infty \\
\hline
\end{array}
$$

Further, since the salesman has to visit all the n cities (C_1, C_2, C_3 —— C_n) the optimal solution remains independent of selection of the starting point.

Example of Travelling Salesman Problem (Simulated annealing)

Simulated annealing quickly gives a reasonable answer close enough to the true minimum path for practical purposes. It starts with the cities connected in a random order, and then considers making random changes in that order. If changing the order of cities leads to a shorter path, we accept that change. If it results a longer path, we give a certain probability of accepting the modification less likely the larger and proposed increase in path length. We then gradually reduce this probability over time, in order to rule out shorter path and path length increases – there by converging toward a path length close to the absolute minimum.

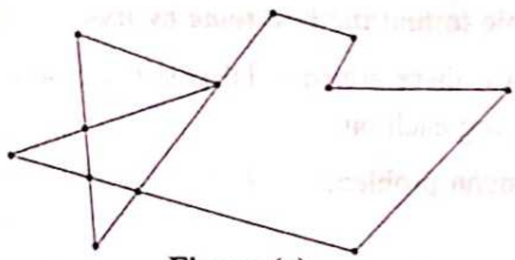


Figure (a)

Let us consider a route with 10 tourism places, the normal coverage of all the cities by a tourist is,

The modified routing after applying simulated annealing concept is shown in following figure.

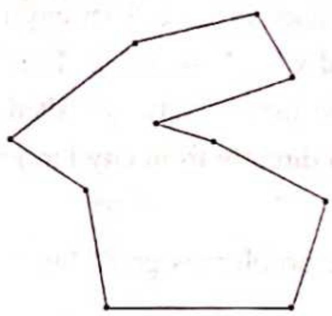


Figure (b)

Conclusion

We can notice that, the route represented in Figure (b) is optimum (shorter) compared to route shown in Figure (a).

Steps in Obtaining Optimization

i. ***Initial trial solution:*** *we may enter any feasible solution (sequence of cities on the tour), perhaps by randomly generating the sequence. It might be helpful to enter a good feasible solution as initial trial solution. Example 1 – 2 – 3 – 4 – 5 – 6 – 7 – 1.*

ii. *Neighborhood structure: An immediate neighbor of the current trial solution is one that is reached by making a sub–tour reversed.*

iii. *We must, however, rule out the sub–tour reversal that simply reverses the direction of tour provided by the current trial solution.*

iv. *Random selection of immediate neighbor: Selecting the sub–tour to be reversed requires selecting the slot in the current sequence of cities where the sub–tour currently begins and then the slot where the sub–tour currently ends.*

v. *The ending slot must be somewhere after the beginning slot, excluding the last slot.*

vi. *We can use random numbers to give equal probabilities to selecting any of the eligible beginning slots and then any of the eligible ending slots.*

vii. *If this selection turns out to be infeasible, then the process is repeated until a feasible selection is made.*

5.18 Tabu Search

Tabu search begins by using a local search procedure as a local improvement procedure in the usual sense to find the local optimum. That is, initially, we usually allow only improving solutions.

As with simulated annealing, the strategy in tabu search is that it continues the search by allowing non–improving moves to the best solutions in the neighborhood of the local optimum solution.

Once a point is reached where better solutions can be found in the neighborhood of the current trial solution, the local improvement procedure can be re–applied to find the new local optimum.

This version of local improvement procedure is sometimes referred to as steepest ascent / mildest descent approach. Each iteration selects the available move that goes furthest up the hill, or, when an upward move is not available, selects a move that drops least down the hill.

The danger with this approach is that after moving away from a local optimum, the process will cycle right back to the same local optimum. To avoid this, a tabu search temporarily forbids moves that would return to (and perhaps toward) a solution previously visited. A tabu list records these forbidden moves, which are referred to as tabu moves.

The only exception to forbidding such a move is if it is found that a tabu move actually is better than the best feasible solution found so far. The use of memory to guide the search by using tabu lists to record some of the recent history of the search is a distinctive feature of tabu search.

This feature comes from artificial intelligence.

Outline of a basic tabu search algorithm

i. *Initialization*

ii. *Start with a feasible initial trial solution*

iii. *Iteration*

iv. *Use an appropriate local search procedure to define the feasible moves into the local neighborhood of the current solution.*

v. *Eliminate from consideration any move on the current tabu list unless the move would result in a better solution that the best trial solution found so far.*

vi *Determine which of the remaining moves provides the best solution.*

vii *Adopt this solution as the next trial solution, regardless of whether it is better or worse than the current trial solution.*

viii. *Update the tabu list to forbid cycling back to what had been the current trial solution.*

ix. *If the tabu list already had been full, delete the oldest member of the tabu list to provide more flexibility for future moves.*

Stopping rule

i. *Use any stopping criteria, such as fixed number of iterations, a fixed amount of CPU time, a fixed number of consecutive iterations without an improvement in the best objective function value.*

ii. *Also stop at any iteration where there are no feasible moves in the local neighborhood of the current trial solution.*

iii. *Accept the best trial solution found an any iteration as the final solution.*

These steps only provide a generic structure and strategy guidelines for developing a specific heuristic method to fit a specific situation.

Example of Travelling Salesman Problem

The classical routing problem (Travelling Salesmen Problem) is often used to understand the functionality of tabu search. The routing problem involves finding a sequence of travel between cities, such that the distance travelled is minimum. Tabu search can be used to find a satisfying solution for this problem. Tabu search starts with an initial solution, which can be generated according to the nearest neighbor algorithm. To obtain new solutions, the order that two cities are visited is swapped. The distance for the total travel between all the cities is used to judge how better one solution compared to another. New solutions continue to be created until some stopping rule is met. After stopping, the best solution is one with shortest distance for the total travel covering all the cities.

Steps in Obtaining Optimization

i. *Local search algorithm*

At each iteration, choose the best immediate neighbor of the current trial solution that is not ruled out by the tabu status.

ii. *Neighbourhood structure*

An immediate neighbor of the current trial solution is one that is reached by making a sub tour reversal. Such a reversal requires adding two links and deleting two other links from the current trial solution.

iii. ***Form of tabu moves***

List the links such that a particular sub tour reversal would be tabu if both links to be deleted in this reversal are on the list.

iv. ***Addition of tabu move***

At each iteration, after choosing the two links to be added to the current solution, also add these two links to the tabu list.

v. ***Maximum size of the tabu list***

Four (two from each of the two most recent iterations). Whenever, a pair of links is added to a full list, delete the two links that already have been on the list the longest.

vi ***Stopping rule***

Stop if after three consecutive iterations there is no improvement in the best objective function value. Also stop at any iteration where the current solution has no immediate feasible neighbor.

5.19 GENETIC ALGORITHMS

Just as simulated annealing is based on a physical phenomenon (the physical annealing process), genetic algorithms are completely based on another natural phenomenon. In this case, the analogy is the biological theory of evolution formulated by Charles Darwin (in mid 19th century).

Each species of plant and animals has great individual variance.

Darwin observed that those individuals with variations that import a survival advantage through improved adaptation to the environment are more likely to survive to the next generation. This phenomenon is popularly known as survival of the fittest. Modern researches in the field of genetics provide explanation of the process of evolution and the natural selection involved in the survival of fittest.

In any species that reproduces by sexual reproduction, each offspring inherits some of the chromosomes of each of the two parents, where the genes within the chromosomes determine the individual features of the child. A child who happens to inherit the better features of the parents is slightly more likely to survive into adulthood and become a parent who passes of these features to the next generation. The population tends to improve slowly over time by this process. A second factor that contributes to this process a random, low level mutation rate in the DNA of the chromosomes.

Thus, a mutation occasionally occurs that changes the features of a chromosomes that a child inherits from a parent. Although most mutations have no impact or are disadvantageous, some mutations provide desirable improvement. Children with desirable mutations are slightly more likely to survive and contribute to the future gene pool of the species. These ideas transfer over to dealing with optimization problem in a rather natural way.

Feasible solutions for a particular problem correspond to members of a particular species, where fitness of each member is measured by the value of the objective function. Rather than processing in a single trial solution at a time (as we did for simulated annealing and tabu search), we now work with an entire population of trial solutions. For each iteration (generation) of a genetic algorithm, the current population consists of the set of trial solutions currently under consideration.

Although the analogy of the process of biological evolution defines the core of any genetic algorithm, it is not necessary to adhere rigidly to this analogy in every detail. For example, some genetic algorithms allow the same trial solutions to be a parent repeatedly over multiple generations (iterations). Thus, the analogy needs to be only a starting points for defining the details of the algorithms to best fit the problem under considerations.

5.19.1 Outline of a basic genetic algorithm

Initialization

Start with initial population of feasible trial solutions, perhaps by generating them randomly. Evaluate the fitness – the objective function value – for each member of this current generation.

Iteration

i. *Use a random process that is biased towards more fit members of the current population to select some of its members to become parents.*

ii. *Pair up the parents randomly and then have each pair of parents give birth to two children – new feasible solutions – whose features (genes) are a random mixture of the feature of the parents.*

iii. *What is the random mixture of features and/or any mutations result in an infeasible solution?*

iv. *These cases are miscarriages – so the process of attempting to give birth is repeated until a child is born that corresponds to a feasible solution.*

v. *Retain the children and enough of the best members of the current population to form the new population of the same size of the next iteration.*

vi We discard the other members of the population.

vii Evaluate the fitness for each new member (the children) in the new populations.

Stopping rule

i. *Use some stopping rule, such as a fixed number of iterations, a fixed amount of CPI time, or a fixed number of consecutive iterations without any improvement in the best trial solutions found so far.*

ii *Use the best trial solution found on any iteration as the final solution.*

Example of Travelling Salesman Problem

Many practical applications can be modeled as a travelling salesmen problem 'or as its' variants.

For example, cable routing, milk truck routing, routing of city/school buses etc., need powerful algorithms for effective planning. Tremendous progress has been made with respect to solving travelling salesman problem.

The constraints in travelling salesmen problem are,

i. *The person should come back to the home city after visiting once and only once all the cities / places assigned.*

ii. *The salesmen can only be in one city at a time.*

Steps in Obtaining Optimization

i. *Our example is probably too simplistic it only has about 10 distinct feasible solutions (if we don't consider sequences in reverse order separate). Hence population size of 10 in such a case won't be possible!*

ii. *We represent the solution by just the sequence in which cities are represented. However, in most of the application of GA, typically the members of populations are coded so that it is easier to generate children, create mutation etc.*

iii. *First task is then to generate population for the initial generation.*

iv. *Starting with home base city (1), random numbers are used to select the next city from amongst those that have a link to the city 1.*

v. *Same process is repeated to select the subsequent cities that would be visited in this tour (member).*

vi. *We stop if all the cities are visited and we are back to the home base city.*

vii. *Or we reach a dead end (because there is no link from the current city to any of the remaining cities that are still not in the tour). In this case, we start the process all over again.*

Limitations of Standard Genetic Algorithm

The following problems have to be addressed to use a standard genetic algorithm.

i. *A binary representation for tours is found such that it can be easily translated into a chromosome.*

ii. *An appropriate fitness function is to be designed, taking the constraints into consideration.*

Genetic algorithms can generate some chromosomes that do not represent valid solutions due to random initialization step of the genetic algorithms.

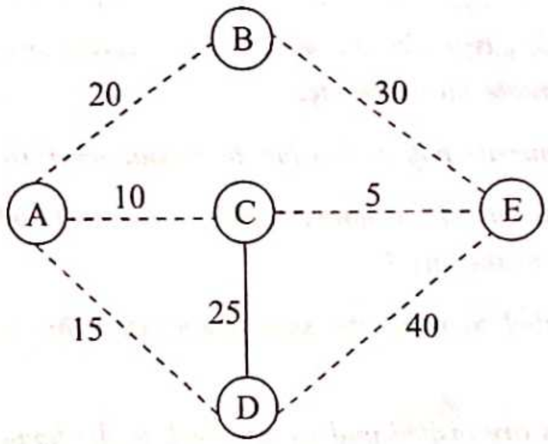
Two tours including the same places in the same order but with different starting points / directions are represented by different matrices and hence by different chromosome.

A proper fitness function is obtained using penalty function method to enforce the constraints.

But, poor results may be obtained by ordinary genetic operators.

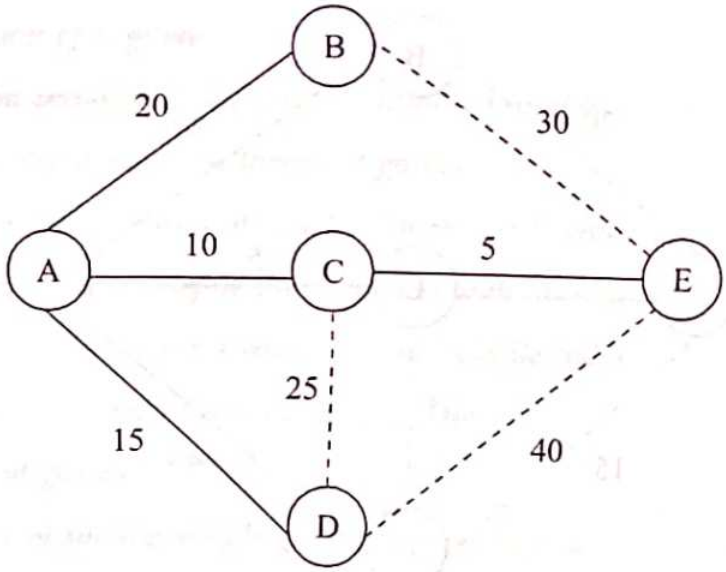
A minimum spanning tree problem with constraints

35 *Use Tabu search algorithm to find the optimal solution of the following illustration.*
Constraint 1: *Link AD can be included only if link DE also included*
Constraint 2: *At most one of the three links AD, CD and AB can be included. Charge a penalty of ₹100 if constraint 1 is violated. Charge a penalty of ₹ 100 if two of the three links specified in constraint 2 are included. Increase this penalty to ₹200 if all three of the links are included*



Solution:

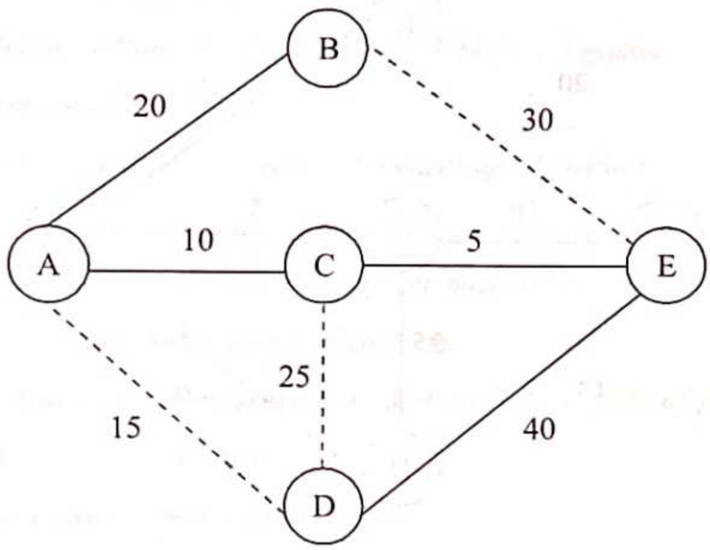
Iteration – 1



Cost = 20 + 10 + 15 + 200 (200 is a penalty as per the given constraints)

= ₹ 250

Iteration – 2



Cost = 20 + 10 + 5 + 40

= ₹ 75